GLOBAL ASYMPTOTIC STABILITY FOR GENERAL
SYMMETRIC RATIONAL DIFFERENCE EQUATIONS

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Abstract
We investigate the global asymptotic stability for positive solutions to a class of general symmetric rational difference equations and prove that the unique positive equilibrium 1 of the general symmetric rational difference equations is globally asymptotically stable. As a special case of our result, we solve the conjecture raised by Berenhaut, Foley and Stević [‘The global attractivity of the rational difference equation $y_n = (y_{n-k} + y_{n-m})/(1 + y_{n-k}y_{n-m})$, Appl. Math. Lett. 20 (2007), 54–58].

Keywords and phrases: rational difference equation, global asymptotic stability, positive solution, equilibrium.

1. Introduction
Recently there has been great interest in studying the qualitative properties of rational difference equations. Some prototypes for development of the basic theory for the global behaviour of nonlinear difference equations of order greater than 1 come from the results for rational equations (see, for example, [1, 4–6] and the references therein).

In 2007, Berenhaut et al. [2] studied the global asymptotic stability for positive solutions to the difference equations

$$y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}}, \quad n \in N_0 = \{0, 1, 2, \ldots\}$$

with $y_{-m}, y_{-m+1}, \ldots, y_{-1} \in (0, \infty)$ and $1 \leq k < m$. At end of the paper, they raised two conjectures as follows.

**Conjecture 1.1.** Suppose that $\{y_i\}$ satisfies

$$y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k}y_{n-l} + y_{n-m}}{y_{n-k}y_{n-l}y_{n-m} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1}, \quad n \in N_0,$$

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with $y_{-m}$, $y_{-m+1}$, \ldots, $y_{-1} \in (0, \infty)$ and $1 \leq k < l < m$. Then the sequence $\{y_i\}$ converges to the unique equilibrium $1$.

**Conjecture 1.2.** Suppose that $v$ is odd and $1 \leq k_1 < k_2 < \cdots < k_v$, and define $S = \{1, 2, \ldots, v\}$. If $\{y_i\}$ satisfies

$$y_n = \frac{f_1(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_v})}{f_2(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_v})}, \quad n \in N_0,$$

where

$$f_1(x_1, x_2, \ldots, x_v) = \sum_{r=1}^{v} \sum_{\{t_1, t_2, \ldots, t_r\} \subseteq S \atop r \text{ odd} \atop t_1 < t_2 < \cdots < t_r} x_{t_1} x_{t_2} \cdots x_{t_r}$$

and

$$f_2(x_1, x_2, \ldots, x_v) = 1 + \sum_{r=2}^{v-1} \sum_{\{t_1, t_2, \ldots, t_r\} \subseteq S \atop r \text{ even} \atop t_1 < t_2 < \cdots < t_r} x_{t_1} x_{t_2} \cdots x_{t_r}$$

with $y_{-k_v}$, $y_{-k_v+1}$, \ldots, $y_{-1} \in (0, \infty)$, then the sequence $\{y_i\}$ converges to the unique equilibrium $1$.

Conjecture 1.1 was solved by Berenhaut and Stević [3]. However, to the best of our knowledge, Conjecture 1.2 has not hitherto been solved.

Let $v \geq 2$ be an arbitrary integer. Denote by $v_o$ ($v_e$) the largest odd (even) number not greater than $v$. We consider the general symmetric rational difference equation

$$y_n = \frac{f_1(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_v})}{f_2(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_v})}, \quad n = 0, 1, 2, \ldots, \quad (1.1)$$

where $1 \leq k_1 < k_2 < \cdots < k_v$ and $y_{-k_v}$, $y_{-k_v+1}$, \ldots, $y_{-1} \in (0, \infty)$, and for $S = \{1, 2, \ldots, v\}$,

$$f_1(x_1, x_2, \ldots, x_v) = \sum_{r=1}^{v_o} \sum_{\{t_1, t_2, \ldots, t_r\} \subseteq S \atop r \text{ odd} \atop t_1 < t_2 < \cdots < t_r} x_{t_1} x_{t_2} \cdots x_{t_r} \quad (1.2)$$

and

$$f_2(x_1, x_2, \ldots, x_v) = 1 + \sum_{r=2}^{v_e} \sum_{\{t_1, t_2, \ldots, t_r\} \subseteq S \atop r \text{ even} \atop t_1 < t_2 < \cdots < t_r} x_{t_1} x_{t_2} \cdots x_{t_r} \quad (1.3)$$

In this paper, we investigate the global asymptotic stability for positive solutions to the general symmetric rational difference equation (1.1) and prove that the unique positive equilibrium $1$ of the general symmetric rational difference equation (1.1) is globally asymptotically stable. As a special case of our result, we solve Conjecture 1.2 raised by Berenhaut et al. [2].
2. Preliminaries

For the sake of simplicity, for $S \subseteq S$ let $o(S)$ denote the largest odd number, and $e(S)$ the largest even number, not greater than the number of the elements in $S$,

$$ f_1(S) = \sum_{r=1 \atop r \text{ odd}}^{o(S)} \left\{ \sum_{t_1, t_2, \ldots, t_r \in S \atop t_1 < t_2 < \cdots < t_r} x_{t_1} x_{t_2} \cdots x_{t_r} \right\} \quad (2.1) $$

and

$$ f_2(S) = 1 + \sum_{r=2 \atop r \text{ even}}^{e(S)} \left\{ \sum_{t_1, t_2, \ldots, t_r \in S \atop t_1 < t_2 < \cdots < t_r} x_{t_1} x_{t_2} \cdots x_{t_r} \right\} \quad (2.2) $$

**Lemma 2.1.** Let

$$ f(x_1, x_2, \ldots, x_v) = \frac{f_1(x_1, x_2, \ldots, x_v)}{f_2(x_1, x_2, \ldots, x_v)}, \quad (2.3) $$

where $f_1$ and $f_2$ are defined by (1.2) and (1.3), respectively. Then

$$ \frac{\partial f}{\partial x_i} = (f_2)^{-2} \prod_{j \neq i} (1 - x_j^2), \quad i = 1, 2, \ldots, v. $$

**Proof.** For any $i \in \{1, 2, \ldots, v\}$, let $S_i = S \setminus \{i\}$. From (2.1) and (2.2), it is easy to obtain that

$$ f_1(S) = x_i f_2(S_i) + f_1(S_i) \quad (2.4) $$

and

$$ f_2(S) = x_i f_1(S_i) + f_2(S_i). \quad (2.5) $$

For any $j \neq i$, set $S_{ij} = S_i \setminus \{j\}$. Then it follows from (2.4) and (2.5) that

$$ f_1(S_i) = x_j f_2(S_{ij}) + f_1(S_{ij}) \quad (2.6) $$

and

$$ f_2(S_i) = x_j f_1(S_{ij}) + f_2(S_{ij}). \quad (2.7) $$

Now from (2.4)–(2.7),

$$ \frac{\partial f_1(S)}{\partial x_i} f_2(S) - \frac{\partial f_2(S)}{\partial x_i} f_1(S) = f_2(S_i)(x_i f_1(S_i) + f_2(S_i)) $$

$$ - f_1(S_i)(x_i f_2(S_i) + f_1(S_i)) $$

$$ = f_2^2(S_i) - f_1^2(S_i) $$

$$ = (x_j f_1(S_{ij}) + f_2(S_{ij}))^2 - (x_j f_2(S_{ij}) + f_1(S_{ij}))^2 $$

$$ = (1 - x_j^2)(f_2^2(S_{ij}) - f_1^2(S_{ij})). $$
Noticing that $j \neq i$ was arbitrary and that $f_2^2(S_{ij}) - f_1^2(S_{ij})$ does not depend on $i$ or $j$, it is easy to see that
\[
\frac{\partial f_1(S)}{\partial x_i} f_2(S) - \frac{\partial f_2(S)}{\partial x_i} f_1(S) = \prod_{j \neq i} (1 - x_j^2)
\]
and so
\[
\frac{\partial f(S)}{\partial x_i} = (f_2(S))^{-2} \prod_{j \neq i} (1 - x_j^2).
\]
This completes the proof.

By (1.1), (2.4) and (2.5), it is easy to obtain that $\bar{x} = 1$ is the unique positive equilibrium of (1.1). The following concepts can be found in [7]. A subsequence $\{y_n\}$ of a solution of (1.1) is called trivial if it is eventually identical to the equilibrium $1$. Otherwise it is nontrivial. The sign of a subsequence $\{y_n\}$ of a solution of (1.1) is defined as the sequence which is composed of the signs of the terms of $\{y_n - 1\}$. If $y_n - 1 = 0$, then the sign of the $n$th term of $\{y_n - 1\}$ is denoted by 0.

Let $\{y_n\}$ be a positive solution of (1.1), $m = k_v$ and $A_i = \{y_{nm+i}\}_{n=-1}^{\infty}$ for $i = 0, 1, \ldots, m - 1$. Then $\{y_n\}$ is divided into $m$ subsequences $A_0, A_1, \ldots, A_{m-1}$. Set $A_i = B_i^1 \cup B_i$ with $B_i^1 = \{y \in A_i | y \geq 1\}$ and $B_i = \{y \in A_i | y < 1\}$. For the sake of simplicity, for $n \geq 0$, $j = 1, 2$ and $i = 0, 1, \ldots, m - 1$, let
\[
f_j(n, i) = f_j(y_{nm+i-k_1}, y_{nm+i-k_2}, \ldots, y_{nm+i-k_v})
\]
and
\[
f_j(n, i, v) = f_j(y_{nm+i-k_1}, y_{nm+i-k_2}, \ldots, y_{nm+i-k_{v-1}}),
\]
where $f_1$ and $f_2$ on the right-hand side of (2.8) (or (2.9)) are given by (2.1) and (2.2) with $S = \bar{S}$ (or $S = S \setminus \{v\}$), respectively. It follows from (2.4), (2.5), (2.8) and (2.9) that, for any $n \geq 0$,
\[
f_1(n, i) = y_{(n-1)m+i} f_2(n, i, v) + f_1(n, i, v)
\]
and
\[
f_2(n, i) = y_{(n-1)m+i} f_1(n, i, v) + f_2(n, i, v).
\]

**Lemma 2.2.** For each $i \in \{0, 1, \ldots, m - 1\}$, there exists some $L_i \geq 1$ such that $B_i^1$ converges to $L_i$ if $B_i^1$ is infinite, and $B_i$ converges to $1/L_i$ if $B_i$ is infinite.

**Proof.** If $A_i$ is trivial, the proof is trivial.

If $A_i$ is nontrivial, we can assert that $y_{nm+i} \neq 1$ for all $n \geq -1$. Otherwise, $y_{nm+i} = 1$ for some $\bar{n} \geq 1$ and by (2.8)–(2.11),
\[
y_{(\bar{n}+1)m+i} = \frac{f_1(\bar{n}+1, i)}{f_2(\bar{n}+1, i)}
\]
\[
= \frac{y_{\bar{n}m+i} f_2(\bar{n}+1, i, v) + f_1(\bar{n}+1, i, v)}{y_{\bar{n}m+i} f_1(\bar{n}+1, i, v) + f_2(\bar{n}+1, i, v)}
\]
\[
= 1,
\]
so that $A_i$ is trivial, a contradiction. It follows that $y_{nm+i} = 1$ for $n \geq \bar{n}$ by induction, which contradicts the fact that $A_i$ is nontrivial. Thus $B^i$ is in fact $\{ y \in A_i | y > 1 \}$, and it follows from (2.8)–(2.11) that, for any $j = 0, 1, \ldots, m - 1$ and $n \geq 0$,

$$
y_{nm+j} - y_{(n-1)m+j} = f_1(n, j) - y_{(n-1)m+j} f_2(n, j)
= y_{(n-1)m+j} f_2(n, j) f_2(n, j) + f_1(n, j, v) - y_{(n-1)m+j} (y_{(n-1)m+j} f_1(n, j, v) + f_2(n, j, v))
= f_1(n, j, v) (1 - y_{(n-1)m+j}) (1 + y_{(n-1)m+j}) \neq 0.
$$

This implies that

$$(y_n - y_{n-m})(y_{n-m} - 1) < 0, \quad n = 0, 1, 2, \ldots \quad (2.12)$$

We suppose that $y_{-m+i} > 1$. The proof of the case $y_{-m+i} < 1$ is analogous, so we omit it. Assume that the sign of $A_i$ is $q_1^+, q_2^-, q_3^+, q_4^- \ldots$, where $q_1^+$ means $q_1$ successive positive signs and $q_2^-$ means $q_2$ successive negative signs. We consider two cases as follows.

**Case 1.** The sign sequence is finite; that is, there exists a positive $N$ such that $q_N = \infty$. Without loss of generality we may assume $N = 1$, that is, $q_1 = \infty$ and $B_i$ is empty. By (2.12),

$$y_{-m+i} > y_i > y_{m+i} > \cdots$$

and so $\{y_{nm+i}\}$ is decreasing with lower bound 1. This implies that $\lim_{n \to \infty} y_{nm+i} = L_i$ for some $L_i \geq 1$.

**Case 2.** The sign sequence is infinite. Then each $q_j$ is a positive integer. Letting

$$s(0) = -1, s(n) = s(n-1) + q_n, \quad n = 1, 2, \ldots,$$

then

$$B^i = \{ y_{s(2n)+j}m+i \mid n \geq 0, j = 0, 1, \ldots, q_{2n+1} - 1 \} \quad (2.13)$$

and

$$B_i = \{ y_{s(2n+1)+j}m+i \mid n \geq 0, j = 0, 1, \ldots, q_{2n+2} - 1 \} \quad (2.14)$$

It is easy to obtain from (2.12) that

$$y_{s(2n)m+i} > y_{s(2n)+1m+i} > \cdots > y_{s(2n+1)-1m+i} > 1 \quad (2.15)$$

and

$$y_{s(2n+1)m+i} < y_{s(2n+1)+1m+i} < \cdots < y_{s(2n+2)-1m+i} < 1 \quad (2.16)$$

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for \( n \geq 0 \). By (2.8)–(2.11) and (2.15),

\[
y_{s(2n+1)m+i} = \frac{f_1(s(2n + 1), i)}{f_2(s(2n + 1), i)} = \frac{y_{s(2n+1-1)m+i}f_2(s(2n + 1), i, v) + f_1(s(2n + 1), i, v)}{y_{s(2n+1-1)m+i}f_1(s(2n + 1), i, v) + f_2(s(2n + 1), i, v)} > \frac{1}{y_{s(2n+1-1)m+i}}
\]

for \( n \geq 0 \). Similarly, by (2.8)–(2.11) and (2.16), we can get

\[
y_{s(2n+2)m+i} < \frac{1}{y_{s(2n+2-1)m+i}}
\]

for \( n \geq 0 \). Define a sequence \( \{x_n\} \) by

\[
x_n = \begin{cases} 
y_n & \text{if } y_n \in B^i, \\
1/y_n & \text{if } y_n \in B_i.
\end{cases}
\]

Then (2.13)–(2.18) imply that \( \{x_n\} \) is decreasing with lower bound 1 and so \( \{x_n\} \) has a limit \( L_i \geq 1 \) as desired. This completes the proof. \( \square \)

For our main result, we also need the following lemma.

**Lemma 2.3** [5, Corollary 1.3.2].

Assume that \( F = F(u_0, \ldots, u_k) \) is a \( C^1 \) function and let \( \bar{x} \) be an equilibrium of the rational difference equations

\[
x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots,
\]

where \( k \geq 0 \) is an integer. If all the roots of the polynomial equation

\[
\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \ldots, \bar{x})\lambda^{k-i} = 0
\]

lie in the open unit disk \( |\lambda| < 1 \), then the equilibrium \( \bar{x} \) of (2.19) is asymptotically stable.

### 3. Main Results

**Theorem 3.1.** The unique positive equilibrium 1 of (1.1) is globally asymptotically stable.
The linearized equation of (1.1) with respect to the positive equilibrium $\bar{x} = 1$ is
\[
y_n = \sum_{i=1}^{v} \frac{\partial f(1, 1, \ldots, 1)}{\partial x_i} y_{n-k_i}, \quad n = 1, 2, \ldots,
\]
where $f = f(x_1, \ldots, x_v)$ is given by (2.3). By Lemma 2.1, we know that
\[
\frac{\partial f(1, 1, \ldots, 1)}{\partial x_i} = 0, \quad i = 1, 2, \ldots, v.
\]
It follows from Lemma 2.3 that $\bar{x} = 1$ is locally asymptotically stable.

Now it is sufficient to prove that $1$ is a global attractor for the positive solutions of (1.1). Moreover, by Lemma 2.2, it is sufficient to prove that $L_i = 1$ for $i = 0, 1, \ldots, m - 1$, where $L_i$ is the same as in Lemma 2.2.

Let $i \in \{0, 1, \ldots, m - 1\}$. Without loss of generality, we may assume that $B^i$ is infinite. Thus, Lemma 2.2 implies that, for any $\varepsilon > 0$ sufficiently small, there exist some $\bar{n}$ and
\[
P_l \in \{L_0, \ldots, L_{m-1}, L_{0}^{-1}, \ldots, L_{m-1}^{-1}\}, \quad l = 0, 1, \ldots, m - 1,
\]
such that $y_{\bar{n}m+i} \in B^i$,
\[
|y_{\bar{n}m+i} - L_i| < \varepsilon, \quad |y_{\bar{n}m+i-k_l} - P_l| < \varepsilon \quad (3.1)
\]
with
\[
|y_{(\bar{n}-1)m+i} - L_i| < \varepsilon \quad \text{if } y_{(\bar{n}-1)m+i} \in B^i \quad (3.2)
\]
and
\[
|y_{(\bar{n}-1)m+i} - 1/L_i| < \varepsilon \quad \text{if } y_{(\bar{n}-1)m+i} \in B_i. \quad (3.3)
\]
For the sake of simplicity, let
\[
f_j(P, \pm \varepsilon) = f_j(P_1 \pm \varepsilon, P_2 \pm \varepsilon, \ldots, P_{v-1} \pm \varepsilon), \quad j = 1, 2, \quad (3.4)
\]
and
\[
f_j(P) = f_j(P_1, P_2, \ldots, P_{v-1}), \quad j = 1, 2, \quad (3.5)
\]
where $f_1$ and $f_2$ on the right-hand side of the above two equations are the same as in (2.9). Notice that $f_1$ and $f_2$ are increasing. If $y_{(\bar{n}-1)m+i} \in B^i$, then (2.8)-(2.11), (3.1), (3.2) and (3.4) imply that
\[
L_i - \varepsilon < y_{\bar{n}m+i}
\]
\[
= \frac{f_1(\bar{n}, i)}{f_2(\bar{n}, i)}
\]
\[
= \frac{y_{(\bar{n}-1)m+i} f_2(\bar{n}, i, v) + f_1(\bar{n}, i, v)}{y_{(\bar{n}-1)m+i} f_1(\bar{n}, i, v) + f_2(\bar{n}, i, v)}
\]
\[
\leq \frac{(L_i + \varepsilon) f_2(P, +\varepsilon) + f_1(P, +\varepsilon)}{(L_i - \varepsilon) f_1(P, -\varepsilon) + f_2(P, -\varepsilon)} \quad (3.6)
\]
and

\[ L_i + \varepsilon > y_{\bar{n}m+i} \]
\[ = \frac{y_{(\bar{n}-1)m+i}f_2(\bar{n}, i, v) + f_1(\bar{n}, i, v)}{y_{(\bar{n}-1)m+i}f_1(\bar{n}, i, v) + f_2(\bar{n}, i, v)} \]
\[ \geq \frac{(L_i - \varepsilon)f_2(P, -\varepsilon) + f_1(P, -\varepsilon)}{L_i + \varepsilon)f_2(P, +\varepsilon) + f_2(P, +\varepsilon)} \tag{3.7} \]

Since \( \varepsilon \) is arbitrary and \( f_1, f_2 \) are continuous, it follows from (3.6), (3.7) and (3.5) that

\[ L_i = \frac{L_i f_2(P) + f_1(P)}{L_i f_1(P) + f_2(P)}, \]

which yields that \( L_i = 1 \). Similarly, if \( y_{(\bar{n}-1)m+i} \in B_i \), by (2.8)–(2.11), (3.1), (3.3)–(3.5) we obtain

\[ L_i = \frac{(1/L_i)f_2(P) + f_1(P)}{(1/L_i)f_1(P) + f_2(P)} \]

This also leads to \( L_i = 1 \). This completes the proof. \( \square \)

Letting \( v = 3 \) in Theorem 3.1, we have the following corollary.

**Corollary 3.2.** Suppose that \( \{y_i\} \) satisfies

\[ y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m}}{y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1}, \quad n \in N_0, \]

with \( y_m, y_{m+1}, \ldots, y_1 \in (0, \infty) \) and \( 1 \leq k < l < m \). Then the sequence \( \{y_i\} \) converges to the unique equilibrium 1.

For \( v \geq 3 \) and \( v \) odd in Theorem 3.1, we have the following corollary.

**Corollary 3.3.** Suppose that \( v \) is odd, \( 1 \leq k_1 < k_2 < \cdots < k_v \), and \( S = \{1, 2, \ldots, v\} \). If \( \{y_i\} \) satisfies

\[ y_n = \frac{f_1(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_v})}{f_2(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_v})}, \quad n \in N_0, \]

where

\[ f_1(x_1, x_2, \ldots, x_v) = \sum_{r=1}^{v} \sum_{\{t_1, t_2, \ldots, t_r\} \subseteq S, t_i \text{ odd}} x_{t_1}x_{t_2} \cdots x_{t_r} \]

and

\[ f_2(x_1, x_2, \ldots, x_v) = 1 + \sum_{r=2}^{v-1} \sum_{\{t_1, t_2, \ldots, t_r\} \subseteq S, t_i \text{ even}} x_{t_1}x_{t_2} \cdots x_{t_r} \]

with \( y_{-k_1}, y_{-k_2+1}, \ldots, y_{-1} \in (0, \infty) \), then the sequence \( \{y_i\} \) converges to the unique equilibrium 1.

**Remark 3.4.** Corollary 3.3 solves Conjecture 1.2 raised by Berenhaut et al. [2].
References


[2] K. S. Berenhaut, J. D. Foley and S. Stević, ‘The global attractivity of the rational difference equation \( y_n = \frac{(y_{n-k} + y_{n-m})}{(1 + y_{n-k} y_{n-m})} \)', *Appl. Math. Lett.* 20 (2007), 54–58.


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