Computable bounds for Rasmussen’s concordance invariant

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Abstract
Given a diagram $D$ of a knot $K$, we give easily computable bounds for Rasmussen’s concordance invariant $s(K)$. The bounds are not independent of the diagram $D$ chosen, but we show that for diagrams satisfying a given condition the bounds are tight. As a corollary we improve on previously known Bennequin-type bounds on the slice genus.

1. Statement of results

1.1 Introduction
In [Ras10], Rasmussen defined a homomorphism on the smooth concordance group of knots $C$, $s : C \to 2\mathbb{Z}$, which he showed had the property that
$$|s(K)| \leq 2g^*(K),$$
where we write $g^*(K)$ for the smooth 4-ball genus (or slice genus) of $K$.

The starting point for this paper is the following theorem of Rasmussen [Ras10].

**Theorem 1.1.** For positive knots $K$ (that is, knots that admit a diagram with no negative crossings)
$$s(K) = 2g^*(K).$$

The point here is that, in the case of positive knots $K$, the computation of $s(K)$ is a triviality and agrees with twice the genus of an obvious candidate for a minimal-genus slicing surface (namely, the one obtained by pushing the Seifert surface given by Seifert’s algorithm into the 4-ball).

The invariant $s(K)$ is equivalent to all the information contained in $\mathcal{F}^jH^i(K)$, where $\mathcal{F}^jH^i$ is the perturbed version of standard Khovanov homology first defined and studied by Lee [Lee05]. There is a spectral sequence with $E_2$ page being the standard Khovanov homology of a knot $K$ and $E_\infty$ page being the bigraded group $\mathcal{F}^jH^i(K)/\mathcal{F}^{j+1}H^i(K)$, and many efforts to compute $s$ for knots other than for positive knots have made use of the existence of spectral sequences (for some nice examples see [Shu07]).

However, since it is known that $\mathcal{F}^jH^i(K) = 0$ for $i \neq 0$, to define $s(K)$ only requires knowledge of the partial chain complex
$$\mathcal{F}^jC^{-1}(D) \xrightarrow{\partial_{-1}} \mathcal{F}^jC^0(D) \xrightarrow{\partial_0} \mathcal{F}^jC^1(D),$$
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where $D$ is a diagram of $K$. In fact, since explicit representatives for a basis of $\mathcal{F}^jH^i(K)$ are known at the chain level, one only needs to know the map

$$\partial_{-1} : \mathcal{F}^jC^{-1}(D) \to \mathcal{F}^jC^0(D).$$

Remark. For a positive diagram $D$, $C^{-1}(D) = 0$. This is what made Theorem 1.1 a trivial corollary once the properties of $s$ were established.

By studying this map we obtain a diagram-dependent upper bound $U(D)$ for $s(K)$. We also give an error estimate $2\Delta(D)$ for this upper bound. The resulting lower bound $U(D) - 2\Delta(D)$ for $s(K)$ improves upon previously known Rudolph–Bennequin-type inequalities. We give a list of particular cases where $\Delta(D)$ vanishes and so $U(D)$ necessarily agrees with $s(K)$.

Just prior to posting on the arXiv, we heard from Kawamura [Forthcoming Paper] that she has independently obtained several of the results in this paper, using entirely different methods. Kawamura’s work is based on Livingston’s axiomatic approach to $s$ and also to the bound $\tau$ coming from Heegaard–Floer homology.

1.2 Results

The following results are stated for knots, since the Rasmussen invariant is most familiar in this setting. Some results, however, admit a generalization to links (via the definition of $s$ for links as found for example in [BW]). We discuss this in §3.

Our results concern an easily computable number $U(D) \in 2\mathbb{Z}$, which is defined from an oriented knot diagram $D$. Postponing an explicit description of how to compute $U(D)$ until Definition 1.8, we begin by giving some results.

**Theorem 1.2.** For any oriented knot diagram $D$,

$$s(D) \leq U(D).$$

Of course, we must remember that $s(D)$ depends only on the isotopy class of the knot represented by $D$, whereas the same is not true of $U(D)$. Hence, in order for the bound of Theorem 1.2 to be a good bound, we should expect to be forced to give some restrictions on diagrams $D$.

**Proposition 1.3.** The bound of Theorem 1.2 is tight for positive diagrams $D$ and for negative diagrams $D$.

**Proposition 1.4.** Let $\varepsilon_i \in \{-1, +1\}$ for $i = 1, 2, \ldots, n$. Then if $w$ is any word in the $n$ letters

$$\{\sigma_1^{\varepsilon(1)}, \sigma_2^{\varepsilon(2)}, \ldots, \sigma_n^{\varepsilon(n)}\}$$

and $B$ is a knot diagram that is the closure of the $(n+1)$-stranded braid represented by $w$, then we have

$$s(B) = U(B).$$

Remark. We note that knots admitting such a braid presentation are known to be fibered [Sta78], so in particular not every knot admits such a presentation.

**Proposition 1.5.** Let $D$ be an alternating diagram of a knot. Then we have

$$s(D) = U(D).$$
Propositions 1.3–1.5 are each consequences of Theorem 1.10 for which we need a few definitions. Given a diagram $D$ we write $O(D)$ for the oriented resolution.

**Definition 1.6.** We form a decorated graph $T(D)$, known as the Seifert graph of $D$, as follows. We start with a node for each component of $O(D)$. Each crossing in $D$, when smoothed, lies on two distinct components of $O(D)$; for each positive (respectively negative) crossing of $D$ we connect the corresponding nodes by an edge decorated with $+$ (respectively $-$).

Note that $T(D)$ by itself is not enough to recover the full Khovanov chain complex of the diagram $D$, but if we added extra data of an ordering of the edges at each node, we would be able to recover the full complex.

**Definition 1.7.** From $T(D)$ we now form two other graphs. We form a subgraph $T^-(D)$ (respectively $T^+(D)$) from $T(D)$ by removing all edges of $T(D)$ decorated with a $+$ (respectively $-$).

**Definition 1.8.** We define the number
\[ U(D) = \#\text{nodes}(T(D)) - 2\#\text{components}(T^-(D)) + w(D) + 1, \]
where $w(D)$ is the writhe of $D$.

**Definition 1.9.** We define the number
\[ \Delta(D) = \#\text{nodes}(T(D)) - \#\text{components}(T^-(D)) - \#\text{components}(T^+(D)) + 1. \]

Then we have the following theorem.

**Theorem 1.10.** If $\Delta(D) = 0$ then $s(D) = U(D)$. In fact we can say more:
\[ U(D) - 2\Delta(D) \leq s(D) \leq U(D). \]

Theorem 1.10 enables us to improve on previously known easily computable combinatorial lower bounds for the slice genus. We have the following corollary.

**Corollary 1.11.**
\[ 2g^+(K) \geq s(K) \geq U(D) - 2\Delta(D) \]
\[ \geq w(D) - \#\text{nodes}(T(D)) + 2\#\text{components}(T^+(D)) - 1, \]
which is stronger than the Rudolph–Bennequin inequalities as proved in [Kaw07, Pla06, Shu07] (for a nice discussion see [Sto07]).

**Proof of Propositions 1.3–1.5.** This is just a matter of checking that the condition $\Delta(D) = 0$ of Theorem 1.10 holds in each case. This is only a non-trivial check for the case of $D$ being alternating.

Suppose $D$ is an alternating diagram. The complement of the oriented resolution $O(D)$ is a number of regions of the plane. If $D$ is not the trivial diagram, there is a unique way to associate to each region either a $+$ or a $-$ such that only positive (respectively negative) crossings of $D$ occur in regions associated with a $+$ (respectively $-$) and such that adjacent regions have different associated signs. See Figure 1 for an example.

Then each region with associated sign $+$ (respectively $-$) corresponds to exactly one component of $T^+(D)$ (respectively $T^-(D)$). Since there is one more region than there are circles of $O(D)$ (or equivalently nodes of $T(D)$) we must have $\Delta(D) = 0$. \qed
We note that Proposition 1.5 gives a combinatorial formula for the Rasmussen invariant of an alternating diagram. It is known [Lee05] that the Rasmussen invariant of an alternating knot agrees with the signature of the knot, and there is also known [Tra04] a combinatorial formula for the signature of an alternating diagram. Proposition 1.5 gives an equivalence between these two results.

There is a nice topological interpretation of $\Delta$, which is useful in computing it by hand.

**Proposition 1.12.** Form a graph $G$ that has a node for each component of $T^-(D)$ and a node for each component of $T^+(D)$. Each circle in $O(D)$ is a member of exactly one component of $T^-(D)$ and exactly one component of $T^+(D)$; for each circle in $O(D)$ let $G$ have an edge connecting the corresponding pair of nodes.

Then $\Delta(D) = b_1(G)$, the first Betti number of $G$.

**Proof.** This follows from the connectedness of $G$ so that we have

\[
b_1(G) = b_0(G) - \chi(G) = 1 - \#\text{nodes}(G) + \#\text{edges}(G)
= 1 - \#\text{components}(T^-(D)) - \#\text{components}(T^+(D)) + \#\text{nodes}(T(D))
= \Delta(D).
\]
2. Proof of main results

We assume familiarity with the definition of the Khovanov chain complex defined from a knot diagram $D$, and with Rasmussen’s paper [Ras10]. We write $\mathcal{F}^j C^i(D)$ for Lee’s perturbed chain complex with complex coefficients (where the topological quantum field theory (TQFT) is induced from the Frobenius algebra $\mathbb{C} \hookrightarrow \mathbb{C}[x]/(x^2-1)$), with the $\mathcal{F}^j$ representing the quantum filtration

$$\ldots \subseteq \mathcal{F}^{j+1} C^i \subseteq \mathcal{F}^j C^i \subseteq \mathcal{F}^{j-1} C^i \subseteq \ldots$$

and the superscript $i$ denoting the homological grading

$$\partial_i : \mathcal{F}^j C^i \to \mathcal{F}^j C^{i+1}, \quad \partial_i \partial_{i-1} = 0.$$ 

Similarly we write $\mathcal{F}^j H^i(D)$ for the homology of the chain complex $\mathcal{F}^j C^i(D)$.

There is a distinguished subspace of $C^0(D)$, which I shall write as $H(O(D))\{w(D)\}$, $O(D)$ being the oriented resolution of $D$ and $\{w(D)\}$ being a shift in the quantum filtration by the writhe of $D$. Here one can think either of $H$ as being Lee’s TQFT functor or of $H(O(D))$ as being the perturbed Khovanov homology of the (0-crossing) diagram $O(D)$.

Remark. Our method of proving Theorem 1.2 is to restrict our attention to the summand $H(O(D))$ of $C^0(D)$. There is a generator for the homology $H^0(D)$ whose filtered degree in the homology determines $s(D)$. This generator lies in the summand $H(O(D))$, so a bound on $s(D)$ can be calculated by looking at the filtered degree of the generator in a certain quotient of $H(O(D))$.

This method will give possibly better (certainly no worse) approximations for $s(D)$ if the subspace $H(O(D))$ is enlarged (for example by taking the direct sum of $H(O(D))$ with a summand corresponding to a different resolution of $D$, which still lies in homological degree zero). In the general case, there is no obvious choice for a useful enlargement, but given a particular class of knots it is possible that better bounds on $s(D)$ can be obtained by a suitable choice of larger summand.

By Lee [Lee05] we know that the following theorem holds.

**Theorem 2.1.** Given a knot diagram $D$ with orientation $o$, there exist $s_o, s_\overline{o} \in H(O(D))\{w(D)\} \subseteq C^0(D)$ such that $\partial_0 s_o = \partial_0 s_\overline{o} = 0$ Furthermore, the homology $\mathcal{F}^j H^i(D)$ is two-dimensional and supported in homological grading $i = 0$ with $H^0(D) = \{[s_o], [s_\overline{o}]\}$.

There is an explicit description of these generators at the chain level.

**Definition 2.2.** The orientation $o$ on $D$ induces an orientation on $O(D)$. For each circle $C$ in $O(D)$ we give an invariant that is the mod 2 count of the number of circles in $O(D)$ separating $C$ from infinity, to which we add 0 (respectively 1) if $C$ has the counter-clockwise (respectively clockwise) orientation. We label $C$ with $v_+ + v_+$ (respectively $v_- - v_+$) if the invariant is 0 (respectively 1) (mod 2). Here $v_+, v_-$ is a basis for the vector space $H(S^1)$, where $H$ is Lee’s TQFT functor; $v_+$ has quantum degree +1 and $v_-$ has quantum degree −1. This determines an element $s_o \in H(O(D))\{w(D)\}$, $s_\overline{o}$ being given in the same way but using the opposite orientation $\overline{o}$ on $D$.

We know that, in Rasmussen’s notation, $s(D) = s_{\min}(D) + 1$ and $s_{\min}(D)$ is the filtration grading of the highest filtered part of $H^0(D)$ to contain $[s_o]$ (or equivalently $[s_\overline{o}]$; this
interchangeability is taken as understood from now on). This is the same as the filtration grading of the highest filtered part of \( \mathcal{C}^0/\text{im}(d_{-1}) \) containing \([s_o]\). The following lemma results.

**Lemma 2.3.** Let \( p: \mathcal{C}^0(D) \rightarrow H(O(D))\{w(D)\} \) be the projection onto the vector space summand. Then

\[
\min(D) \leq L(D),
\]

where \( L(D) \) is the filtration grading in \( H(O(D))\{w(D)\}/\text{im}(p \circ d_{-1}) \) of the highest filtered part containing \([s_o]\).

**Proof of Theorem 1.2.** Given a knot diagram \( D \) with orientation \( o \), we write \( n_+, n_- \) for the number of positive, negative crossings of \( D \), respectively, so that the writhe \( w(D) = n_+ - n_- \).

Form the diagram \( D^- \) by taking the oriented resolution at each of the positive crossings. Note that diagram \( D^- \) is also oriented with writhe \(-n_-\). Suppose there are \( l \) components \( D^-_1, D^-_2, \ldots, D^-_l \) of \( D^- \) (where we mean components as a subset of the plane, so that the standard 2-crossing diagram of the Hopf link would be considered as a single component, for example) and suppose that \( D^-_r \) has \( n_r \) crossings for \( 1 \leq r \leq l \).

We observe that, up to quantum filtration shift by \( \{n_+\} \), the map

\[
p \circ d_{-1}: C^{-1}(D) \rightarrow H(O(D))\{w(D)\} \subseteq \mathcal{C}^0(D)
\]

can be identified with the map

\[
d_{-1}: C^{-1}(D^-) \rightarrow \mathcal{C}^0(D^-) = H(O(D^-))\{-n_-\}.
\]

This latter map is in fact \( \bigoplus_{r=1}^l d_{-1}^r \otimes \text{id}^r \), where

\[
d_{-1}^r: C^{-1}(D^-_r) \rightarrow \mathcal{C}^0(D^-_r) = H(O(D^-_r))\{-n_r\}
\]

is the \((-1)\)th differential in the chain complex \( \mathcal{C}^*(D^-_r) \) and

\[
\text{id}^r: H(O(D^- \setminus D^-_r))\{-n_- + n_r\} \rightarrow H(O(D^- \setminus D^-_r))\{-n_- + n_r\}
\]

is the identity map.

Inductively on \( r \) we observe a canonical identification

\[
\text{coker} \left( \bigoplus_{r=1}^l (d_{-1}^r \otimes \text{id}^r) \right) = \bigotimes_{r=1}^l \text{coker}(d_{-1}^r) = \bigotimes_{r=1}^l (H^0(D^-_r)).
\]

Now \( s_o = s_1 \otimes s_2 \otimes \cdots \otimes s_l \), where \( s_r \in \mathcal{C}^0(D^-_r) \) is either the element \( s_{o'} \) or \( s_{\overline{o'}} \), where we use \( o' \) to stand for the induced orientation on the oriented resolution of \( D^-_r \). This is because the mod 2 invariant associated to each circle \( C \subset O(D^-_r) \) via Definition 2.2 differs by 0 or 1 from the invariant associated to \( C \subset O(D) \) via Definition 2.2, and it is the same difference for all circles of \( O(D^-_r) \).

Suppose that the number of components of \( O(D^-_r) \) is \( e_r \). We observe that \( \mathcal{F}^{e_r - n_r} \mathcal{C}^0(D^-_r) \) is the highest filtered part of \( \mathcal{C}^0(D^-_r) \) to be non-zero and is one-dimensional. By [Ras10, Lemma 3.5], we know that \([s_r]\) cannot be of top filtered degree in \( H^0(D^-_r) \). Therefore \([s_r]\) has filtered degree less than or equal to \( e_r - n_r - 2 \) in \( H^0(D^-_r) \).
We compute for $L(D)$ in Lemma 2.3:

$$L(D) \leq n_+ + \sum_{r=1}^{l} (e_r - n_r - 2)$$

$$= n_+ - n_- + \#\text{nodes}(T(D)) - 2\#\text{components}(T^-(D))$$

$$= \#\text{nodes}(T(D)) - 2\#\text{components}(T^-(D)) + w(D).$$

Hence we have

$$s(D) = s_{\min}(D) + 1 \leq L(D) + 1 \leq \#\text{nodes}(T(D)) - 2\#\text{components}(T^-(D)) + w(D) + 1 = U(D).$$

**Proof of Theorem 1.10.** Given an oriented knot diagram $D$, let $\overline{D}$ be the mirror image of $D$. It is then easy to check that

$$2\Delta(D) = U(D) + U(\overline{D}).$$

So we have

$$s(D) = -s(\overline{D}) \geq -U(\overline{D}) = U(D) - 2\Delta(D).$$

3. Generalizations to links

Given an $r$-component link $L \subset S^3$, let $G(L)$ be the genus of a connected minimal-genus smooth surface in the 4-ball that has $L$ as boundary. We extend the definition of the slice genus $g^*$ to links by defining

$$g^*(L) = G(L) + \frac{1}{2} - \frac{r}{2} \in \frac{1}{2}\mathbb{Z}.$$  

The definition of the $s$-invariant for links as found in [BW] is such that the proof of Theorem 1.2 carries through unchanged to this setting. Also by [BW] we know that:

(i) $s(L) \leq 2g^*(L)$; and

(ii) $s(L) + s(\overline{L}) \geq 2 - 2r$.

Hence we also obtain a version of Corollary 1.11 for links.

**Corollary 3.1.** Suppose $D$ is a diagram of an $r$-component link and $T(D)$ and $T^+(D)$ are the associated graphs, then

$$2g^*(D) \geq w(D) - \#\text{nodes}(T(D)) + 2\#\text{components}(T^+(D)) - 2r + 1.$$ 

**Proof.** We have

$$2g^*(D) \geq s(D) \geq 2 - 2r - s(\overline{D}) \geq 2 - 2r - U(\overline{D}) = w(D) - \#\text{nodes}(T(D)) + 2\#\text{components}(T^+(D)) - 2r + 1. \quad \square$$

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REFERENCES


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