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INDEPENDENCE RESULTS CONCERNING DEDEKIND-FINITE SETS

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A Dedekind-finite set is one not equinumerous with any of its proper subsets; it is well known that the axiom of choice implies that all such sets are finite. In this paper we show that in the absence of the axiom of choice it is possible to construct Dedekind-finite sets which are large, in the sense that they can be mapped onto large ordinals; we extend the result to proper classes. It is also shown that the axiom of choice for countable sets is not implied by the assumption that all Dedekind-finite sets are finite.

1. Preliminaries

We work throughout in Zermelo-Fraenkel (ZF) set theory, without the axiom of choice but with the axiom of foundation.

NOTATIONS. *Let $f: X \to Y$ be a function.*

$$\text{If } A \subseteq X,\ f''A = \{y : (\exists x \in A)(f(x) = y)\}.$$

$$\text{If } B \subseteq Y,\ f^{-1}(B) = \{x : (\exists y \in B)(f(x) = y)\}.$$

$$X \ {}^*\!\geqq Y \text{ means that } X \text{ can be mapped onto } Y.$$

We write $|X|$ for the cardinal of X, $S(X)$ for the power set of X, $S_\kappa(X)$ for $\{Y \subseteq X : |Y| < \kappa\}$, $\mathrm{Seq}(X)$ for the set of finite sequences of elements of X. ${}^A B$ is the set of all functions from A into B; $B^A = |{}^A B|$. V_α is the set of sets of rank $\leqq \alpha$, ON the class of all ordinals. 'x is finite' means that x has n elements for some $n < \omega$.

FORCING. We follow the approach of Shoenfield (1971), but adopt a different convention for names in the forcing language. In general we say that if $x \in M[G]$

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$\boldsymbol{x}$ shall be a name for x, where $\boldsymbol{x}$ is a symbol which is an element of M. However if in fact $x \in M$ we use x as a name for x. This does not introduce any ambiguity as if $x \in M$ $\boldsymbol{x}$ is interpreted in all $M[G]$ as x.

We note the following general symmetry lemma.

LEMMA. *Let M be a countable transitive model of ZFC, $P \in M$ a notion of forcing, $\pi \in M$ an automorphism of P and $\phi(v_0, \cdots, v_n)$ a ZF-formula. Then*

$$p \Vdash \phi(G, x_1, \cdots, x_n) \leftrightarrow \pi^{-1}p \Vdash \phi(\pi''G, x_1, \cdots, x_n),$$

where $x_1, \cdots, x_n \in M$ and $p \in P$.

We borrow from Shoenfield the notation $H_\kappa(A, B)$ for

$$\{f : \mathrm{dom}(f) \in S_\kappa(A),\ \mathrm{ran}(f) \subseteq B\}.$$

WEAK AXIOMS OF CHOICE. C^ω (axiom of choice for countable sets). If A is a countable set of non-empty sets then A has a choice function.

wC^ω (weak C^ω). If A is a set of non-empty sets such that $|A| = \omega$ there exists $B \subseteq A$, $|B| = \omega$, such that B has a choice function.

RELATIVE CONSTRUCTIBILITY. Let $\mathfrak{A}$ be a relational structure. We define $D(\mathfrak{A})$, the set of definable subsets of $\mathfrak{A}$, thus.

$D(\mathfrak{A}) = \{\{x \in A : \phi[x]\} : \phi(v_0)$ a formula of the language of $\mathfrak{A}$ with v_0 its only free variable$\}$.

THEOREM. *Let N be a transitive class such that*

(i) *$N = \bigcup \{N_\alpha : \alpha \in ON\}$, where each N_α is a set*

(ii) *$(N_\alpha)_{\alpha \in ON}$ is increasing.*

(iii) *α a limit ordinal $\rightarrow N_\alpha = \bigcup \{N_\beta : \beta < \alpha\}$*

(iv) *$D(\langle N_\alpha, \in, (y)_{y \in N_\alpha}\rangle) \subseteq N$, for each α.*

Then N is a proper class and a model of ZF.

We give an application. Let X be a transitive set. Set $N_0 = X$, $N_{\alpha+1} = D(\langle N_\alpha, \in, (y)_{y \in N_\alpha}\rangle)$. In this case we write $L(X)$ for N. It can be shown that $L(X)$ is the smallest transitive proper class which contains X and satisfies ZF. Further there exists a canonical functional

$$F_0 : ON \times \mathrm{Seq}(X) \rightarrow L(X)$$

with the properties that F_0 is onto, F_0 is defined from X and the definition of F_0 is absolute for any transitive proper class which contains X and satisfies ZF.

If X is not transitive, by $L(X)$ we mean $L(TC(X))$, where

$$TC(X) = \{X\} \cup X \cup (\cup X) \cup (\cup \cup X) \cup \cdots$$

$TC(X)$ (the transitive closure of X) is the smallest transitive set with X as a member.

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2. Some positive results

By 'x is *DF*' we mean that x is Dedekind-finite but not finite. It is well known that x is Dedekind-finite if and only if $\omega \nleqq |x|$, so x is *DF* if and only if $\omega \nleqq |x|$ and $|x| \nleqq \omega$. This shows that comparing the cardinal of a *DF* set with the alephs won't give us any information. However we may get some idea of the 'size' of a *DF* set by seeing which alephs it can be mapped onto.

The following theorem is due to Kuratowski (see Tarski (1924), pages 94–95).

THEOREM 2.1. *Here x is an arbitrary set.*

(i) *If x is infinite then $\omega \leqq^* 2^x$.*

(ii) $\omega \leqq^* x \leftrightarrow \omega \leqq 2^x$.

(iii) *If x is DF, $\omega \leqq^* x$ if and only if 2^x is not DF.*

COROLLARY 2.2 *If there are DF sets there are DF sets which can be mapped onto ω.*

THEOREM 2.3 (J. L. Hickman, unpublished).

$$wC^{\omega} \rightarrow \text{there are no } DF \text{ sets.}$$

PROOF. (from 2.2) Suppose x is *DF* and $f: x \rightarrow \omega$ is onto. Then no infinite subsequence of $(f^{-1}(n))_{n<\omega}$ can have a choice function. (For such a choice function would map an infinite subset of ω 1: 1 into x.)

wC^{ω} is the weakest axiom known to me which implies that there are no *DF* sets. In §5 we will show that the implication of 2.3 cannot be reversed.

3. Large Dedekind-finite sets

Our aim in this section is, given an arbitrary aleph, to show that it is consistent that there exist *DF* sets which can be mapped onto that aleph. The model we shall construct is a generalization of the model used by Halpern and Lévy to show that the Boolean prime ideal theorem does not imply the axiom of choice. A treatment of the Halpern-Lévy model is given by Felgner (1971) in Chapter IV, sections C (page 96) and G (page 128). Although Felgner's approach differs a little from ours, his proofs can be carried over to our case.

Let M be a countable transitive model of $ZF + V = L$, κ a (regular aleph)^M. We take as our notion of forcing $(H_{\kappa}(\kappa \times \kappa, 2))^M$; if p and q are in the notion we say $p \leqq q$ if and only if $p \supseteq q$. Let G be $(H_{\kappa}(\kappa \times \kappa, 2))^M$-generic over M. For $i < \kappa$ set $G_i = \bigcup \{p(i): p \in G\}$ and set $G^* = \{G_i: i < \kappa\}$. (Note that for $p \in H_{\kappa}(\kappa \times \kappa, 2)$ by $p(i)$ we mean $\{\langle j, k\rangle : \langle\langle i, j\rangle, k\rangle \in p\}$.)

THEOREM 3.1. *Write* $\mathrm{cf}(\alpha)$ *for the cofinality of* α.

(i) $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

(ii) *M and $M[G]$ have the same initial ordinals.*

(iii) *For any $\alpha < \kappa$ and $x \in M$ $({}^{\alpha}x)^M = ({}^{\alpha}x)^{M[G]}$.*

PROOF. We assume the terms 'λ-closed' and 'λ-chain condition' from Schoenfield (1971) § 10. It follows from Schoenfield (1971) lemma 10.3 that $H_\kappa(\kappa \times \kappa, 2)$ satisfies the κ^+-chain condition, and $H_\kappa(\kappa \times \kappa, 2)$ is κ-closed by a remark on page 372 of Schoenfield (1971). Our theorem then follows from Schoenfield (1971) lemma 10.2 and lemma 10.6 (and corollary).

LEMMA 3.2. *In* $M[G]$

(i) *each* G_i *is a member of* ${}^\kappa 2$

(ii) *the sequence* $(G_i)_{i<\kappa}$ *is* 1: 1.

(iii) *for each* $h \in (H_\kappa(\kappa, 2))^M$ *there are infinitely many* $i < \kappa$ *such that* $G_i \supseteq h$.

LEMMA 3.3 (Continuity lemma). *Let* ϕ *be a ZF-formula,* $x \in M$. *Then* $M[G] \models$ *if* $r_1, \cdots, r_n, r \in G^*$ *and* $\phi(G^*, r_1, \cdots, r_n, x, r)$ *and* $r \notin \{r_1, \cdots, r_n\}$ *then there is* $h \in (H_\kappa(\kappa, 2))^M$ *such that* $r \supseteq h$ *and* $(\forall s \in G^*)\ (s \supseteq h \rightarrow \phi(G^*, r_1, \cdots, r_n, x, s))$.

PROOF. This lemma is similar to the continuity lemma given by Felgner (1971) page 133. We give here only an indication of the form Felgner's proof takes in our situation.

We work in $M[G]$. Suppose that

$M[G] \models r_1, \cdots, r_n, r \in G^*$ and $\phi(G^*, r_1, \cdots, r_n, x, r)$. Then there are $i_1, \cdots, i_n$, $i \in \kappa$ such that

$$M[G] \models r_1 = G_{i_1}, \cdots, r_n = G_{i_n} \text{ and } r = G_i,$$

so there is $p \in G$ such that

$$p \Vdash \phi(G^*, G_{i_1}, \cdots, G_{i_n}, x, G_i).$$

Set $h = p(i)$. We note that since $p \in M$, $h \in M$. I claim that

$$(1) \qquad p \Vdash (\forall s \in G^*)(s \notin \{G_{i_1}, \cdots, G_{i_n}\} \text{ and } s \supseteq h \rightarrow \phi(G^*, G_{i_1}, \cdots, G_{i_n}, x, s)).$$

Now (1) is equivalent to

$$(2) \quad (\forall s)(\forall q \leqq p)(q \Vdash s \in G^* - \{G_{i_1}, \cdots, G_{i_n}\} \text{ and } s \supseteq h \rightarrow (\exists q' \leqq q)(q' \Vdash \phi(G^*, G_{i_1}, \cdots, G_{i_n}, x, s)).$$

Suppose then that $q \leqq p$ and that

$$q \Vdash s \in G^* - \{G_{i_1}, \cdots, G_{i_n}\} \text{ and } s \supseteq h$$

for some name s of the forcing language. There is certainly $q' \leqq q$ such that $q' \Vdash s = G_j$ for some $j \notin \{i_1, \cdots, i_n\}$. In fact this q' will do as the q' in (2). This shown as in Felgner's proof, using the symmetry lemma of §1 and the following easily established restriction lemma (*).

(*) If $p \in (H_\kappa(\kappa \times \kappa, 2))^M$ and $p \Vdash \psi(G^*, G_{j_1}, \cdots, G_{j_m}, x)$

then $p \,|\, \{j_1, \cdots, j_m\} \Vdash \psi(G^*, G_{j_1}, \cdots, G_{j_m}, x)$, for $x \in M$.

We now set $N = (L(G^*))^{M[G]}$; N is the model we are interested in. We recall from §1 that there is in $M[G]$ a canonical functional

$$F_0 \colon ON \times \mathrm{Seq}(TC(G^*)) \to L(G^*),$$

definable from $TC(G^*)$ (and thence G^*). Now if $x \in TC(G^*)$, $x = G^*$ or $x = G_i$ $(i < \kappa)$ or $x \in ON \times 2$ or $x \in ON$. So by appropriate coding we may replace F_0 by

$$F_1 \colon ON \times \mathrm{Seq}(G^*) \to L(G^*).$$

Now $G^* \subseteq {}^\kappa 2$, so G^* has a natural linear order. Whence by further coding we may replace F_1 by

$$F \colon ON \times S_\omega(G^*) \to L(G^*).$$

The functional F has all the properties of F_0; in particular F is in N, indeed F is definable in N from G^*, and $N \models F$ is onto. If $x = F(\alpha, a)$ we say x is *constructed by* G^*, α and a or *constructible from* G^* and a. Thus

$$N \models \text{every set is constructible from } G^* \text{ and some } a \in S_\omega(G^*).$$

THEOREM 3.4. (i) *M, N and $M[G]$ have the same cf function and the same initial ordinals.*

(ii) *For any $\alpha < \kappa$ and $x \in M$ $({}^\alpha x)^M = ({}^\alpha x)^N = ({}^\alpha x)^{M[G]}$.*

PROOF. This is an immediate consequence of Theorem 3.1 and the fact that $M \subseteq N \subseteq M[G]$.

THEOREM 3.5. *$N \models G^*$ is DF.*

PROOF. $M[G] \models |G^*| = \kappa$, so certainly

$$N \models G^* \text{ is infinite.}$$

Suppose $N \models G^*$ is not DF. Then there is $f \in N$ such that $f \colon \omega \to G^*$ and f is 1:1. Now f is constructible from G^* and $\{r_1, \cdots, r_n\}$, say, in $S_\omega(G^*)$. Choose $k \in \omega$ and $r \in G^* - \{r_1, \cdots, r_n\}$ such that $f(k) = r$. We may write

$$\text{‘}M[G] \models f(k) = r\text{’}$$

as

$$M[G] \models \phi(G^*, r_1, \cdots, r_n, x, r) \qquad (\text{some } x \in M).$$

By 3.3 there is some $h \in (H_\kappa(\kappa, 2))^M$ such that

$$M[G] \models (\forall s \in G^*)(s \supseteq h \to \phi(G^*, r_1, \cdots, r_n, x, s)).$$

By 3.2 (iii) this implies that f takes infinitely many values at k, a contradiction.

LEMMA 3.6 (Support lemma). *If x is constructible from G^* and a and from G^* and b, then x is constructible from G^* and $a \cap b$.*

PROOF. This lemma corresponds to Felgner's support lemma (page 137) and may be proved similarly.

For $x \in N$ we define $\operatorname{supp}(x)$ (the *support* of x) as the smallest $a \in S_\omega(G^*)$ such that x is constructible from G^* and a; by 3.6 this makes sense. We define α_x as the least α such that $x = F(\alpha, \operatorname{supp}(x))$. We note that since F is in N the functionals $\{\langle x, \operatorname{supp}(x)\rangle : x \in N\}$ and $\{\langle x, \alpha_x\rangle : x \in N\}$ are in N.

The next theorem is the main result of this section.

THEOREM 3.7. *In N*

(i) *there are at least κ distinct DF cardinals.*

(ii) *there is only a set of distinct DF cardinals.*

(iii) *if x is not well-orderable x can be mapped onto κ.*

(iv) *no DF set can be mapped onto κ^+.*

PROOF. We work in N.

(i) Define $f : G^* \to \kappa$ thus: $f(r)$ is the least ordinal α such that $r(\beta) = 0$ for $\beta < \alpha$ and $r(\alpha) = 1$. f is onto; in fact from 3.2 (iii) it follows that $f^{-1}(\{\alpha\})$ is infinite for $\alpha < \kappa$. So if $X_\beta = f^{-1}(\beta)$, $(X_\beta)_{\beta < \kappa}$ is a strictly increasing sequence of DF sets, so $(|X_\beta|)_{\beta<\kappa}$ is a strictly increasing sequence of DF cardinals.

(ii) Suppose that in N X is DF. For each $a \in S_\omega(G^*)$ set $A_a = \{x \in X : \operatorname{supp}(x) = a\}$. (Possibly $A_a = \emptyset$.) A_a can clearly be well-ordered by setting $x < y \leftrightarrow \alpha_x < \alpha_y$, and so A_a is finite. The well-ordering in fact provides an embedding of A_a into ω, and since the procedure is uniform in a combining all the embeddings gives us a single embedding of X into $S_\omega(G^*) \times \omega$. So all DF cardinals are $\leqq |S_\omega(G^*) \times \omega|$.

(iii) Suppose X is not well-orderable. As in (ii) we may embed X into $S_\omega(G^*) \times ON$, so let $Y \subseteq S_\omega(G^*) \times ON$ be equinumerous with X, and suppose Y is constructible from G^* and $\{r_1, \cdots, r_n\}$. Now $\{a \in S_\omega(G^*) : (\exists\alpha)(\langle a, \alpha\rangle \in Y)\}$ is infinite (for otherwise there is an obvious well-ordering of Y), so there is $b \in S_\omega(G^*)$ such that $b \nsubseteq \{r_1, \cdots, r_n\}$ and $\langle b, \beta\rangle \in Y$ for some $\beta \in ON$. Suppose $b = \{s_1, \cdots, s_m\}$, where $s_1 \notin \{r_1, \cdots, r_n\}$. The sentence '$\langle b, \beta\rangle \in Y$' is of the form

$$\phi(G^*, r_1, \cdots, r_n, x, s_1, \cdots, s_m) \qquad (\text{some } x \in M).$$

From 3.3 there is $h \in (H_\kappa(\kappa, 2))^M$ such that

$$N \models (\forall t \in G^*)(t \supseteq h \to \langle\{t, s_2, \cdots, s_m\}, \beta\rangle \in Y).$$

But it is easily shown (as in (*i*)) that $\{t \in G^* : t \supseteq h\}$ can be mapped onto κ. So Y, and thus X, can be mapped onto κ.

(iv) We note that from (ii) if X is DF,

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$$N \models |X| \leqq |S_\omega(G^*) \times \omega|.$$

Also $M[G] \models |S_\omega(G^*) \times \omega| = \kappa$. So in $M[G]$ there is no function mapping X onto κ^+, whence there is no such function in N either.

Let τ be any term of ZF such that $ZF \vdash \tau$ is an aleph, and such that τ is absolute with respect to transitive models of ZF with the same ordinals and alephs. Then we have certainly shown that

$$ZF + (\exists x)(x \text{ is } DF \text{ and } x^* \geqq \tau)$$

is consistent.

We observe also that if $\lambda < \kappa$

$N \models$ every set of subsets of ${}^{\lambda}2$ has a choice function (for from 3.4 (ii) and the fact that $M \models ZFC$ we have that $N \models {}^{\lambda}2$ can be well-ordered). So this particular weak axiom of choice does not imply that there are no DF sets.

4. Dedekind-finite proper classes

For this section we must step outside the bounds of ZF; we define $ZF(K)$ to be the theory with language that of ZF plus the additional one-place predicate K and axioms those of ZF plus replacement for formulae involving K. Clearly if ZF is consistent so is $ZF(K)$.

Let M be a countable transitive model of $ZF + V = L$. We define a class notion of forcing in M thus. Set, for κ a regular cardinal

$$X_\kappa = \{\langle \lambda, \alpha, \beta \rangle : \lambda \text{ regular}, \lambda \leqq \kappa \text{ and } \alpha, \beta < \lambda\}.$$

Set $X = \bigcup \{X_\kappa : \kappa \text{ a regular cardinal}\}$. Let C be the class of functions p mapping some subset of X to 2 such that $|\mathrm{dom}(p) \cap X_\kappa| < \kappa$ for all regular κ. Define $\leqq$ on C by $p \leqq q \leftrightarrow p \supseteq q$. Then $\langle C, \leqq \rangle$ is our class notion of forcing. Let G be C-generic over M.

THEOREM 4.1. (i) *$M[G]$ is a model of $ZFC(G)$ and the fundamental theorem of forcing holds for $M[G]$.*

(ii) *M and $M[G]$ have the same cf function and hence the same alephs.*

(iii) *G is a proper class in $M[G]$.*

PROOF. The notion of forcing is obtained from Schoenfield (1971) page 376 by setting $H(\kappa) = \kappa$, and the proofs in Schoenfield (1971) §12 still apply. In Schoenfield (1971) it is shown only that $M[G]$ is a model of ZFC, not of $ZFC(G)$, but it is easy to add G as a predicate in the definition of strong forcing.

Set $F = \bigcup G$; F is easily seen to be a functional from X to 2. We observe that $F(\kappa, \beta): \kappa \to 2$. We set $K_\kappa = \{F(\kappa, \beta): \beta < \kappa\}$ and $K = \bigcup \{K_\kappa : \kappa \text{ regular}\}$. Each K_κ is non-void, so K is a proper class in $M[G]$. By standard arguments we have

LEMMA 4.2. (i) *If* $\beta, \gamma < \kappa$ *and* $\beta \neq \gamma$ *then* $F(\kappa, \beta) \neq F(\kappa, \gamma)$.

(ii) *Suppose* κ *is regular in* M *and* $f \in (H_\kappa(\kappa, 2))^M$. *Then there are* κ *distinct* $F(\kappa, \alpha) \in K_\kappa$ *such that* $F(\kappa, \alpha) \supseteq f$.

The appropriate analogue of 3.3 is

LEMMA 4.3. *Let* $\phi(K, v_1, \cdots, v_{n+2})$ *be a formula of the language of* $ZF(K)$. *If* $x \in M$, $F(\kappa, \beta) \notin \{F(\kappa_1, \beta_1), \cdots, F(\kappa_n, \beta_n)\}$ *and*

$$M[G] \models \phi(K, F(\kappa_1, \beta_1), \cdots, F(\kappa_n, \beta_n), x, F(\kappa, \beta))$$

then there is $p \in (H_\kappa(\kappa, 2))^M$ *such that*

$$F(\kappa, \gamma) \supseteq p \rightarrow M[G] \models \phi(K, F(\kappa_1, \beta_1), \cdots, F(\kappa_n, \beta_n), x, F(\kappa, \gamma)).$$

Lemma 4.3 can be proved in two ways: we can either generalize directly the proof of 3.3 (using automorphisms of the classs notion C), or we can proceed via the reflection principle to reduce the problem to consideration of ϕ relativized to some V_α and then work in one of the models $M[G_\lambda]$ (see Schoenfield (1971) pages 376–377). A continuity lemma is easily obtained for $M[G_\lambda]$.

We define in $M[G]$ a sequence of sets $(N_\alpha)_{\alpha \in ON}$ thus:

$$N_0 = \varnothing$$

$$N_\alpha = \bigcup_{\beta < \alpha} N_\beta \text{ if } \alpha \text{ is a limit ordinal}$$

$$N_{\alpha+1} = \begin{cases} D(\langle N_\alpha, \in, K \cap N_\alpha, (a)_{a \in N_\alpha} \rangle) \text{ if } \alpha \neq cf(\alpha) \\ D(\langle N_\alpha, \in, K \cap N_\alpha, (a)_{a \in N_\alpha} \rangle) \cup TC(K_\alpha) \text{ if } \alpha = cf(\alpha). \end{cases}$$

(For the notation $D(\mathfrak{A})$ see §1.) Set $N = \bigcup \{N_\alpha : \alpha \in ON\}$. We will use '$N$' indifferently for the class and the structure $\langle N, \in, K \rangle$.

THEOREM 4.4. (i) *N is a transitive model of* $ZF(K)$.

(ii) M, N and $M[G]$ *have the same cf function and the same alephs.*

PROOF. (i) is an application of the theorem quoted under the heading 'Relative constructibility' in §1; (ii) follows from 4.1 (ii) and the fact that $M \subseteq N \subseteq M[G]$.

We observe that in N K has a natural linear order $<_K$ defined as follows. $F(\kappa, \beta) <_K F(\lambda, \gamma)$ if $\kappa < \lambda$ or $\kappa = \lambda$ and $F(\kappa, \beta)$ precedes $F(\kappa, \gamma)$ in the lexicographic ordering of ${}^\kappa 2$.

THEOREM 4.5. $N \models K$ *is a DF proper class.*

PROOF. K is a proper class in N because it is so in $M[G]$. Suppose that K is not DF, so that in N there is $f : \omega \rightarrow K$, f being 1 : 1. By arguing as in §3 we may say that f is constructible from K and $F(\kappa_1, \beta_1), \cdots, F(\kappa_n, \beta_n)$ say. Choose $k \in \omega$

such that $f(k) \notin \{F(\kappa_1, \beta_1), \cdots, F(\kappa_n, \beta_n)\}$, and suppose $f(k) = F(\kappa, \beta)$. We may write the statement '$f(k) = F(\kappa, \beta)$' as

$$M[G] \models \phi(K, F(\kappa_1, \beta_1), \cdots, F(\kappa_n, \beta_n), x, F(\kappa, \beta)),$$

where $x \in M$. By 4.3 $F(\kappa, \beta)$ may be replaced by infinitely many other members of K_κ, so f takes infinitely many values at k, a contradiction.

THEOREM 4.6. *$N \models$ there is a DF proper class which can be mapped onto the universe.*

PROOF. We work in N. Our proper class is K', the class of finite subsets of K. We use the linear order $<_K$ on K to embed K' into the class of 1 : 1 finite sequences of elements of K, which we call K''. It is easy to see that since K is DF, so is K'' (this appears to be due to Tarski; see Lévy (1965) page 225). Since K'' is DF, K' is DF as well.

We define a functional $\phi: K \to ON$ thus:

$$\phi(F(\kappa, \beta)) \text{ is the least ordinal } \alpha \text{ such that}$$
$$(F(\kappa, \beta))(\alpha) = 1.$$

By 4.2 (ii) $\phi'' K_\kappa = \kappa$ for each regular κ. We now define $\Phi: K' \to N$ thus. If $a \in K'$, let r be the greatest element of a under $<_K$, and set $a' = a - \{r\}$. Then $\Phi(a)$ is that element of N constructed by a', $\phi(r)$ and K. Φ is clearly onto.

5. The assumption that there are no *DF* sets.

In this section I construct a model in which there are no DF sets but wC^ω fails, thus showing that the implication of 2.3 cannot be reversed.

Let M be a countable transitive model of $ZF + V = L$. We take as our notion of forcing $(H_{\omega_1}(\omega \times \omega_1 \times \omega_1, 2))^M$; let G be $(H_{\omega_1}(\omega \times \omega_1 \times \omega_1, 2))^M$-generic over M. As in the proof of 3.1 we have that M and $M[G]$ have the same cofinality function and initial ordinals, and that for $\alpha < \omega_1$ and $x \in M$ $({}^\alpha x)^M = ({}^\alpha x)^{M[G]}$. In particular every element of ${}^\omega 2$ in $M[G]$ is also in M.

Write K for $\bigcup G$; then $K: \omega \times \omega_1 \times \omega_1 \to 2$, so if $i < \omega$ and $j < \omega_1$ then $K(i,j): \omega_1 \to 2$. Set

$$X_i = \{K(i,j): j < \omega_1\},$$
$$Y_i = \{f : \text{dom}(f) = \omega, \text{ran}(f) \subseteq X_i \text{ and } f \text{ is } 1:1\}$$
$$Z = (Y_i)_{i<\omega}.$$

By standard arguments if $\langle i,j \rangle \neq \langle i',j' \rangle$, then $K(i,j) \neq K(i',j')$. Further if $h \in H_{\omega_1}(\omega_1, 2)$ and $i < \omega$ there are ω_1 j's such that $K(i,j) \supseteq h$.

The appropriate continuity lemma is

THEOREM 5.1. *Let ϕ be a ZF-formula, and suppose that*

$$M[G] \models \phi(Z, s_1, \cdots, s_m, x, t_1, \cdots, t_n),$$

where $x \in M$, $s_1 \in Y_{i_1}, \cdots, s_m \in Y_{i_m}, t_1 \in Y_{j_1}, \cdots, t_n \in Y_{j_n}$ *and* $\{i_1, \cdots, i_m\} \cap \{j_1, \cdots, j_n\} = \emptyset$. *Then there are functions* $h_{kl} \in H_{\omega_1}(\omega_1, 2)$ $(1 \leqq k \leqq n, l < \omega)$ *such that* $l \neq l' \rightarrow h_{kl}$ *is incompatible with* $h_{kl'}$ *and such that if*

$$A_{kl} = \{x \in X_{j_k} : x \supseteq h_{kl}\}, \text{ then } t_k(l) \in A_{kl} \text{ and}$$

$$M[G] \models (\forall t_1', \cdots, t_n')[(\forall k \leqq n)(\forall l < \omega)(t_k'(l) \in A_{kl} \rightarrow \phi(Z, s_1, \cdots, s_n, x, t_1', \cdots, t_n')].$$

This continuity lemma differs from our others in that instead of having one 'movable' element t, we have a finite number $t_1, \cdots, t_n$. This however makes no essential difference to the proof; indeed Felgner's continuity lemma for the Halpern-Lévy model (Felgner (1971) page 133) is of this more general form.

The model we consider is $N = (L(Z))^{M[G]}$. The theorem we quoted on relative constructibility shows that in $M[G]$ there is a canonical functional

$$F_0 : ON \times \mathrm{Seq}(TC(Z)) \rightarrow L(Z).$$

By analyzing the members of $TC(Z)$ we may, as in §3, replace F_0 by a functional

$$F : ON \times A \rightarrow L(Z)$$

where $A = \{f : \mathrm{dom}(f) \in \omega, \mathrm{ran}(f) \subseteq \bigcup_{i<\omega} Y_i$ and if $j \neq k$ and

$$f(j) \in Y_{i_j}, f(k) \in Y_{i_k} \text{ then } i_j \neq i_k\}.$$

If $F(\alpha, \langle s_1, \cdots, s_n \rangle) = z$ we say z is constructed by Z, α and $s_1, \cdots, s_n$, or constructible from Z and $s_1, \cdots, s_n$. F is defined from Z alone and is onto N.

THEOREM 5.2. *wC^ω fails in N.*

PROOF. Consider the ω-sequence $(Y_i)_{i<\omega} = Z$; suppose f is a choice function for an infinite subsequence of Z. Then f is constructible from Z and elements $s_1, \cdots, s_n$ of $Y_{i_1}, \cdots, Y_{i_n}$ say. Choose n such that $f(n)$ is not in any of the $Y_{i_1}, \cdots, Y_{i_n}$; suppose $f(n) = t$. It follows from an application of 5.1 to the sentence '$f(n) = t$' that f is not a function, contradiction.

THEOREM 5.3. *$N \models$ there are no DF sets.*

PROOF. Suppose $N \models X$ is DF. Since $X \in N$, X is constructible from Z and $s_1, \cdots, s_n$ (where $s_k \in Y_{i_k}$). Now the class of sets constructible from Z and $s_1, \cdots, s_n$ is well-orderable in N, so if we set

$$X' = \{x \in X : x \text{ not constructible from } Z \text{ and } s_1, \cdots, s_n\},$$

then X' is DF. Further X' is constructible from Z and $s_1, \cdots, s_n$.

Take $a \in X'$. Now a is constructed by Z, α and $s_1, \cdots, s_n, t_1, \cdots, t_m$ say (where $t_k \in Y_{j_k}$). We write $a(t_1, \cdots, t_m)$ for a. By applying 5.1 to the sentence '$a \in X'$' we find there are functions $h_{kl} \in (H_{\omega_1}(\omega_1, 2))^M$ (for $1 \leqq k \leqq m$, $l < \omega$) such that if $A_{kl} = \{x \in X_{j_k}: x \supseteq h_{kl}\}$ then the A_{kl} are pairwise disjoint, $t_k(l) \in A_{kl}$ and

$$(\forall t_1', \cdots, t_m')(t_k'(l) \in A_{kl} \rightarrow a(t_1', \cdots, t_m') \in X').$$

(Here $a(t_1', \cdots, t_m')$ is the set constructed by Z, α and $s_1, \cdots, s_n, t_1', \cdots, t_m'$.)

Set $A = \{a(t_1', \cdots, t_m'): t_k'(l) \in A_{kl}$ for $1 \leqq k \leqq m$ and $l < \omega\}$.

We observe that A is constructible from Z and $s_1, \cdots, s_m$. For this is clear once we know that the double sequence $(h_{kl})_{1 \leqq k \leqq m, l < \omega}$ is an element of M. But any countable ordinal can be coded by a single element of ${}^{\omega}2$, so any countable sequence of countable ordinals can be coded by a single element of ${}^{\omega}2$. It follows that $(h_{kl})_{1 \leqq k \leqq m, l < \omega}$ can be coded by a single element of ${}^{\omega}2$, and all the elements of ${}^{\omega}2$ in $M[G]$ are in M. We note also that $a \in A$ and $A \subseteq X'$.

If A is a singleton, a is constructible from Z and $s_1, \cdots, s_n$, a contradiction.

If $N \models A$ is infinite, then $M[G] \models A$ has a countable subset, say

$$\langle a(t_1^{(1)}, \cdots, t_m^{(1)}), a(t_1^{(2)}, \cdots, t_m^{(2)}), \cdots \rangle \tag{1}$$

Now the ω-sequences $\langle t_1^{(1)}, t_1^{(2)} \cdots \rangle, \langle t_2^{(1)}, t_2^{(2)}, \cdots \rangle, \cdots, \langle t_m^{(1)}, t_m^{(2)}, \cdots \rangle$ are codable into single elements of $Y_{j_1}, \cdots, Y_{j_m}$. For

$$Y_i = (\{f: \mathrm{dom}(f) = \omega, \mathrm{ran}(f) \subseteq X_i \text{ and } f \text{ is } 1:1\})^{M[G]}.$$

And any element of $({}^{\omega}X_i)^{M[G]}$ can be coded by an element of ${}^{\omega}2$ and an element of Y_i. So the sequence (1) is in fact in N, whence $N \models X'$ is not DF, a contradiction.

Suppose finally that A is finite. Recall that $a(t_1, \cdots, t_m) \in A$. Choose t_k' ($1 \leqq k \leqq m$) such that $\mathrm{ran}(t_k) \cap \mathrm{ran}(t_k') = \emptyset$ and $t_k'(l) \in A_{kl}$ for $l < \omega$. (Since $M[G] \models |A_{kl}| = \omega_1$, this is clearly possible.) We distinguish two cases.

CASE I.

$$a(t_1, \cdots, t_m) = a(t_1', \cdots, t_m'). \tag{2}$$

If r, s are two ω-sequences define $r * s$ as the sequence given by

$$(r * s)(2i) = r(i); \ (r * s)(2i + 1) = s(i).$$

Now set $t_k'' = t_k * t_k'$. Then (2) can be written in the form

$$\phi(Z, s_1, \cdots, s_n, z, t_1'', \cdots, t_m''),$$

where ϕ is a ZF-formula and $z \in M$. By applying 5.1 to this statement we obtain sets $B_{kl}, B_{kl}' \subseteq X_{j_k}$ ($1 \leqq k \leqq m$, $l < \omega$) such that

$$N \models a(r_1, \cdots, r_m) = a(r_1', \cdots, r_m') \text{ for all } r_k$$

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such that $r_k(l) \in B_{kl}$ and all r_k' such that

$r_k'(l) \in B'_{kl}$.

So if we set

$$A' = \{a(r_1, \cdots, r_m): r_k \text{ such that } r_k(l) \in B_{kl}\},$$

A' is a singleton and constructible from Z and $s_1, \cdots, s_n$. It follows that X' contains an element constructible from Z and $s_1, \cdots, s_n$ (namely the unique element of A'): this is a contradiction.

CASE II.

$$a(t_1, \cdots, t_m) \neq a(t_1', \cdots, t_m').$$

By proceeding as in case I we find $B_{kl}, B'_{kl} \subseteq X_{j_k}$ such that

$$N \vDash a(r_1, \cdots, r_m) \neq a(r_1', \cdots, r_m') \text{ for all } r_k \text{ such}$$

that $r_k(l) \in B_{kl}$ and all r_k' such that

$r_k'(l) \in B'_{kl}$.

So if we set

$$A' = \{a(r_1, \cdots, r_m): r_k \text{ such that } r_k(l) \in B_{kl}\},$$

A' is constructible from Z and $s_1, \cdots, s_n$ and is a non-empty proper subset of A. After the construction is repeated a finite number of times we reach a singleton, which gives us a contradiction as before. This completes the proof of the theorem.

Acknowledgements

After this paper was written I was informed by Dr J. L. Hickman that he has obtained results related to those of §3. Hickman's model contains a *DF* set X such that the set of two-element subsets of X has cardinality greater than κ, for κ a fixed but arbitrary aleph. I should like to thank the referee for his comments.

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