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Necessary and sufficient conditions for a maximal ergodic theorem along subsequences

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Abstract. Let T be an ergodic measure preserving point transformation from a probability space X onto itself. Assume that $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence of subsets of the positive integers. Conditions are given which are sufficient for the ergodic maximal function associated with these subsets to be weak type (p, p). These conditions are shown to be both necessary and sufficient for a larger two-sided maximal function. The conditions are in the form of covering lemmas for the integers.

Let (X, Σ, m) denote a complete non-atomic probability space, and let $T: X \to X$ be an invertible measure-preserving point transformation mapping X onto itself.

Let $\{S_n\}_{n=1}^{\infty}$ be a collection of finite subsets of integers. Define

$$f_N^*(x) = \sup_{n \leq N} \frac{1}{|S_n|} \sum_{k \in S_n} f(T^k x),$$

where $|S_n|$ denotes the number of elements in the set S_n. Also define

$$M_N f(x) = \sup_{n \leq N} \sup_{s \in S_n} \frac{1}{|S_n|} \sum_{k \in S_n - s} f(T^k x).$$

If $S_n = \{0, 1, 2, 3, \ldots, n-1\}$, then f_N^* is the classical ergodic maximal function and $M_N f$ is the standard two-sided maximal function. With this choice of $\{S_n\}_{n=1}^{\infty}$ the classical statement of the maximal ergodic theorem is:

$$\int_{\{f_N^* > 0\}} f(x)\, dm(x) \geq 0 \qquad \text{for all } f \in L^1(X).$$

If we let $f = g - \lambda$ then $f^*(x) = g^*(x) - \lambda$, and the inequality can be written

$$\int_{\{g_N^* > \lambda\}} g(x) - \lambda \, dm(x) \geq 0,$$

or

$$\int_{\{g_N^* > \lambda\}} g(x)\, dm(x) \geq \lambda m\{g^* > \lambda\}.$$

The goal is to prove a generalization of the above maximal inequality with more general $\{S_n\}_{n=1}^{\infty}$, and also with the larger maximal function $M_N f$. In what follows the subscript N will be dropped unless it is needed for clarity. In general we will want to obtain estimates that do not depend on N, and then let N go to infinity.

We will actually prove a weak type (p, p) inequality, i.e.

$$m\{x \mid f^*(x) > \lambda\} \le \frac{c}{\lambda^p} \int_X |f(x)|^p \, dm(x),$$

where c depends only on $\{S_n\}_{n=1}^{\infty}$, and the same inequality with f^* replaced by Mf. We will also assume that $f \ge 0$. If not, replace $f(x)$ in the definition of $f^*(x)$ and $Mf(x)$ by $|f(x)|$. The convergence theorem for $f \in L^p$ will follow from the weak type (p, p) result if we can establish convergence on some dense subset of L^p. For example, in the classical case, the dense subset of L^1 that is used is

$$\{f(x) + g(x) - g(Tx) \mid f \in L^1 \text{ is invariant, } g \in L^{\infty}\}.$$

As an aside, in the study of the ergodic theorem along subsequences, it was only recently discovered (by A. Bellow in [**1**]) that there are cases which are delicate in the following sense: If we look at a sequence that was previously known to be 'bad', such as $S_n = \{1, 2, 4, 8, 16, \ldots, 2^n\}$ then a 'bad' f could be found which was even in $L^{\infty}(X)$. If we look at the previously known 'good' sequences, they worked for all $f \in L^1(X)$. However, there are many examples of convergence problems where more delicate methods are needed. For example, in Fourier analysis, if $f \in L^p$, $p > 1$, then the Fourier series of f converges almost everywhere, but there exists an f in L^1 such that the Fourier series of f diverges everywhere. Bellow has shown that it is necessary, when considering certain subsequences, to use these more delicate methods in ergodic theory too, i.e. to look at $f \in L^p(X)$, $p > 1$. The necessary inequality will then be the weak type (p, p) inequality:

$$m\{f^* > \lambda\} < \frac{c}{\lambda^p} \int_X |f(x)|^p \, dm(x).$$

The fact that the first cases to require the L^p inequality for $p > 1$ have so recently been discovered seems to indicate our state of ignorance about this situation.

There are several special subsequences for which complete information is known. These will be discussed later in the paper.

Theorem 1 below will give necessary and sufficient conditions on the sequence of sets, $\{S_n\}_{n=1}^{\infty}$, for the weak type (p, p) inequality to hold for the maximal function Mf.

The following definition will use notation analogous to that in [**3**].

Definition. The sequences of sets $\{S_n\}_{n=1}^{\infty}$ *has the property* V_q, $1 \le q \le \infty$, if the following is satisfied:

There exist constants $C < \infty$ and $c > 0$, depending only on the sets $\{S_n\}_{n=1}^{\infty}$, such that if U is a finite subset of the integers with the property that for each $k \in U$ we have an associated set $S_{n(k)} + k - s(k)$ where $s(k) \in S_{n(k)}$, then we can select a subset I of U, such that:

(1) $\|\sum_{i \in I} \chi_{\{S_{n(i)} + i - s(i)\}}\|_q \le C \, |\bigcup_{i \in I} \{S_{n(i)} + i - s(i)\}|^{1/q}$ with $1/p + 1/q = 1$; and

(2) $|\bigcup_{i \in I} \{S_{n(i)} + i - s(i)\}| > c|U|$.

(Here $\|f\|_q$ is the norm in l^q.)

These properties say that if we know that the size of the union of $\{S_{n(i)}+i-s(i)\}_{i\in I}$ is larger than a certain fraction of U, and we know an estimate for the degree of overlap, then we know a good estimate for the size of the set U. The first property says that there is only limited overlap. In fact, the case $q=\infty$ says that no point is in more than C of the sets. The second property says there are enough points to cover a given percent of U.

THEOREM 1. *The maximal function Mf is weak type (p, p) if and only if the sequence of sets $\{S_n\}_{n=1}^{\infty}$ has the property V_q, $1/p+1/q=1$, $1<p<\infty$.*

Proof. First assume that the sequence of sets satisfies the property V_q. Because T is measure preserving, we can write

$$\int_X |f(x)|^p\,dm(x)=\frac{1}{2L+1}\sum_{k=-L}^{k=L}\int_X |f(T^kx)|^p\,dm(x)$$
$$=\int_X \frac{1}{2L+1}\sum_{k=-L}^{k=L}|f(T^kx)|^p\,dm(x).$$

If we could show that

$$\sum_{k=-L}^{k=L}|f(T^kx)|^p>c\lambda^p\sum_{k=-(L-N^*)}^{k=(L-N^*)}\chi_{\{M_Nf>\lambda\}}(T^k(x)), \tag{$*$}$$

where $N^*=\sup_{k\le N}\sup_{j\in S_k}|j|$ and L is as large as desired, then we would have

$$\int_X |f(x)|^p\,dm(x)>\frac{1}{2L+1}\int_X c\lambda^p\sum_{k=-(L-N^*)}^{k=(L-N^*)}\chi_{\{M_Nf>\lambda\}}(T^k(x))\,dm(x)$$
$$\ge c\lambda^p\frac{2(L-N^*)+1}{2L+1}\,m\{Mf>\lambda\}.$$

Since L could be taken as large as necessary, this would prove the desired weak type (p, p) inequality.

To prove that $(*)$ holds, define

$$U=U_x=\{k\in(-(L-N^*),\,L-N^*)\,|\,Mf(T^kx)>\lambda\}.$$

Thus $(*)$ becomes

$$\sum_{k=-L}^{k=L}|f(T^kx)|^p>c\lambda^p|U|.$$

If $k\in U$ then there exists $n(k)<N$ and an integer $s(k)\in S_{n(k)}$ such that

$$\frac{1}{|S_{n(k)}|}\sum_{j\in S_{n(k)}-s(k)}|f(T^k(T^jx))|>\lambda,$$

which can be rewritten as

$$|S_{n(k)}+k-s(k)|<\frac{1}{\lambda}\sum_{j\in S_{n(k)}+k-s(k)}|f(T^j(x))|.$$

Now select from $\{S_{n(k)}+k-s(k)\,|\,k\in U\}$ a sub-collection $\{S_{n(i)}+i-s(i)\,|\,i\in I\}$ where

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$I \subset U$, satisfying (1) and (2) above. We then have by (2) that $|U| < c^{-1} |\bigcup_{i\in I} S_{n(i)} + i - s(i)|$ and

$$\left|\bigcup_{i\in I} S_{n(i)} + i - s(i)\right| \le \sum_{i\in I} |S_{n(i)} + i - s(i)|$$
$$\le \sum_{i\in I} \frac{1}{\lambda} \sum_{j\in S_{n(i)}+i-s(i)} |f(T^j x)|)$$
$$\le \sum_{i\in I} \frac{1}{\lambda} \sum_{j=-L}^{L} |f(T^j x)| \chi_{\{S_{n(i)}+i-s(i)\}}(j)$$
$$\le \frac{1}{\lambda} \sum_{j=-L}^{L} f(T^j x) \sum_{i\in I} \chi_{\{S_{n(i)}+i-s(i)\}}(j)$$

(now use Holder's inequality)

$$\le \frac{1}{\lambda}\left[\sum_{j=-L}^{L} |f(T^j x)|^p\right]^{1/p} \left[\sum_{j=-L}^{L} \left|\sum_{i\in I} \chi_{\{S_{n(i)}+i-s(i)\}}(j)\right|^q\right]^{1/q}$$
$$\le \frac{1}{\lambda}\left[\sum_{j=-L}^{L} |f(T^j x)|^p\right]^{1/p} C \left|\bigcup_{i\in I} S_{n(i)} + i - s(i)\right|^{1/q}.$$

The last step uses property (1) of our sequence $\{S_n\}_{n=1}^{\infty}$. Dividing both sides by $|\bigcup_{i\in I} S_{n(i)} + i - s(i)|^{1/q}$, and recalling that $1 - 1/q = 1/p$, we have

$$\left|\bigcup_{i\in I} S_{n(i)} + i - s(i)\right|^{1/p} \le C \frac{1}{\lambda}\left[\sum_{j=-L}^{L} |f(T^j x)|^p\right]^{1/p};$$

now raise both sides to the pth power and use the estimate for $|U|$ from condition (2) to obtain

$$|U| \le c^{-1} C^p \frac{1}{\lambda^p} \sum_{j=-L}^{L} |f(T^j x)|^p,$$

the required inequality. □

The proof of the converse requires the following lemma, the proof of which follows that given by A. Cordoba and R. Fefferman in [3] for the case of rectangles in $\mathbb{R}^n$.

LEMMA. *Let Mf be an operator on the integers in the interval $(-L, L)$ defined by*

$$Mf(j) = \sup_{n\le N} \sup_{m\in S_n} \frac{1}{|S_n|} \sum_{k\in S_n - m} f(j+k),$$

where $f(j)$ is assumed to be zero for j not in $(-L, L)$. If $Mf: L^p \to L(p, \infty)$, i.e. Mf maps L^p into weak L^p, with the operator norm bounded independent of L, then the sequence of sets $\{S_n\}_{n=1}^{\infty}$ satisfies the property V_q, where $1/p + 1/q = 1$ and $1 < p < \infty$.

Proof. First assume that we are given a set U and a sub-collection $\{S_{n(i)} + i - s(i) \mid i \in I\}$ of the family of sets $\{S_{n(k)} + k - s(k) \mid k \in U\}$, with the property that for each $i \in I$,

(P1) $|(S_{n(i)} + i - s(i)) \cap (\bigcup_{j<i, j\in I} (S_{n(j)} + j - s(j)))| \le \frac{1}{2}|S_{n(i)}|$.

We claim that the sets $\{S_{n(i)} + i - s(i) \mid i \in I\}$ also satisfy

$$\left\|\sum_{i\in I} \chi_{S_{n(i)}+i-s(i)}\right\|_q \le C \left|\bigcup_{i\in I} (S_{n(i)} + i - s(i))\right|^{1/q}.$$

To see this, let

$$E_i=(S_{n(i)}+i-s(i))-\Bigg(\bigcup_{\substack{j<i\\ j\in I}}(S_{(j)}+j-s(j))\Bigg).$$

By property P1 we have $|E_i|\geq\frac{1}{2}|S_{n(i)}|$. Define the linear operator σ on $(-L, L)$ by

$$\sigma(f)(u)=\sum_{i\in I}\left\{\frac{1}{|S_{n(i)}|}\sum_{j\in S_{n(i)}+i-s(i)}f(j)\right\}\chi_{E_i}(u).$$

Note that if $u\in E_i$ then $u=s+i-s(i)$ for some $s\in S_{n(i)}$. Consequently $i=u-s+s(i)$, and we have that $\sigma(f)(u)\leq Mf(u)$. The adjoint operator is then given by

$$\sigma^*(f)(u)=\sum_{i\in I}\left\{\frac{1}{|S_{n(i)}|}\sum_{j\in E_i}f(j)\right\}\chi_{S_{n(i)}+i-s(i)}(u).$$

Notice that

$$\sigma^*(\chi_{\bigcup_{i\in I}(S_{n(i)}+i-s(i))})\geq\tfrac{1}{2}\sum_{i\in I}\chi_{(S_{n(i)}+i-s(i))}.$$

Because σ is bounded from L^p to $L(p,\infty)$ it follows that σ^* is bounded from $L(q,1)$ to L^q. Consequently,

$$\left\|\sum_{i\in I}\chi_{S_{n(i)}+i-s(i)}\right\|_q\leq C\left|\bigcup_{i\in I}(S_{n(i)}+i-s(i))\right|^{1/q}.$$

The constant C depends only on the bound of the operator Mf and does not depend on L, or the choice of the process of selection for the sets $\{S_{n(i)}+i-s(i)\}_{i\in I}$.

To see that it is always possible to select a subset I of U with the required disjointness property, P1, consider the following: Let k_1 be the first element in U. (Where elements are ordered by the usual ordering.) Put k_1 in I. Assume that $k_1, k_2, \ldots, k_j$ have been placed in I. Place k in I (and call it k_{j+1}) if k is the smallest element in I such that $k>k_j$ and

$$\left|(S_{n(k)}+k-s(k))\cap\Bigg(\bigcup_{\substack{j<k\\ j\in I}}(S_{n(j)}+j-s(j))\Bigg)\right|\leq\tfrac{1}{2}|S_{n(k)}|.$$

Continue the process until it is no longer possible to make a further selection.

To see that the selected sets cover enough of U, let $k\in U$ and assume that k was not selected. Then

$$\left|(S_{n(k)}+k-s(k))\cap\Bigg(\bigcup_{j\in I}(S_{n(j)}+j-s(j))\Bigg)\right|>\tfrac{1}{2}|S_{n(k)}|.$$

Consequently, if we write E for $\bigcup_{j\in I}(S_{n(j)}+j-s(j))$, then

$$\begin{aligned}M\chi_E(k)&\geq\frac{1}{|S_{n(k)}|}\sum_{j\in S_{n(k)}+k-s(k)}\chi_E(j)\\&=\frac{1}{|S_{n(k)}|}|(S_{n(k)}+k-s(k))\cap E|\\&>\frac{1}{|S_{n(k)}|}\cdot\tfrac{1}{2}|S_{n(k)}|\geq\tfrac{1}{2}.\end{aligned}$$

Therefore, $|U|<|\{M\chi_E>\frac{1}{2}\}|\leq c\|\chi_E\|_p^p=c|E|$. □

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With this lemma, we can now easily complete the proof of the theorem. If Mf defined on $(-L, L)$ fails to be weak type (p, p) then the ergodic maximal function Mf defined on (X, Σ, m) by T and $\{S_n\}_{n=1}^{\infty}$ also fails to be weak type (p, p). To see this assume that there exists L_N and f_N such that

$$|\{(-L_N, L_N)\,|\,Mf_N > 1\}| \geq N\|f_N\|_p^p.$$

Construct a Rohlin tower of height $2L_N+1$ and error ε. On the jth step above the base define $g_N(x) = f_N(-L_N+j)$, and define g_N to be zero off the tower. Then clearly we have $m\{Mg_N > 1\} \geq N\|g_N\|_p^p$, i.e. the ergodic maximal function fails to be weak type (p, p).

THEOREM 2. *If Mf is weak type $(1, 1)$ then it is possible to select a subset I of U such that the disjointness property P1 holds and such that*

$$\left\|\exp\left(\sum_{i\in I} \chi_{(S_{n(i)}+i-s(i))}\right)\right\|_1 \leq C\left|\bigcup_{i\in I} S_{n(i)}+i-s(i)\right|$$

for an absolute constant $C < \infty$.

The proof of this result follows as in the proof of theorem 1 above.

THEOREM 3. *If the sequence of sets $\{S_n\}_{n=1}^{\infty}$ satisfies the property V_q with $1/p+1/q=1$, $1 \leq p < \infty$, and with $s(k)$ identically zero, then the ergodic maximal function f^* is weak type (p, p).*

Proof. The proof is exactly the same as the first part of theorem 1, with $s(k)$ replaced by zero. □

With the above result, it is easy to prove the maximal inequality for f^* in the special case $S_n = \{0, 1, 2, 3, \ldots, (n-1)\}$. Start at $-(L-N^*)$ and move to the right. Select the first point in U, call it k_0, and put it in the set I. Continue moving to the right and select the next point not in $S_{n(k_0)}+k_0$ that is in U, call it k_1, and put it in I. In general, select k_{i+1} the next point in U to the right of $S_{n(k_i)}+k_i$, and put it in I. Continue until we have reached $L-N^*$. By construction the sets are disjoint so property (1) is true with $q=\infty$. Also by construction, U is contained in $\bigcup_{i\in I} S_{n(i)}+i$. Hence property (2) is true.

An argument similar to the one above works for the special case $S_n = \{0, k, 2k, 3k, \ldots, (n-1)k\}$. Simply make k passes through the region. On the jth pass look only at the points of the form $j \pm nk$.

The following set of sufficient conditions for f^* to be weak type $(1, 1)$ were introduced by Templeman:

(a) $|S_n - S_n| < K|S_n|$; and

(b) $S_n \subset S_{n+1}$.

We now show that a related condition is a sufficient condition for Mf to be weak type $(1, 1)$.

THEOREM 4. *If the conditions*

(a′) $|S_n - S_n + S_n - S_n| < C|S_n|$; *and*

(b′) $S_n \subset S_{n+1}$

are satisfied, then Mf is weak type $(1, 1)$.

Proof. We can show that these conditions are sufficient for the maximal function Mf to be weak type (1, 1) by showing that if (a′) and (b′) above are true then $\{S_n\}_{n=1}^{\infty}$ satisfies property V_∞.

Partially order the sets $S_{n(k)}+k-s_{n(k)}$ by

$$S_{n(j)}+j-s(j)<S_{n(k)}+k-s(k) \qquad \text{if } |S_{n(j)}|<|S_{n(k)}|.$$

Select the subcovering of U by the following procedure: First select the largest possible set, $S_{n(k_1)}+k_1-s(k_1)$. If there are two or more sets of the largest size, select any one of them. Future selections are made in sequence by selecting at each stage the largest possible remaining set (based on the above order) that does not intersect any previously selected set. The process must stop because we have only a finite number of sets.

By construction, the covering is disjoint. To see that a fraction of U is covered, consider the following: Let u be a point in U that is not in any of the selected sets. Then by construction, $S_{n(u)}+u-s(u)\cap S_{n(j)}+j-s(j)\neq\emptyset$ for some j which was selected. We also know by construction that $S_{n(u)}\subset S_{n(j)}$. Let $t_u\in S_{n(u)}$ and $t_j\in S_{n(j)}$ such that $t_u+u-s(u)=t_j+j-s(j)$. Then $u=t_j-s(j)+s(u)-t_u+j$. In other words,

$$u\in S_{n(j)}-S_{n(j)}+S_{n(j)}-S_{n(j)}+j,$$

or

$$U\subset\bigcup\{-S_{n(j)}+S_{n(j)}-S_{n(j)}+S_{n(j)}+j \mid j \text{ in selected set } I\},$$

which says

$$\begin{aligned}|U|&\leq\sum_{j\in I}|-S_{n(j)}+S_{n(j)}-S_{n(j)}+S_{n(j)}|\\ &\leq C\sum_{j\in I}|S_{n(j)}|\\ &\leq C\left|\bigcup_{j\in I}S_{n(j)}+j-s(j)\right|.\end{aligned}$$

which is just condition (2). □

The above condition can be used to give a proof of the fact that the maximal function Mf is weak type (1, 1) for the block sequences discussed by Bellow and Losert [**2**]. They define a block sequence by first defining a pair of increasing sequences of integers $\{n_k\}$ and $\{l_k\}$ with $l_k<n_{k+1}-n_k$ and the growth condition $l_k\geq Cn_{k-1}$. The block sequence generated by $\{n\}$ and $\{l\}$ is defined to be the sequence

$$n_1, n_1+1, \ldots, n_1+l_1, \ldots, n_k, n_k+1, \ldots, n_k+l_k, \ldots.$$

To see that the maximal function Mf is weak type (1, 1) we will show that it satisfies the Templeman-like condition above.

By definition of the block sequence, a typical set S will satisfy

$$S\subset[0, n_{k-2}+l_{k-2}]\cup[n_{k-1}, n_{k-1}+l_{k-1}]\cup[n_k, n_k+e]$$

where $e\leq l_k$ denotes the number of terms from the last block which are used in the set. Note that all of the last block will not necessarily be used.

Using the growth condition, $l_k>Cn_{k-1}$, we have

$$S\subset[0,(1+C)l_{k-1}]\cup[n_{k-1}, n_{k-1}+l_{k-1}]\cup[n_k, n_k+e].$$

If $e \geq l_{k-1}$ then l_{k-1} can be replaced by e in the above containment. In any case S will be contained in the union of three intervals. It is easy to see that $S-S+S-S$ will be contained in the union of at most 16 intervals. The longest of these intervals will have a length which is no more than a fixed multiple of the length of the longest of the three intervals that contain S. Thus $|S-S+S-S|<C|S|$.

If we look at the sequence

$$1, 4, 9, 16, 25, \ldots, n^2, \ldots,$$

Fernando Soria [4] has pointed out that maximal function Mf associated with this sequence is not weak type (p, p) for any $p<2$. To see this, it is enough to first see the result on the integers and then construct a Rohlin tower as in the proof of theorem 1. Define f_N on the integers to be one on the interval from 0 to N^2 and zero elsewhere. At each positive integer k define $s(k)$ to be the largest perfect square less than k. Then

$$Mf(k) > \frac{1}{\sqrt{s(k)}} \sum_{j^2 \leq s(k)} \chi_{(k+j^2-s(k)<N^2)} \simeq N/\sqrt{k},$$

if k is larger than N. Note that

$$|\{Mf(k)>1/N\}| \simeq |\{N/\sqrt{k} \geq 1/N\}| = |\{N^2 \geq \sqrt{k}\}| \simeq N^4.$$

To be weak type (p, p) we must have

$$|\{Mf(k)>1/N\}| \leq \frac{c}{(1/N)^p} N^2 = cN^{2+p}.$$

For p less than two, this is impossible by the above example.

A similar argument shows that the maximal function Mf associated with the sequence $\{n^p\}_{n=1}^{\infty}$ can be no better than weak type (p, p).

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