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DYNAMICS OF SLOWLY GROWING ENTIRE FUNCTIONS

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Dedicated to George Szekeres on his 90th birthday

For a transcendental entire function f let $M(r)$ denote the maximum modulus of $f(z)$ for $|z| = r$. Then $A(r) = \log M(r)/\log r$ tends to infinity with r. Many properties of transcendental entire functions with sufficiently small $A(r)$ resemble those of polynomials. However the dynamical properties of iterates of such functions may be very different. For instance in the stable set $F(f)$ where the iterates of f form a normal family the components are preperiodic under f in the case of a polynomial; but there are transcendental functions with arbitrarily small $A(r)$ such that $F(f)$ has nonpreperiodic components, so called wandering components which are bounded rings in which the iterates tend to infinity. One might ask if all small functions are like this.

A striking recent result of Bergweiler and Eremenko shows that there are arbitrarily small transcendental entire functions with empty stable set—a thing impossible for polynomials. By extending the technique of Bergweiler and Eremenko, an arbitrarily small transcendental entire function is constructed such that F is nonempty, every component G of F is bounded, simply-connected and the iterates tend to zero in G. Zero belongs to an invariant component of F, so there are no wandering components. The Julia set which is the complement of F is connected and contains a dense subset of "buried" points which belong to the boundary of no component of F. This behaviour is impossible for a polynomial.

1. INTRODUCTION AND RESULTS

For an entire function f, which will always be assumed to be different from a linear polynomial, we denote the n-th iterate by f^n, $n \in \mathbb{N}$, and the Fatou set, which is the maximal open set in which $\{f^n\}$ is a normal family, by $F(f)$. The complement of $F(f)$ in $\mathbb{C}$ is the Julia set in $J(f)$. The Julia set is non-empty, perfect and has the property of complete invariance under f, that is $z \in J(f)$ if and only if $f(z) \in J(f)$, see [**15**].

$M(r, f) = \max\{|f(z)| : |z| = r\}$ measures the growth of f. If $\log M(r, f) = O(\log r)$ as $r \to \infty$, then f is a polynomial. Since $\log M(r, f)$ is a convex increasing function of $\log r$ the claim that there are transcendental entire functions of arbitrarily slow growth with a property P is the assertion that for a given function $A(r)$ which is positive and

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increases to ∞ as $r \to \infty$ on $\mathbb{R}_+$, there is a transcendental entire function such that f has the property P and satisfies

$$\log M(r, f) < A(r) \log r, \quad r > r_0. \tag{1}$$

Sufficiently slowly growing transcendental entire functions share many properties with polynomials, for example such functions are large outside small neighbourhoods of their zeros (see for example, [**16**]). Thus it is natural to compare the iteration theory for these two classes. We survey some results of this comparison and add to their number in Theorem 1.

For a polynomial f the Fatou set $F(f)$ has a component F_∞ which includes a neighbourhood of ∞, and is completely invariant, multiply-connected in $\mathbb{C}$, where $f^n \to \infty$. All other components of $F(f)$ are simply-connected and their union together with $J(f)$ is called the filled Julia set $K(f)$ $(= F_\infty^c)$. Thus $K(f)$ is bounded. For transcendental entire functions, however, J is unbounded by Picard's theorem.

Multiply-connected Fatou components. The maximum modulus theorem implies that in any multiply-connected component G of $F(f)$ the iterates $f^n \to \infty$. Examples of transcendental entire functions with this property were constructed in [**2**] and it was shown later, in [**4**], that G could be assumed bounded.

Wandering Fatou components. For each component G of $F(f)$ the image $f^n(G)$ lies in some component G_n. If all G_n are different, G is a "wandering component", otherwise G is "preperiodic". The behaviour of iterates in a preperiodic component falls into a small number of reasonably well-understood cases. For this reason Sullivan's proof [**25**] that for rational functions there are no wandering components gave a great impetus to rational dynamics. For transcendental entire functions, however, wandering components can occur, and occur among the examples of [**2, 4**] above.

Theorem A. [**3, 4, 6, 7, 17**] *If f is a transcendental entire function and G is a multiply-connected component of $F(f)$ then G is bounded and wandering, and $f^n \to \infty$ in G. There are examples of arbitrarily slow growth.*

In Theorem A, if G_n is the Fatou component which contains $f^n(G)$, then, for large n, G_{n+1} is contained in the unbounded component of $(G_n)^c$.

For transcendental entire functions many other types of wandering components may occur. For simply-connected types see for example, [**6, 14**].

Bounded Fatou components. For small transcendental entire functions the argument of [**5**, pp 493-4] shows:

Theorem B. *If f is a transcendental entire function and for some $1 < p < 3$,*

$$\log M(r, f) = O\{(\log r)^p\} \quad \text{as} \quad r \to \infty, \tag{2}$$

then there is a sequence R_n such that $R_{n+1} = M(R_n, f)$, which tends to ∞ as $n \to \infty$, and no component of $F(f)$ can meet both $C(0, R_n)$ and $C(0, R_{n+1}^3)$, where $C(0, r) = \{z : |z| = r\}$.

Thus all Fatou components of f are bounded. Stallard [25] has improved (2) to

$$\log\log M(r,f) < \frac{(\log r)^{1/2}}{(\log\log r)^c}, \quad c \in (0,1), \quad r > r_0$$

and further improvements are possible if one puts regularity conditions on the growth [1, 23].

FUNCTIONS WITH $J = \mathbf{C}$. For polynomials $J \neq \mathbf{C}$. Fatou [15] conjectured that $J(e^z) = \mathbf{C}$, first proved by Misiurewicz [19]. Can $J(f) = \mathbf{C}$ hold for f of arbitrarily small growth? Recently Bergweiler and Eremenko gave the answer.

THEOREM C. [10] *There are transcendental entire functions of arbitrarily slow growth such that* $J(f) = \mathbf{C}$.

EXAMPLES WITH $J \neq \mathbf{C}$. The only very small functions whose iteration has been discussed seem to be those constructed to prove Theorems A and C (Theorem B is proved by contradiction arising from the existence of an unbounded component). This raises the question: Are there transcendental entire functions of arbitrarily slow growth such that $J \neq \mathbf{C}$ and all Fatou components are simply-connected, or more generally such that ∞ is not a limit function of iterates in any Fatou component?

THEOREM 1. *There is a transcendental entire function of arbitrarily slow growth such that*

(i) *zero is an attracting fixed point (so* $J \neq \mathbf{C}$*),*
(ii) *every component* G *of* $F(f)$ *is simply-connected and bounded,*
(iii) $f^n \to 0$ *in* G *and*
(iv) $J(f)$ *is a connected subset of* $\mathbf{C}$.

In fact statement (iv) follows from (ii) by Theorem 2 in [20]. However, the only explicitly given example of transcendental entire functions with connected $J(f)$ seems to be $\sin z$ [12].

Theorem 1 also shows that it is not necessary for a small transcendental entire functions with $J(f) \neq \mathbf{C}$ to have a wandering domain.

One may ask whether there are arbitrarily small transcendental entire functions such that all Fatou components are simply-connected and $f \to \infty$ in at least one component.

RESIDUAL JULIA SETS. The residual Julia set $J_r(f)$ consists of those points in $J(f)$ which do not lie on the boundary of any component of $J^c = F(f)$. Such points are also called "buried" and a component of $J(f)$ which consists entirely of buried points is a "buried component". If $F(f)$ has a completely invariant component G, then $J(f) = \partial G$ and there can be no buried points. This is the case for polynomials. If $J_r(f) \neq \emptyset$, then it is residual in $J(f)$ in the sense of category theory [8, 12, 21, 22].

THEOREM D. [12] *If* f *is a transcendental entire function such that* $F(f)$ *has a multiply-connected component, then* $J(f)$ *has an everywhere dense set of singleton components, which are buried components of* $J(f)$.

The singletons in Theorem D are in fact limits of a shrinking "nest" of different small multiply-connected Fatou component. By Theorem A, f may have arbitrarily slow growth.

What of f in Theorem 1? Here $J(f)$ is connected so there is no buried component of $J(f)$. If K is any Fatou component and H is the component which contains 0, for large n we have $f^n(K) \subset H$, which shows that $\left\{\bigcup_n f^n(K)\right\}$ has bounded closure. It is shown in [13] that (for any transcendental f) $J(f)$ contains points z such that $f^n(z) \to \infty$. Clearly such points are in $J_r(f)$. Similarly we may show that any periodic point in J which does not belong to ∂H (and such points are dense in $J(f)$) is in $J_r(f)$.

The proof of Theorem 1 follows very closely that of Theorem C by Bergweiler and Eremenko [10] with some small changes and some further development to suit our different goal.

2. The main lemma

The main work in the inductive construction used to prove Theorem 1 is done by the following lemma.

We shall use $D(0, R)$ for the open disc of radius R, $U(R)$ for $\{z : |z| > R\}$ and for $0 < r < s$, the annulus $\{z : r < |z| < s\}$ will be denoted by $A(r, s)$.

Lemma 1. *Let P be a polynomial of degree $d \geqslant 2$ and such that $P(0) = 0, P(1) = 1, P'(0) = 0$ and $P''(0) \neq 0$. Suppose that there are $(d-1)$ critical points c_i of P, with $c_1 = 0$, such that the P orbits of c_i, $i > 1$, tend to ∞.*

Suppose further that $\varepsilon > 0$, $R > 0$ and points z_i, $1 \leqslant i \leqslant k$ are given such that $z_i \in K(P)$, $1 \leqslant i \leqslant k-1$, while $z_k \in F_\infty(P) = K(P)^c$, and

$$K(P) \cup \{z_k, c_1, \ldots, c_{d-1}\} \subset D(0, R/2).$$

Then there exists a polynomial Q of degree $(d+1)$ such that

(i) *$Q(0) = 0$, $Q(1) = 1$, $Q'(0) = 0$ and $Q''(0) \neq 0$;*

(ii) *One critical point of Q is $c'_1 = 0$ while the remaining d critical points c'_i have Q orbits which tend to ∞, the orbit of c_d lies in $U(2R)$; indeed if $|P^n(c_i)| > T + \varepsilon$, where $0 < T < R$, holds for $n \geqslant n_0$, then $|Q^n(c'_i)| > T$ for $n \geqslant n_0$;*

(iii) *There are points z'_i, $1 \leqslant i \leqslant k$, with $|z_i - z'_i| < \varepsilon$, $1 \leqslant i \leqslant k$, whose Q orbits lie in $K(Q)$;*

(iv) *For $i < k$, $|P^n(z_i) - Q^n(z_i)| < \varepsilon$ for all n. If $P^m(z_i) = 0$, then $Q^m(z'_i) = 0$. The Q orbit of z_k terminates in zero and contains a point in $U(R)$;*

(v) *$|P(z) - Q(z)| < \varepsilon$ for $z \in D(0, R)$;*

(vi) *If $a_1, \ldots, a_r$ are the zeros of P of multiplicities $m_1, \ldots m_r$, then Q has an m_j fold zero in each disc $|z - a_j| < \varepsilon$ and a single zero in $\{z : |z| > 2R\}$;*

(vii) *There is a point of $J(Q)$ in $D(z_k, 2\varepsilon)$.*

Preliminaries. We require some ideas about quasiconformal surgery as introduced in dynamics by Douady and Hubbard [11] and Shishikura [24].

A homeomorphism ϕ of a domain D to a domain D' is quasiconformal if it is absolutely continuous on almost all horizontal and vertical lines and if the complex dilatation

$\mu = \mu_\phi(z)$ defined almost everywhere in D by $\mu_\phi(z) = \phi_{\bar{z}}/\phi_z$ satisfies $\|\mu\|_\infty < 1$. Two quasiconformal maps ϕ_i of D with the same μ almost everywhere differ by a conformal map $f : \phi_1 = f \circ \phi_2$.

If in the above we drop the requirement that ϕ be a homeomorphism, the map is called quasiregular.

LEMMA 2. [10] *For every $\delta > 0$ and $R > 0$ there exists $\eta > 0$ such that every quasiconformal homeomorphism $\phi : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ which fixes $0, 1, \infty$ and such that $\|\mu\|_\infty < \eta$ holds, satisfies $|\phi(z) - z| < \delta$ for $z \in D(0, R)$.*

LEMMA 3. [10] *For every positive integer d and $\eta > 0$, there exists $\gamma \in (0, 1/2)$ with the following property: Let h_1, h_2 be holomorphic functions in the annulus $A(r/2, 4r)$ with $|h_i| \leqslant \gamma$. Then if we set*

$$h(z) = \left(2 - \frac{|z|}{r}\right)h_1(z) + \left(\frac{|z|}{r} - 1\right)h_2(z),$$

the map $\phi(z) = z^d(1 + h(z))$ is a quasiregular local homeomorphism $A(r, 2r) \to \mathbf{C}$ with boundary values

$$\phi(z) = z^d(1 + h_1(z)), \quad |z| = r,$$
$$\phi(z) = z^d(1 + h_2(z)), \quad |z| = 2r.$$

Further, the complex dilatation μ of ϕ satisfies $\|\mu\|_\infty < \eta$. In fact γ depends only on d, η and is independent of r, $r \geqslant 1$. We note also that $|h(z)| < 2\gamma$ in $A(r, 2r)$.

3. PROOF OF LEMMA 1

First one constructs a quasiregular map S of $\mathbf{C}$ which has most of the properties desired for Q, which agrees with P in some $D(0, r)$ and with a polynomial of degree $(d+1)$ in $U(2r)$, with $S^n A(r, 2r) \subset U(2r)$ for all n. This will permit us to find a suitable quasiconformal conjugation ("change of conformal structure") which converts S to the desired Q.

CONSTRUCTION OF S. Following [10] note that the Green's function U for $F_\infty(P)$ with pole ∞ (so $U(z) = \log|z| + O(1)$ as $z \to \infty$) satisfies

$$U(z) = \lim_{n\to\infty} d^{-n} \log|P^n(z)|. \tag{3}$$

Thus $U(z_k) > 0$ and $U(c_j) > 0$, $2 \leqslant j \leqslant d - 1$, and we can assume, by making a small change of z_k if necessary, that

$$U(z_k) \neq d^p U(c_j), \quad 2 \leqslant j \leqslant d - 1, \quad p \in \mathbf{Z}. \tag{4}$$

This implies that there is a constant $\alpha > 0$ such that $|U(z_k) - d^p U(c_j)| > \alpha$, $2 \leqslant j \leqslant d-1$, $p \in \mathbf{Z}$ and so

$$|d^n U(z_k) - d^q U(c_j)| > \alpha d^n, \quad 2 \leqslant j \leqslant d - 1, \quad q \in \mathbf{Z}, \ n \in \mathbf{Z}_+. \tag{5}$$

From (3) we see that for a fixed j and large n the minimum over p of $\Big|\log\{|P^n(z_k)/|P^p(c_j)|\}\Big|$ is achieved for some large value p and $\sim$ $|d^nU(z_k)-d^pU(c_j)|>\alpha d^n$. Hence

(6) $$\min_{p,j\neq 1}\big|\log|P^n(z_k)/P^p(c_j)|\big|\to\infty \text{ as } n\to\infty.$$

Similarly

(7) $$\min_{j\leqslant n}|P^n(z_k)/P^j(z_k)|\to\infty \text{ as } n\to\infty.$$

We choose $\delta>0$ so that $\delta<\varepsilon/2$ and that $|z-z'|<\delta$ implies $|P(z)-P(z')|<\varepsilon/2$ for all z,z' in $D(0,R+\varepsilon)$. We then choose η to satisfy Lemma 2 in $D(0,R')$ which includes $D(0,R+\varepsilon)$ and $P\big(D(0,R+\varepsilon)\big)$, and a $\gamma\in(0,1/2)$ for which Lemma 3 holds. Choose n so large that with r defined by

(8) $$r=\gamma|P^n(z_k)|/4$$

and $P(z)=ez^d+O(z^{d-1})$ for large z, we have

(9) $$r>R>1,\ \ 8er>2,\ \ e\big(|P^n(z_k)|^{1/2}-d\big)>2,$$

(10) $$er^d(1-2\gamma)>2|P^n(z_k)|>2e^{-1}(d+1)^{d+1}d^{-d},$$

(11) $$(1-\gamma/2)|P^n(z_k)|^{1/2}>2,\ \ 16|z_k|<(\varepsilon/2)(1-\gamma/2)|P^n(z_k)|^{1/2},$$

(12) $$\min_{p\in\mathbf{Z}}\Big|\log|P^n(z_k)/P^p(c_j)|\Big|\geqslant\log\frac{4}{\gamma},\ \ j\neq 1,$$

(13) $$\min_{p<n}|P^n(z_k)/P^p(z_k)|>\frac{4}{\gamma},$$

and

(14) $$|z^{-d}e^{-1}P(z)-1|<\gamma \text{ for } |z|\geqslant\frac{r}{2}.$$

Write $W=P^n(z_k)$ and define $h_1(z)=z^{-d}e^{-1}P(z)-1$, $h_2(z)=-z/W$ and S by

(15) $$S(z)=P(z)=ez^d\big(1+h_1(z)\big),\ \ z\in D(0,r)$$

(16) $$S(z)=P_1(z)=ez^d\big(1+h_2(z)\big),\ \ z\in U(2r)$$

(17) $$S(z)=e\phi(z) \text{ in } A(r,2r),$$

where ϕ is as in Lemma 3.

By (8) and (14) $h_1(z),h_2(z)$ satisfy the conditions of Lemma 3, and S is quasiregular in $\mathbf{C}$ with $\|\mu_S\|\leqslant\eta$ (outside $A(r,2r)$, $\mu=0$).

Properties of S. The set $V=U\big(2|W|\big)\subset U(16r)$ by (8). By (9) we have $S(V)=P_1(V)\subset V$ and $S^n\to\infty$ in V. By (10) S maps A into V.

The only points where S fails to be a local homeomorphism are the critical points c_i, $1\leqslant i\leqslant d-1$ of P, and one further point $c_d=dW/(d+1)$ in $U(2r)$. The P orbit of c_i, $2\leqslant i\leqslant d-1$, tends to ∞ and by (12) does not meet $A\big(r,8|W|\big)$. Thus the S orbit of c_i tends to ∞. By the second inequality of (10) $S(c_d)=P_1(c_d)\in V$ and so $S^n(c_d)\to\infty$ as $n\to\infty$.

By (13) we have $S^j(z_k) = P^j(z_k) \in D(0,r)$, $j < n$, $S^n(z_k) = P^n(z_k)$, $S^{n+1}(z_k) = P_1(P^n(z_k)) = 0$. For $i < k$ the S orbit of z_i is the same as the P orbit and lies in $K(P)$, since $r > R$.

P^{-n} is univalent in $D(W,\rho)$, where $\rho = W(1-\gamma/2)$. If $(P^{-n})'(W) = \lambda$, then by the 1/4-theorem $P^{-n}(D(W,\rho) \supset D(z_k, |\lambda|\rho/4)$. Since this latter set does not contain zero we have

$$|\lambda|\rho < 4|z_k|. \tag{18}$$

If $\rho' = |W|^{1/2}$ we have from the last inequality (9) that $|S(z)| > e|W|^{d-1}(\rho' - d) > 2|W|$ for $z \in C = C(W,\rho')$. Thus $S(C) \subset V$, $S^n \to \infty$ on C. By (11) $\rho'/\rho < 1/2$. By the distortion theorem and (11), (18) for $z \in C$

$$|P^{-n}(z) - z_k| \leqslant \frac{|\lambda\rho|(\rho'/\rho)}{(1-(\rho'/\rho))^2} \leqslant \frac{16|z_k|}{(1-(\gamma/2))|W|^{1/2}} < \frac{\varepsilon}{2}, \tag{19}$$

and the S orbit of $P^{-n}(z)$ tends to ∞.

THE QUASICONFORMAL CONJUGATION $\Phi : \mathbf{C} \to \mathbf{C}$. We define Φ by giving its complex dilatation μ, which we compute, starting with $\mu = 0$ in V (the "standard complex structure"), and elsewhere by "pulling this μ back" in such a way as to make $\Phi S \Phi^{-1}$ analytic, that is making ΦS, Φ have the same dilatation. In fact this means taking $\mu = 0$ except in $A \bigcup_{n=1}^{\infty} S^{-n}A$, and

$$\mu = \mu(\phi) = \frac{\phi_{\bar z}}{\phi_z} \text{ in } A$$
$$\mu(z) = \mu(S^n(z))\frac{\overline{(S^n)'(z)}}{(S^n)'(z)}, \; z \in P^{-n}(A).$$

Thus $\sup_{\mathbf{C}} |\mu| = \sup_{\mathbf{A}} |\mu(\phi)| \leqslant \eta$, noting that S^n is analytic on $S^{-n}(A)$.

By the measurable Riemann mapping theorem [**18**] there is a unique quasiconformal homeomorphism Φ of $\widehat{\mathbf{C}}$ which fixes $0, 1, \infty$ and has dilatation μ. The function $Q = \Phi S \Phi^{-1}$ fixes $0, 1, \infty$, is ∞ only at ∞ and is holomorphic in $\mathbf{C}$. Thus Q is a polynomial.

PROPERTIES OF Q. All this is much as in [**10**]. The valency of Q at zero is two since Φ is a homeomorphism. The remaining critical points are $c_i' = \Phi(c_i)$, $2 \leqslant i \leqslant d$, and their orbits are $\Phi(S^m(c_i)) \to \infty$. Thus if $|P^n(c_i)| > T + \varepsilon$ holds for $n \geqslant n_0$ we have $|S^n(c_i)| > T + \varepsilon$ in the construction and hence $|Q^n(c_i')| = \Phi(S^n(c_i)) > T$. $\Phi(S^m(c_d))$ in $\Phi U(2r) \subset U(R)$ for all m. We put $z_i' = \Phi(z_i)$ whose Q orbit is $\Phi\{S^m(z_i)\}$ which is bounded and thus in $K(Q)$. We have $|z_i' - z'| < \varepsilon$ because $|\Phi(z) - z| < \delta < \varepsilon/2$ in $D(0,R) \supset \{z_i\}$.

For $i < k$ the S orbit and the P orbit of z_i are the same and the Q orbit z_i' is given by $Q^m(z_i') = \Phi(P^m(z_i))$ so that $|P^m(z_i) - Q^m(z_i)| < \varepsilon$ for all m and if the P orbit of z_i is in $D(0,A)$ then the Q orbit of z_i' is in $D(0, A+\varepsilon)$. If $P^m(z_i) = 0$, then $Q^m(z_i') = 0$. The Q orbit of z_k also terminates in zero and contains a point outside $D(0,R)$.

Since Φ^{-1} has dilatation bounded by η we have for $z \in D(0,R)$ that $|\Phi^{-1}(z) - z| < \delta$, $|P \circ \Phi^{-1}(z) - P(z)| < \varepsilon/2$, $|\Phi \circ P \circ \Phi^{-1}(z) - P \circ \Phi^{-1}(z)| < \delta$ and hence $|Q(z) - P(z)| < \varepsilon$.

The zeros of Q are $\Phi(a_i)$, where a_i are the zeros of S and the multiplicities are the same. (vi) is clear.

There is a point p with $|p - z_k| < \varepsilon/2$ such that $S^m(p) \to \infty$ as $m \to \infty$. Then $p' = \Phi(p)$ satisfies $Q^m(p') = \Phi(S^m(p)) \to \infty$. $|p' - p| < \varepsilon/2$. Since $Q^m(z_k) = 0$, z_k' is in a component of $F(Q)$ where $Q^m \to 0$ and the segment $[z_k', p']$ contains some point of $J(Q)$, whose distance from z_k is at most 2ε. The proof is complete. $\square$

4. Proof of Theorem 1

Let $A(r)$ be the given increasing positive function for which f is to satisfy (1). We take sequences ε_n, R_n, $n \geqslant 0$ which are to satisfy

$$0 < \varepsilon_{n+1} < \frac{1}{2}\varepsilon_n, \tag{20}$$

$$8 < R_n < R_{n+1}, \quad R_n > 2^n, \quad A(R_n) > 2n + 2, \tag{21}$$

and certain conditions to be imposed as we proceed. Let $\{\alpha_n\}$, $n \geqslant 0$, be a countable dense sequence in $D(3,1) = \Delta$. We shall construct a sequence of polynomials P_n of degree $(n+2)$ whose limit is the required function f.

Begin with $P_0(z) = z^2$, so that $K(P_0) = \overline{D(0,1)}$. Make ε_0 smaller if necessary so that $D(\alpha_0, 4\varepsilon_0) \subset \Delta$, and R_0 larger so that $P = P_0$, $\varepsilon = \varepsilon_0$, $k = 1$, $z_1 = \alpha_0$ satisfy the conditions of Lemma 1. Then there exists a polynomial P_1 of degree 3 with a double zero at zero, $P_1(1) = 1$, a critical point c_1' whose orbit lies in $U(2R_0)$ and tends to ∞, a point z_1' with $|z_1' - z_1| < \varepsilon_0$ whose P_1 orbit terminates in zero. Further $|P_1(z) - P_0(z)| < \varepsilon_0$ in $D(0, R_0)$ and there is a point of $J(P_1)$ and hence also a point of $K(P_1)^c$ in Δ.

Let $n(1)$ be the smallest j such that $\alpha_j \notin K(P_1)$. Set $z_{0,0} = \alpha_0 = z_1$, $z_{0,1} = z_1'$, $z_{1,1} = \alpha_1, \dots, z_{n(1)-1,1} = \alpha_{j-1}$, $z_{n(1),1} = \alpha_j$. Make ε_1 smaller so that $D(\alpha_i, 4\varepsilon_1) \subset \Delta$, $i \leqslant n(1)$.

Enlarge R_1 so that Lemma 1 applies to $P = P_1$, $R = R_1$, $\varepsilon = \varepsilon_1$ with $k = n(1)$, $\{z_1, \dots, z_k\} = \{z_{0,1}, \dots, z_{n(1),1}\}$, $z_k = z_{n(1),1} = \alpha_j$. Let P_2 be the Q of Lemma 1.

Continuing in this way we obtain polynomials P_k of degree $(k+2)$ with a double zero at 0, $P_k(1) = 1$, integers $n(k)$ increasing to ∞, sequences ε_n, R_n satisfying (20), (21) and points $z_{s,k}$, $0 \leqslant s \leqslant n(k)$, such that

$$z_{s,k} \in K(P_k),\ 0 \leqslant s < n(k), \tag{22}$$

$$z_{s,k} = \alpha_s,\ n(k-1) < s \leqslant n(k), \tag{23}$$

$$n(k) = \min j \ \text{ with } \ \alpha_j \notin K(P_k). \tag{24}$$

For $0 \leqslant s \leqslant n(k)$, $D(\alpha_s, 4\varepsilon_k) \subset \Delta$. For $0 \leqslant s \leqslant n(k)$,

$$|z_{s,k} - z_{s,k+1}| < \varepsilon_k \tag{25}$$

There is a point of $J(P_k)$ and hence also a point of $\{K(P_k)\}^c$ in $D(z_{n(k-1)}, 4\varepsilon_{k-1})$.

$$|P_{k+1}(z) - P_k(z)| < \varepsilon_k, \text{ for } z \in D(0, R_k). \tag{26}$$

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The P_k orbit of $z_{i,k}$ is close to the P_{k-1} orbit of $z_{i,k-1}$ for $i \leqslant n(k-1)$:

(27) $$\left|P_k^j(z_{i,k}) - P_{k-1}^j(z_{i,k-1})\right| < \varepsilon_{k-1}, \quad j \in \mathbf{N}.$$

If the P_{k-1} orbit of $z_{i,k-1}$ terminates in zero, (which happens for $i = n(k-1)$), then so does the P_k orbit of $z_{i,k}$ after the same number of steps. The P_{k-1}-orbit of $z_{n(k-1),k-1}$ contains a point in $U(2R_{k-1})$.

The critical points of P_k other than zero are $c_{i,k}$, $1 \leqslant i \leqslant k$, where $0 < |c_{1,k}| < |c_{2,k}| \cdots < |c_{k,k}|$ and $|c_{k,k}| > 2R_{k-1}$. The orbit of each of these under P_k tends to ∞. Further

(28) $$|c_{i,k-1} - c_{i,k}| < \varepsilon_{k-1}, \quad 1 \leqslant i \leqslant k-1.$$

If $\left|P_{k-1}^n(c_{i,k-1})\right| > S$, where $S \leqslant R_{k-1}$, for $n \geqslant N$, then $\left|P_k^n(c_{i,k})\right| > S - \varepsilon_{k-1}$, $n \geqslant N$. Apart from zero the zeros of P_k are simple,

$$0 < |a_{1,k}| < |a_{2,k}| \cdots < |a_{k,k}|,$$

where

$$|a_{k,k}| > 2R_{k-1} \text{ and } |a_{j,k-1} - a_{j,k}| < \varepsilon_{k-1}, \quad 1 \leqslant j \leqslant k.$$

By definition $R_{k-1} > 2|a_{j,k-1}|$, so that we may assume

(29) $$2 < |a_{i,k}| \leqslant \frac{1}{2}|a_{i+1,k}|, \quad |a_{i,k}| > R_{i-1}.$$

By (26) P_k converges locally uniformly in $\mathbf{C}$ to an entire function f. First we estimate $M(r, f)$.

For a fixed k write $a_{i,k} = a$ so that $P_k(z) = ez^2 \prod_1^k (1 - z/a_i)$. The condition $P_k(1) = 1$ and (29) gives $|e| \leqslant B$ where $B^{-1} = \prod_1^\infty (1 - 2^{-j})$, and also

$$M(r, P_k) \leqslant BB'r^2 \prod_1^n \left(1 + r/|a_j|\right), \quad n = \max\{j : |a_j| < r\}, \quad B' = 2\prod_1^\infty (1 + 2^{-n}).$$

Thus

$$\log M(r, P_k) < \log(BB'') + 2\log r + n\log(2r) < 2n\log r, \quad r > r_0.$$

But $r \geqslant a_n > R_{n-1}$ by (29) and $A(r) > 2n$ by (21). Thus $\log M(P_k, r) < A(r)\log r$, $r > r_0$ and the same estimate holds for all values of k.

Thus f satisfies (1). Since we may assume that $A(r) < (\log r)^2$ we have from Theorem B that all components of $F(f)$ are bounded.

Suppose that $\alpha_s = z_{sk}$ as in (23). Then for any $l > k$, $|z_{sk} - z_{sl}| < 2\varepsilon_k$ by (20) and (25). Thus as $k \to \infty$ we have $z_{sk} \to \xi_s$ in $\overline{D}(\alpha_j, 2\varepsilon_k) \subset \Delta$, and $P_k^j(z_{sk}) \to f^j(\xi_s)$ for any fixed j. The P_{k+1} orbit of z_{sk} is in $K(P_{k+1})$ and by (27) the P_l orbit is in an

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$\varepsilon_{k+1}+\varepsilon_{k+2}+\cdots+\varepsilon_{l-1}<\varepsilon'$ neighbourhood of this, where $\varepsilon'=\sum_0^\infty \varepsilon_j$. Thus the f orbit of ξ_s is bounded. Moreover, for $s=n(k)$ the P_l orbit, $l \geqslant k$, of $z_{s,l}$ terminates in 0 after a fixed number, say $m(k)$ of steps and so $f^{m(k)}(\xi_s)=0$, while the f orbit of $\xi_{n(k-1)}$ contains a point in $U(2R_{k-1}-\varepsilon)$. Thus $\bigcup_n f^n(\Delta)$ is unbounded. Also $\{\xi_s\}$ is dense in Δ since $\varepsilon_n \to \infty$ as $n\to\infty$.

We have $f(0)=0$, $f'(0)=0$ so that 0 is an attracting fixed point of f and $F(f)$ has a bounded invariant component H which contains 0.

If $\Delta \subset F(f)$, then since there are in Δ some points whose orbits terminate in 0, we have $f^m(\Delta)\subset H$ for some m and $\bigcup_0^\infty f^n(\Delta)$ lies in the bounded set $H \cup \left(\bigcup_{j=0}^{m-1} f^j(\Delta)\right)$. Thus in fact Δ meets $J(f)$ in say a point β.

Suppose now that there is some component G of $F(f)$ where $f^n\to\infty$. Any neighbourhood of β contains preimages of G and hence Δ meets a component G_1 without loss of generality G, in which $f^n\to\infty$. This contradicts the density of $\{\xi_s\}$ in Δ.

Thus there is no Fatou component where $f^n\to\infty$ and it follows that all components are simply-connected.

From (28) and preceding inequalities the critical points of P_k can be taken to satisfy $R_{i-1}<c_{i,k}<(1/2)R_i$.

Since f has order 0 its inverse has no finite singularities other than critical values. These correspond to critical points 0 and $\eta_i=\lim_{k\to\infty} c_{i,k}$, so we have $R_{i-1}<\eta_i<(1/2)R_i$. We claim that for each fixed i, $\lim_{k\to\infty} f^n(c_i)=\infty$.

For given $T>0$ choose $k>i$ so that $R_k>T+\varepsilon'$. Then there is an integer N such that $n>N$ implies $|P_k^n(c_{i,k})|>R_k$. Thus for $n\geqslant N$: $|P_{k+1}^n(c_{i,k+1})|>R_k-\varepsilon_k$. Indeed, for all p we have $|P_{k+p}^n(c_{i,k+p})|>R_k-\varepsilon'$ for $n\geqslant N$, and so $|f^n(\eta_i)|\geqslant R_k-\varepsilon'\geqslant T$.

The postcritical set of f is $P=\{0, f^n(\eta_i),\ n\geqslant 0,\ i\geqslant 0\}$. $\overline{P}$ does not divide the plane which is known to imply that there are no Fatou domains with non constant limit functions. The only possible constant limit functions are critical points which are fixed points or else limit points of P [9]. Thus only $0,\infty$ come into question and ∞ has been excluded above, so the proof is complete.

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