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A NON-ABELIAN AUTOMORPHISM GROUP WITH ALL AUTOMORPHISMS CENTRAL

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This paper provides an example of a non-abelian 2-group with a non-abelian automorphism group in which every automorphism is central.

An automorphism θ of a group G is said to be central if θ commutes with every inner automorphism of G, or equivalently if $g^{-1}g^{\theta} \in Z(G)$, for all $g \in G$. The central automorphisms form a normal subgroup, denoted $\mathrm{Aut}_c G$, of $\mathrm{Aut}\, G$. Various authors ([1], [6], [7], [8]) have considered non-abelian p-groups with abelian automorphism groups in which necessarily all automorphisms are central. Miller's group (see [7]) of order 2^6 with elementary abelian automorphism group of order 2^7 was the first such group found. This group became group number 99 in the Hall-Senior tables [3]. The parameter t_3 of these tables is used to denote the order of the group of automorphisms of the central quotient induced by the automorphisms of the group. Thus $t_3 = 1$ precisely when every automorphism is central. Only two other groups, numbers 91 and 92, appear in these tables with the property that every automorphism is central. In fact, the full automorphism groups of both are elementary abelian of order 2^9.

More recently, the combined results of Heineken and Liebeck [4] and

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Hughes [5] shows that for any finite group K, there exists for each prime p a p-group G of nilpotency class 2 and exponent p^2 such that $\text{Aut}\, G/\text{Aut}_c G$ and K are isomorphic. When $K = 1$ however (and every automorphism is central) the groups G have elementary abelian automorph automorphism groups.

In these examples then $\text{Aut}\, G$ is abelian when $\text{Aut}_c G = \text{Aut}\, G$. The question thus arises whether this necessary condition for $\text{Aut}\, G$ to be abelian is also sufficient. This paper provides a negative answer for the prime 2 and the example below is clearly a minimum counter-example in this case.

The notation is standard and that of Gorenstein [2].

Let M be the Miller group of order 2^6 and let

$$G = M \times Z_2 = \langle a, b, c, d \mid a^8 = b^4 = c^2 = d^2 = 1, a^b = a^5, b^c = b^{-1}, [a, c] = [a, d] = [b, d] = [c, d] = 1 \rangle .$$

PROPOSITION. *The non-abelian group G of order 2^7 has a non-abelian automorphism group of order 2^{12} in which every automorphism is central.*

Proof. We first list some straightforward properties of G:

(i) $Z(G) = \langle a^2, b^2, d \rangle$; $\Omega_1(Z(G)) = \langle a^4, b^2, d \rangle$;

(ii) $A = \langle a, b^2, c, d \rangle$ is a maximal abelian characteristic subgroup of G ; $\Omega_1(A) = \langle a^4, b^2, c, d \rangle$; $\mho^1(A) = \langle a^2 \rangle$;

(iii) $G' = \langle a^4, b^2 \rangle$;

(iv) the non-trivial elements of G and their orders are respectively:

$a^{2i+1}b^j c^k d^l$, $0 \le i, j \le 3$, $0 \le k, l \le 1$, 8 ,

$a^{4i-2}b^j c^k d^l$, $0 \le j \le 3$, $0 \le i, k, l \le 1$, 4 ,

$a^{4i}b^{2j+1}d^l$, $0 \le i, j, k \le 1$, 4 ,

$a^{4i}b^{j}cd^{l}$, $0 \le j \le 3$, $0 \le i, l \le 1$, 2 ,

$a^{4i}b^{2j}d^{l}$, $0 \le i, j, l \le 1$, $(i, j, k) \ne (0, 0, 0)$; 2 ;

(v) b^2 is the commutator of every pair of non-commuting elements of order 2 ;

(vi) $$[a^{i}b^{j}c^{k}d^{l}, b] = \begin{cases} a^{4i} , & k = 0 , \\ a^{4i}b^{2} , & k = 1 , \end{cases}$$ $(0 \le i \le 7, 0 \le j \le 3, 0 \le l \le 1)$;

(vii) $$(a^{i}b^{j}c^{k}d^{l})^{2} = \begin{cases} a^{2i} , & j \text{ even}, \\ a^{-2i}b^{2j(k+1)} , & j \text{ odd}, \end{cases}$$ $(0 \le i \le 7, 0 \le j \le 3, 0 \le k, l \le 1)$.

We now show that if θ is an arbitrary automorphism of G then $g^{-1}g^{\theta} \in Z(G)$, for all $g \in G$, and thus θ is central. It clearly suffices to show this property for the generators, so let $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ denote the images of a, b, c and d respectively under θ .

By considering the characteristic subgroups $\Omega_1(Z(G))$ and $\Omega_1(A)$ in (i) and (ii) above, $\bar{c}$ must be of the form $a^{4i}b^{2j}cd^{l}$, $0 \le i, j, l \le 1$; that is, $\bar{c} = cz_c$, where $z_c = a^{4i}b^{2j}d^{l} \in Z(G)$.

From (v), b^2 is fixed by θ , so, by (iv) and (vii), $\bar{b}$ is of form $a^{4i}b^{2j+1}d^{l}$, $0 \le i, j, k \le 1$; that is, $\bar{b} = bz_b$, where $z_b = a^{4i}b^{2j}d^{l} \in Z(G)$.

From (iv), $\bar{a} = a^{2i+1}b^{j}c^{k}d^{l}$ for some i, j, k and l with $0 \le i, j \le 3$, $0 \le k, l \le 1$. But from (ii) and (v) respectively a^4 and b^2 are fixed by θ . Thus since $\bar{b} = bz_b$ it follows from (vi) that $k = 0$. Also, from (ii), $\langle a^2 \rangle$ is invariant under θ , so, by (vii), j is even. Thus $\bar{a}$ is of the form $a^{2i+1}b^{2j}d^{l}$, $0 \le i \le 3$,

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$0 \le j, l \le 1$; that is, $\bar{a} = az_a$, where $z_a = a^{2i}b^{2j}d^{l} \in Z(G)$.

Lastly, by considering the characteristic subgroups $\Omega_1(Z(G))$ and G' in (i) and (iii) above, $\bar{d}$ must be of the form $a^{4i}b^{2j}d$, $0 \le i, j \le 1$; that is, $\bar{d} = dz_d$, where $z_d = a^{4i}b^{2j} \in Z(G)$. Thus θ is central as claimed.

Further since $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ clearly satisfy the same relations as a, b, c and d , all the possibilities listed above for $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ occur as the images of automorphisms of G . Thus $|\text{Aut } G| = 2^{12}$.

Finally Aut G is non-abelian. For let θ be the automorphism which maps a to ad and fixes the remaining generators and let ψ be the automorphism which maps d to db^2 and fixes the remaining generators. Then $a^{\psi\theta} = ad$ but $a^{\theta\psi} = adb^2$. So $\theta\psi \ne \psi\theta$.

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