Canad. Math. Bull. Vol. **32** (1), 1989

# A NOTE ON THE DUBOIS-EFROYMSON DIMENSION THEOREM

BY
WOJCIECH KUCHARZ

ABSTRACT. Let $X \subset \mathbf{R}^n$ be an irreducible nonsingular algebraic set and let $Z$ be an algebraic subset of $X$ with $\dim Z \leqq \dim X - 2$. In this paper it is shown that there exists an irreducible algebraic subset $Y$ of $X$ satisfying the following conditions: $\dim Y = \dim X - 1$, $Z \subset Y$ and that the ideal of regular functions on $X$ vanishing on $Y$ is principal.

1. **Introduction.** We shall consider algebraic varieties over the field $\mathbf{R}$ of real numbers. First of all, to prevent confusion, we shall fix some terminology.

Given a Zariski closed subset $V$ of $\mathbf{R}^n$, we denote by $\mathcal{O}_V$ the sheaf of regular functions on $V$. Recall that if $U$ is a Zariski open subset of $V$, then a function $f: U \to \mathbf{R}$ is said to be regular if for each point $x$ in $U$ there exist polynomial functions $\varphi_x$ and $\psi_x$ on $\mathbf{R}^n$ and a Zariski neighborhood $U_x$ of $x$ in $U$ such that $\psi_x$ does not vanish on $U_x$ and $f = \varphi_x/\psi_x$ on $U_x$. It is well known [4], [8], [11] that

$$\mathcal{O}_V(U) = \{\varphi/\psi \mid \varphi, \psi \text{ are polynomial functions on } \mathbf{R}^n \text{ and } \psi \text{ does not vanish on } U\}.$$

An affine real algebraic variety is a locally ringed space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of rings of $\mathbf{R}$-valued functions on $X$, such that $(X, \mathcal{O}_X)$ is isomorphic as a locally ringed space to $(V, \mathcal{O}_V)$ for some Zariski closed subset $V$ of $\mathbf{R}^n$, $n \geqq 0$. A ringed space $(X, \mathcal{O}_X)$ is called a real algebraic variety if there exists a finite open covering $\{U_i\}$ of $X$ such that $(U_i, \mathcal{O}_X | U_i)$ is an affine real algebraic variety for all $i$. Morphisms between real algebraic varieties and such concepts as dimension, irreducibility, nonsingularity, etc., can be defined in the standard way (cf. [4], [8], [11]). Every real algebraic variety can be endowed with the *strong topology* induced from the Euclidean topology on the reals.

Given a subvariety $Y$ of $X$, we shall denote by $I_X(Y)$ the ideal of $\mathcal{O}_X(X)$ of regular functions vanishing on $Y$.

Our aim is the following

Received by the editors October 21, 1986, and, in revised form, September 10, 1987.
AMS Subject Classification (1980): 14G30, 14J99.

THEOREM 1. *Let $X$ be an affine irreducible nonsingular real algebraic variety and let $Z$ be a closed algebraic subvariety of $X$ of codimension at least* 2. *Then there exists a closed algebraic subvariety $Y$ of $X$ such that*

(i) *$Y$ is irreducible*, codim $Y = 1$, $Z \subset Y$,

(ii) *the ideal $I_X(Y)$ of $\mathcal{O}_X(X)$ is principal.*

Condition (i) has been proved by Dubois and Efroymson [5] for varieties over any real closed field without the assumption that $X$ is nonsingular; condition (ii) is new. Notice that, in general, there are many essentially different subvarieties $Y$ satisfying (i). For instance, let $X = S^1 \times S^1$, where $S^1$ is the unit circle, and let $Z$ consist of a single point $z = (z_1, z_2)$. Clearly, both $Y_1 = \{z_1\} \times S^1$ and $Y_2 = S^1 \times \{z_2\}$ satisfy (i) but neither of them satisfies (ii). From a geometric or topological point of view there is an essential difference between $Y_1$ and $Y_2$. Indeed, the divisors represented by $Y_1$ and $Y_2$ are not linearly equivalent and the homology classes of $Y_1$ and $Y_2$ in $H_1(S^1 \times S^1, \mathbf{Z}/2\mathbf{Z})$ are distinct. Of course, it is easy to find a subvariety $Y$ of $S^1 \times S^1$ satisfying (i) and (ii). It suffices to take any nonsingular subvariety of $S^1 \times S^1$ containing $z$ whose homology class in $H_1(S^1 \times S^1, \mathbf{Z}/2\mathbf{Z})$ vanishes and which is $C^\infty$ diffeomorphic to $S^1$ (cf. for example [3], Theorem 3). The meaning of (ii) is that $Y$ is, in a certain sense, the simplest subvariety of $X$ satisfying (i). Indeed, (ii) implies that the divisor represented by $Y$ is linearly equivalent to zero and the homology class represented by $Y$ in $H_{d-1}(X, \mathbf{Z}/2\mathbf{Z})$, $d = \dim X$, vanishes (here we use the homology built on infinite locally finite chains if $X$ is not compact in the strong topology). It should be mentioned that the proof of Theorem 1 will not be obtained by refining methods of [5] (the author does not know if that is possible) but by completely different techniques in which algebraic blowing-ups play the main role.

2. **Proof of Theorem 1.** Let $X$ be an affine real algebraic variety and let $Y$ be its closed subvariety. We shall denote by $\pi(X, Y): B(X, Y) \to X$ the algebraic blowing-up of $X$ along $Y$ (cf. [2] or [4] for elementary properties of this construction in the context suitable for this paper). Since real projective space with its natural structure of an abstract algebraic variety is actually an affine variety (cf. [2], [4] or [11]), it follows that $B(X, Y)$ is an affine real algebraic variety. Moreover, $\pi(X, Y)$ is a proper map in the strong topology and induces an algebraic isomorphism between $B(X, Y)\setminus\pi(X, Y)^{-1}(Y)$ and $X\setminus Y$. If both $X$ and $Y$ are nonsingular, then $B(X, Y)$ also is nonsingular and $\pi(X, Y)$ is surjective. If $Z$ is a closed subvariety of $Y$, then $B(Y, Z)$ will be considered as a subvariety of $B(X, Z)$. Notice that $B(Y, Z)$ is just the Zariski closure of $\pi(X, Z)^{-1}(Y\setminus Z)$ in $B(X, Z)$. We shall be using these properties later on without explicitly referring to them.

By a multiblowup of $X$ along $Y$ we mean a real algebraic morphism $\pi: \tilde{X} \to X$ together with a sequence of real algebraic morphisms

$$\widetilde{X} = X_k \xrightarrow{\pi_k} X_{k-1} \to \ldots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

and a sequence of closed nonsingular subvarieties $Z_{i-1} \subset X_{i-1}$, $\dim Z_{i-1} < \dim Y$, $i = 1, \ldots, k$, such that $\pi = \pi_1 \circ \ldots \circ \pi_k$, $Z_0 \subset Y$, $X_1 = B(X_0, Z_0)$, $\pi_1 = \pi(X_0, Z_0)$ and for each $i = 2, \ldots, k$, $X_i = B(X_{i-1}, Z_{i-1})$, $\pi_i = \pi(X_{i-1}, Z_{i-1})$. If $V$ is a closed subvariety of $X$, then the subvariety $\widetilde{V}$ of $\widetilde{X}$ obtained by the following process: $V_1 =$ the Zariski closure of $\pi_1^{-1}(V \backslash Z_0)$ in $X_1, \ldots, V_k =$ the Zariski closure of $\pi_k^{-1}(V_{k-1} \backslash Z_{k-1})$ in $X_k$, $\widetilde{V} = V_k$, is called the strict preimage of $Z$ under $\pi$. To simplify the exposition, we shall refer to $\pi: \widetilde{X} \to X$ as a multiblowup, tacitly assuming that there are also associated with $\pi$ the sequences $\{\pi_i\}$ and $\{Z_{i-1}\}$, $i = 1, \ldots, k$.

Given a real algebraic variety $X$, we shall denote by $\operatorname{Sing}(X)$ its set of singular points. The following deep result, which is a version of Hironaka's desingularization theorem, will be crucial.

THEOREM 2 [6]. *Let $X$ be an affine real algebraic variety and let $Y$ be a closed subvariety of $X$. Then there exists a multiblowup $\pi: \widetilde{X} \to X$ of $X$ along $Y$ such that the strict preimage $\widetilde{Y}$ of $Y$ under $\pi$ is a nonsingular subvariety of $\widetilde{X}$ and $\pi$ induces an algebraic isomorphism from $\widetilde{X} \backslash \pi^{-1}(\operatorname{Sing}(Y))$ onto $X \backslash \operatorname{Sing}(Y)$.*

We shall also need the following technical result.

LEMMA 3. *Let $X$ be an affine nonsingular real algebraic variety and let $Z$ be a closed subvariety of $X$ of codimension at least 2. Let $I$ be a locally principal ideal of $\mathcal{O}_X(X)$ such that the ideal $I\mathcal{O}_X(X \backslash Z)$ of $\mathcal{O}_X(X \backslash Z)$ is principal. Then the ideal $I$ is principal, provided that $X$ is compact in the strong topology.*

PROOF. We shall consider $\mathcal{O}_X(X)$ as a subring of the ring $\mathscr{C}(X)$ of continuous (in the strong topology) real-valued functions on $X$. Clearly, $I\mathscr{C}(X)$ is a projective $\mathscr{C}(X)$-module of rank 1. Let $\xi$ be a continuous real line bundle over $X$ associated with $I\mathscr{C}(X)$ in the standard way. By assumption, $\xi$ is topologically trivial over $U_0 = X \backslash Z$. Pick a continuous nonvanishing section $s_0$ of $\xi$ over $U_0$. One easily finds a finite collection $\{U_i\}_{i=1,\ldots,k}$ of open subsets of $X$ and a collection $\{s_i\}_{i=1,\ldots,k}$, $s_i: U_i \to \xi$, of nonvanishing continuous sections of $\xi$ such that $Z \subset U_1 \cup \ldots \cup U_k$, each $U_i$ is homeomorphic to a ball and $s_i = \lambda_i s_0$ on $U_i \backslash Z$, where $\lambda_i$ is a continuous positive function on $U_i \backslash Z$ (notice that $U_i \backslash Z$ is connected, codim $Z$ being at least 2). Let $\{\varphi_j\}_{j=0,\ldots,k}$ be a continuous partition of unity subordinate to the open covering $\{U_j\}_{j=0,\ldots,k}$ of $X$. Define the section $s$ of $\xi$ by $s = \varphi_0 s_0 + \ldots + \varphi_k s_k$. Obviously, $s$ vanishes nowhere. Thus the line vector bundle $\xi$ is topologically trivial and hence the ideal $I\mathscr{C}(X)$ of $\mathscr{C}(X)$ is principal. Let $f_1 g_1 + \ldots + f_\ell g_\ell$ be a generator of $I\mathscr{C}(X)$, where $f_r \in I$ and $g_r \in \mathscr{C}(X)$. It follows from the Weierstrass approximation theorem that there exists $h_r \in \mathcal{O}_X(X)$ arbitrarily close to $g_r$, $r = 1, \ldots, \ell$. By [10], Lemma 1.3, we

https://doi.org/10.4153/CMB-1989-004-0 Published online by Cambridge University Press

may assume that $f = f_1h_1 + \ldots + f_\ell h_\ell$ generates $I\mathscr{C}(X)$. Let $\mathscr{J} = (f)\mathscr{O}_X(X)$. Then $\mathscr{J}\mathscr{C}(X) = I\mathscr{C}(X)$ and in virtue of [10], Theorem 2.2(a), $\mathscr{J} = I$.

Given a real algebraic variety $X$ and its subvariety $Y$, to simplify notation, we shall write $\mathscr{O}(X)$ for $\mathscr{O}_X(X)$ and $I(Y)$ for $I_X(Y)$. Assume that $X$ is nonsingular. A family $\{Y_i\}_{i=1,\ldots,c}$ of nonsingular subvarieties of $X$ is said to be in general position if for each $x$ in $Y_1 \cup \ldots \cup Y_c$, the family $\{T_x(Y_i)\}_{i\in\Lambda(x)}$, where $\Lambda(x) = \{i | x \in Y_i\}$, of vector subspaces of $T_x(X)$ ($T_x(\cdot)$ = Zariski tangent space at $x$) is in general position, i.e.,

$$\operatorname{codim} \bigcap_{i\in\Lambda(x)} T_x(Y_i) = \sum_{i\in\Lambda(x)} \operatorname{codim} T_x(Y_i).$$

Proof of Theorem 1. If $V$ is an irreducible component of $Z$, then $V$ is contained in a closed irreducible subvariety $W$ of $X$ with $\dim W = \dim V + 1$. Thus we can assume that all irreducible components of $Z$ have the same dimension. Let $c = \operatorname{codim} Z$.

Step 1. Assume that $X$ is compact in the strong topology, $Z$ is nonsingular and there exist closed nonsingular subvarieties $Y_1, \ldots, Y_c$ of $X$ of condimension 1 which are in general position and satisfy $Z \subset Y_1 \cap \ldots \cap Y_c$.

By [2], Proposition 4.3, there exists an arbitrarily small $C^\infty$ isotopy $f_t: Z \to X$, $0 \leqq t \leqq 1$, such that $f_0$ is the inclusion map and $Z' = f_1(Z)$ is a closed nonsingular algebraic subvariety of $X$ transverse to $Z$. It follows from [2], Lemma 3.1 that the normal vector bundle of $B(Z, Z \cap Z')$ in $B(X, Z \cap Z')$ splits a $C^\infty$ trivial line bundle. Thus there exists a $C^\infty$ function $g: B(X, Z \cap Z') \to \mathbf{R}$ transverse to 0 in $\mathbf{R}$ and vanishing on $B(Z, Z \cap Z')$. By the relative version of the Weierstrass approximation theorem (cf. [1], Proposition 2.1) one can find a regular function $h: B(X, Z \cap Z') \to \mathbf{R}$ vanishing on $B(Z, Z \cap Z')$ and arbitrarily close to $g$ in the $C^\infty$ topology. Clearly, one can assume that $h$ is transverse to 0 in $\mathbf{R}$. Let $\varphi_1, \ldots, \varphi_k$ be generators of the ideal $I(B(Z, Z \cap Z'))$ of $\mathscr{O}(B(X, Z \cap Z'))$. We claim that for a generic $s = (s_1, \ldots, s_k)$ in $\mathbf{R}^k$ (i.e. for all $s$ in $\mathbf{R}^k \backslash \Sigma$, where $\Sigma$ is a certain closed subvariety of $\mathbf{R}^k$, $\dim \Sigma < k$), the function $h_s = h + s_1\varphi_1 + \ldots + s_k\varphi_k$ is transverse to 0 in $\mathbf{R}$ and $V_s \backslash B(Z, Z \cap Z')$ is an irreducible variety, where $V_s$ is the set of zeros of $h_s$. Now we shall finish the proof of Step 1 assuming that the claim is true. Notice that $V_s$ is irreducible. Indeed, as one easily sees, $V_s$ is the Zariski closure of $V_s \backslash B(Z, Z \cap Z')$ in $B(X, Z \cap Z')$. Let $Y$ be the Zariski closure of $\pi(X, Z \cap Z')(V_s \backslash \pi(X, Z \cap Z')^{-1}(Z \cap Z'))$ in $X$. Clearly, $Y$ is irreducible and $\operatorname{codim} Y = 1$. Also notice that $Z \backslash (Z \cap Z') \subset Y$ and hence $Z \subset Y$. Since the ideal $I(V_s)$ of $\mathscr{O}(B(X, Z \cap Z'))$ is generated by $h_s$, the ideal $I(Y)\mathscr{O}(X \backslash (Z \cap Z'))$ of $\mathscr{O}(X \backslash (Z \cap Z'))$ is principal. By Lemma 3, the ideal $I(Y)$ of $\mathscr{O}(X)$ is principal.

To prove the claim, notice that its first part follows from a standard transversality argument. Thus it remains to show that $V_s \backslash B(Z, Z \cap Z')$ is an irreducible variety for a generic choice of $s$. Since $B = B(X, Z \cap Z') \backslash B(Z, Z \cap Z')$ is an affine variety (any open subvariety of an affine real algebraic variety is affine!), we can embed $B$ as a closed subvariety in $\mathbf{R}^N$ for some $N > 0$. Let $B_{\mathbf{C}}$ be the complexification of $B$, i.e., the smallest closed complex subvariety of $\mathbf{C}^N$ containing $B$. It follows from the description of $\mathcal{O}(B)$ given at the beginning of Section 1 that one can find a Zariski open neighborhood $U$ of $B$ in $B_{\mathbf{C}}$ and complex regular functions $h_{\mathbf{C}}: U \to \mathbf{C}$, $\varphi_{i\mathbf{C}}: U \to \mathbf{C}$, $i = 1, \ldots, k$ such that $h_{\mathbf{C}}|B = h|B$, $\varphi_{i\mathbf{C}}|B = \varphi_i|B$ and the set of common zeros of $\varphi_{1\mathbf{C}}, \ldots, \varphi_{k\mathbf{C}}$ is empty. Since $B$ is irreducible so is $U$. By Bertini's theorem (cf. [7], p. 586 and also [9], Theorem 1.4), there exists a closed complex subvariety $\Sigma_{\mathbf{C}} \subset \mathbf{C}^k$, $\dim \Sigma_{\mathbf{C}} < k$, such that for each $u = (u_1, \ldots, u_k)$ in $\mathbf{C}^k \backslash \Sigma_{\mathbf{C}}$, the variety

$$\{z \in U | h_{\mathbf{C}}(z) + u_1\varphi_{1\mathbf{C}}(z) + \ldots + u_k\varphi_{k\mathbf{C}}(z) = 0\}$$

is irreducible. If for some $s$ in $\mathbf{R}^k$, $h_s$ is transverse to 0 in $\mathbf{R}$, then the ideal $I(V_s \backslash B(Z, Z \cap Z'))$ of $\mathcal{O}(B)$ is generated by $h_s|B$. Hence $V_s \backslash B(Z, Z \cap Z')$ is irreducible provided that, in addition, $s$ belongs to $\mathbf{R}^k \backslash \Sigma_{\mathbf{C}}$.

STEP 2. Assume only that $X$ is compact in the strong topology.

By Theorem 2, there exists a multiblowup $\pi_1: X_1 \to X$ of $X$ along $Z$ such that the strict preimage $Z_1$ of $Z$ under $\pi_1$ is a nonsingular subvariety of $X_1$ and $\pi_1$ restricted to $X_1 \backslash \pi_1^{-1}(\mathrm{Sing}(Z))$ is an algebraic isomorphism onto $X \backslash \mathrm{Sing}(Z)$. It follows from [2], Theorem 3.5 that one can find a multiblowup $\pi_2: X_2 \to X_1$ of $X_1$ along $Z_1$ such that if $Z_2$ is the strict preimage of $Z_1$ under $\pi_2$, then $Z_2 \subset V_1 \cap \ldots \cap V_c$, where $V_1, \ldots, V_c$ are closed nonsingular subvarieties of $X_2$ of codimension 1 which are in general position. Step 1 implies the existence of a closed irreducible subvariety $Y_2$ of $X_2$ with $Z_2 \subset Y_2$ and the ideal $I(Y_2)$ of $\mathcal{O}(X_2)$ being principal. By construction, there exists a closed subvariety $Z_0$ of $Z$ such that $\dim Z_0 < \dim Z$ and $\pi_1 \circ \pi_2$ induces the algebraic isomorphism between $X_2 \backslash (\pi_1 \circ \pi_2)^{-1}(Z_0)$ and $X \backslash Z_0$. Let $Y$ be the Zariski closure of $(\pi_1 \circ \pi_2)(Y_2 \backslash (\pi_1 \circ \pi_2)^{-1}(Z))$ in $X$. Obviously, $Y$ is irreducible and codim $Y = 1$. Since $Z \backslash Z_0 \subset Y$ and all irreducible components of $Z$ are of the same dimension, we have $Z \subset Y$. Notice that the ideal $I(Y)\mathcal{O}(X \backslash Z_0)$ of $\mathcal{O}(X \backslash Z_0)$ is principal. Thus, by Lemma 3, the ideal $I(Y)$ of $\mathcal{O}(X)$ is principal.

STEP 3. $X$ arbitrary.

We can assume that $X$ is a closed algebraic subvariety of $\mathbf{R}^n$, $n > 0$. Let $S^n$ be the unit $n$-dimensional sphere and let $\rho: S^n \backslash \{a\} \to \mathbf{R}^n$ be the stereographic projection from $a = (0, \ldots, 0, 1)$. Denote by $X^*$ the Zariski closure of $\rho^{-1}(X)$ in $S^n$. Obviously, $X^*$ is compact in the strong topology and $\mathrm{Sing}(X^*) \subset \{a\}$. Let

$\pi : X' \to X^*$ be a multiblowup of $X^*$ such that $X'$ is nonsingular and $\pi$ induces an algebraic isomorphism from $X' \backslash \pi^{-1}(a)$ onto $X^* \backslash \{a\}$ (Theorem 2). Let $Z'$ be the Zariski closure of $\pi^{-1}(\rho^{-1}(Z))$ in $X'$. Since $X'$ is compact in the strong topology, it follows from Step 2 that there exists a closed irreducible subvariety $Y'$ of $X'$ such that $Z' \subset Y'$, codim $Y' = 1$ and the ideal $I(Y')$ of $\mathcal{O}(X')$ is principal. Let $Y$ be the Zariski closure of $\rho(\pi(Y' \backslash \pi^{-1}(a)))$ in $X$. Clearly, $Y$ is irreducible, codim $Y = 1$ and $Z \subset X$. Moreover, since $\rho \circ \pi : X' \backslash \pi^{-1}(a) \to X$ is an algebraic isomorphism, the ideal $I(Y)$ of $\mathcal{O}(X)$ is principal.

## References


1. S. Akbulut, H. King, *The topology of real algebraic sets*, L'Enseignement Math. **29** (1983), pp. 221-261.

2. ———, *Submanifolds and homology of nonsingular real algebraic varieties*, Amer. J. Math. **107** (1985), pp. 45-83.

3. J. Bochnak, W. Kucharz, M. Shiota, *The divisor class groups of some rings of global real analytic, Nash or rational regular functions*, Lect. Notes in Math. **954**, Springer 1981, pp. 218-248.

4. J. Bochnak, M. Coste, M. F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Math., Vol. 12, 1987, Springer.

5. D. Dubois, G. Efroymson, *A dimension theorem for real primes*, Can. J. Math. **25** (1974), pp. 108-114.

6. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), I: pp. 109-203, II: pp. 205-326.

7. N. Mohan Kumar, M. Pavaman Murthy, *Algebraic cycles and vector bundles over affine three-folds*, Ann. of Math. **116** (1982), pp. 579-591.

8. R. Silhol, *Géométrie algébrique sur un corps non algébriquement close*, Comm. Alg. **6** (1978), pp. 1131-1155.

9. R. G. Swan, *A cancellation theorem for projective modules in the metastable range*, Invent. Math. **27** (1974), pp. 23-43.

10. ———, *Topological examples of projective modules*, Trans. Amer. Math. Soc. **230** (1977), pp. 201-234.

11. A. Tognoli, *Algebraic geometry and Nash functions*, Inst. Math. Vol. III, Acad. Press London and New York 1978.



Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
USA