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THE COHOMOLOGY OF $(\Lambda^2X, \Delta X)$

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1. Introduction. Let X be any connected C.W. complex. Let Λ^2X be the 2-fold symmetric product of X, the set of unordered pairs (not necessarily distinct) of elements of X. Let $\Delta X \subset X^2$ be the diagonal, and also (by a slight abuse of notation) let $\Delta X \subset \Lambda^2X$ also denote the set of unordered pairs $\{x, x\}$. The purpose of this paper is to describe the cohomology, and twisted cohomology, of the pair $(\Lambda^2X, \Delta X)$. If X is of finite type and if the reduced cohomology of X is explicitly given in terms of Moore generators (in effect, an isomorphism between the cohomology of X and the cohomology of a wedge product of Moore spaces), then the cohomology of $(\Lambda^2X, \Delta X)$, with both twisted and untwisted coefficients, is explicitly given, also in terms of Moore generators (cf. Theorem 20).

In a later paper, R. D. Rigdon and the author will show how, if M is a manifold immersed in Euclidean space, one can define obstructions to regular homotopy of that immersion with an embedding, taking values in the cohomology of $(\Lambda^2M, \Delta M)$; basically an application of Haefliger's work [**2**]. These obstructions are sufficient to settle the question in the metastable range.

2. The Mod 2 cohomology of $(\Lambda^2X, \Delta X)$**.** In this section, all coefficients will be in Z_2.

Now if $\pi : (K, L) \to (K', L')$ is any relative 2-1 covering of C.W. complexes, i.e., $\pi^{-1}L' = L$ and $\pi|K - L$ is a 2-1 covering, we have a long exact Thom-Gysin sequence:

$$\ldots \longrightarrow H^n(K', L') \xrightarrow{\pi^*} H^n(K, L) \xrightarrow{\theta} H^n(K', L') \xrightarrow{m\cdot} H^{n+1}(K', L') \longrightarrow \ldots$$

where $m \in H^1(K' - L')$ is w_1 of the 0-sphere bundle $\pi|K - L$. The multiplication by m is given by the composition:

$$H^n(K', L') \otimes H^1(K' - L') \cong H^n(K' - L', N - L') \otimes H^1(K' - L') \to H^{n+1}(K' - L', N - L') \cong H^{n+1}(K', L')$$

where N is some closed neighborhood of L' such that L' is a strong deformation retract of N; both isomorphisms are given by excision and homotopy.

LEMMA 1. *If $x \in H^*(K', L')$ and $y \in H^*(K, L)$, then $\theta((\pi * x)y) = x\theta y$.*

Proof. This follows immediately from the definitions of θ and cup product at the cochain level. We leave the details to the reader.

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LEMMA 2. *If K/L is k-connected for some integer k, then K'/L' is also k-connected.*

Proof. We use induction on k. Suppose K/L is k-connected and K'/L' is $(k - 1)$-connected. Then K'/L' is k-connected, immediately from the Thom-Gysin sequence.

Henceforth, let $\pi : (\Lambda^2X, \Delta X) \to (X^2, \Delta X)$ be the quotient map, a relative 2-1 covering.

We pick a basepoint $* \in X$. Let also $*$ denote $(*, *) \in X^2$. The pair $(X^2 \times S^\infty, S^\infty)$ is then of the homotopy type of $(X^2, *)$: we shall freely identify their cohomology. Let $\Gamma X = (X^2 \times S^\infty)/T$, where T is the (free) action which exchanges coordinates of X^2 and which is the antipodal map on S^∞. Letting $P_\infty = S^\infty/T$, we then have a commutative diagram, where each row is an exact Thom-Gysin sequence:

$$(1)\qquad \begin{array}{ccccccccc} \cdots \to & H^k(\Lambda^2X, \Delta X) & \xrightarrow{\pi^*} & H^k(X^2, \Delta X) & \xrightarrow{\theta} & H^k(\Lambda^2X, \Delta X) & \xrightarrow{m} & H^{k+1}(\Lambda^2X, \Delta X) & \to \cdots \\ & \downarrow Q^* & & \downarrow P^* & & \downarrow Q^* & & \downarrow Q^* & \\ \cdots \to & H^k(\Gamma X, P_\infty) & \xrightarrow{\pi^*} & H^k(X^2, *) & \xrightarrow{\theta} & H^k(\Gamma X, P_\infty) & \xrightarrow{m} & H^{k+1}(\Gamma X, P_\infty) & \to \cdots \end{array}$$

where $P : X^2 \times S^\infty \to X^2$ is the projection and $Q : \Gamma X \to \Lambda^2X$ is the corresponding map on the quotient spaces.

Henceforth, let $K_n = K(Z_2, n)$ for any n, and let ι_n be the fundamental class of K_n. Now $K_n{}^2/\Delta K_n$ is $(n - 1)$-connected, hence, $\Lambda^2K_n/\Delta K_n$ is $(n - 1)$-connected also; we then have that $\pi^* : H^n(\Lambda^2K_n, \Delta K_n) \to H^n(\Lambda^2K_n/\Delta K_n)$ is mono. Let $\Lambda\iota_n = (\pi^*)^{-1}(\iota_n \otimes 1 + 1 \otimes \iota_n)$, and let $\Gamma\iota_n = Q^*\Lambda\iota_n \in H^n(\Gamma K_n, P_\infty)$.

LEMMA 3. (i) *$H^0(K_0{}^2, *) \cong Z_2 + Z_2 + Z_2$, generated by $\iota_0 \otimes 1$, $1 \otimes \iota_0$, and $\iota_0 \otimes \iota_0$.*

(ii) *If $n > 0$, $H^n(K_n{}^2, *) \cong Z_2 + Z_2$, generated by $\iota_n \otimes 1$ and $1 \otimes \iota_n$.*

(iii) *$H^0(K_0{}^2, \Delta K_0) \cong Z_2 + Z_2$, generated by $\iota_0 \otimes 1$ and $1 \otimes \iota_0$.*

(iv) *If $n > 0$, $H^n(K_n{}^2, \Delta K_n) \cong Z_2$, generated by $\iota_n \otimes 1 + 1 \otimes \iota_n$.*

(v) *$H^n(\Lambda^2K_n, \Delta K_n) \cong Z_2$, generated by $\Lambda\iota_n$.*

(vi) *$H^0(\Gamma K_0, P_\infty) \cong Z_2 + Z_2$, generated by $\Gamma\iota_0$ and $(\pi^*)^{-1}\iota_0 \otimes \iota_0$.*

(vii) *If $n > 0$, $H^n(\Gamma K_n, P_\infty) \cong Z_2$, generated by $\Gamma\iota_n$.*

Proof. Parts (i), (ii), (iii), and (iv) are well-known. The other parts follow immediately from these, using diagram (1).

If $x \in H^n(X)$, choose a function $f : X \to K_n$ which classifies x; then define Λx to be $(\Lambda^2 f)^*(\Lambda\iota_n) \in H^n(\Lambda^2X, \Delta X)$ and Γx to be $(\Gamma f)^*(\Gamma\iota_n)H^n(\Gamma X, P_\infty)$.

LEMMA 4. *For any $x \in H^n(X)$ and any $n \geqq 0$:*

(i) $\pi^*\Lambda x = 1 \otimes x + x \otimes 1 \in H^n(X^2, \Delta X)$.

(ii) $Q^*\Lambda x = \Gamma x$.

(iii) $\pi^*\Gamma x = 1 \otimes x + x \otimes 1 \in H^n(X^2, *)$.

(iv) $\theta(x \otimes 1) = \theta(1 \otimes x) = \Gamma x$.

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Proof. We need only consider the universal example: $X = K_n$, $x = \iota_n$. If $n = 0$, the lemma follows trivially. If $n > 0$, a dimensionality argument is needed for (iv); we leave the details to the reader.

LEMMA 5. *$\Lambda : \tilde{H}^*(X) \to H^*(\Lambda^2 X, \Delta X)$ and $\Gamma : H^*(X) \to H^*(\Gamma X, P_\infty)$ are monomorphisms.*

Proof. Assume, for the moment, that Λ is a homomorphism. Now $\Gamma = Q^* \circ \Lambda$, hence Γ is also a homomorphism. Suppose that $x \neq 0$. Then $\pi^* \Gamma x = x \otimes 1 + 1 \otimes x \neq 0$, hence Γ is mono, hence Λ is mono.

To prove that Λ is a homomorphism, it is sufficient to consider the universal example: $U = K_n \times K_n$, $u = \iota_n \otimes 1$, $v = 1 \otimes \iota_n$: we need only show that $\Lambda(u + v) = \Lambda u + \Lambda v$. By Lemma 2, $\Lambda^2 U/\Delta U$ is $(n-1)$-connected, hence $\pi^* : H^n(\Lambda^2 U, \Delta U) \to H^n(U^2, \Delta U)$ is mono. Since

$$\pi^* \Lambda(u + v) = (u + v) \otimes 1 + 1 \otimes (u + v) = \pi^*(\Lambda u + \Lambda v),$$

we are done.

LEMMA 6. *For any $x \in H^n(X)$, $\delta x = m\Lambda x$, where $\delta : H^n(X) = H^n(\Delta X) \to H^{n+1}(\Lambda^2 X, \Delta X)$ is the connecting homomorphism.*

Proof. It is sufficient to consider the universal example: $X = K_n$, $x = \iota_n$. If $n = 0$, we are done, since $H^1(\Lambda^2 K_0, \Delta K_0) = 0$. Assume $n > 0$. We have a commutative diagram with the bottom row exact, where each δ is the appropriate connecting homomorphism:

$$\begin{array}{ccccc} & & H^n(K_n) = H^n(\Delta K_n) \cong Z_2 & & \\ & & \downarrow \delta & \searrow \delta = 0 & \\ Z_2 \cong H^n(\Lambda^2 K_n, \Delta K_n) & \xrightarrow{m\cdot} & H^{n+1}(\Lambda^2 K_n, \Delta K_n) & \xrightarrow{\pi^*} & H^{n+1}(K_n{}^2, \Delta K_n) \end{array}$$

Hence $x \in mH^n(\Lambda^2 K_n, \Delta K_n)$. According to Dold [**1**], $H^n(\Lambda^2 K_n) \cong Z_2$, hence, by the exact cohomology sequence of the pair $(\Lambda^2 K_n, \Delta K_n)$ and Lemma 3, we have that $\delta x \neq 0$. Thus $\delta x = m\Lambda x$.

LEMMA 7. *If $x \in H^n(X)$ and $y \in H^p(X)$, for any $n, p \geqq 0$, then:*
(i) $\theta(xy \otimes 1 + x \otimes y) = \Lambda x \Lambda y$.
(ii) $m\Lambda x \Lambda y = 0$.

Proof. To prove (i), it is sufficient to consider the universal example: $U = K_n \times K_p$, $u = \iota_n \otimes 1$, $v = 1 \otimes \iota_p$. Let $B = \Delta U \cup (K_n \vee K_p)^2 \subset U^2$. Note that U^2/B is $(n + p - 1)$-connected, hence $\Lambda^2 U/\pi B$ is also. Consider the commutative diagram with exact rows:

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$$
\begin{array}{ccccccccc}
0 \longrightarrow H^{n+p}(\Lambda^2 U, \pi B) & \xrightarrow{\pi^*} & H^{n+p}(U^2, B) & \xrightarrow{\theta_1} & H^{n+p}(\Lambda^2 U, \pi B) & \xrightarrow{m\,\cdot} & H^{n+p+1}(\Lambda^2 U, \pi B) \\
\downarrow j^* & & \downarrow j^* & & \downarrow j^* & & \downarrow j^* \\
H^{n+p}(\Lambda^2 U, \Delta U) & \xrightarrow{\pi^*} & H^{n+p}(U^2, \Delta U) & \xrightarrow{\theta_2} & H^{n+p}(\Lambda^2 U, \Delta U) & \xrightarrow{m\,\cdot} & H^{n+p+1}(\Lambda^2 U, \Delta U) \\
\downarrow Q^* & & \downarrow P^* & & \downarrow Q^* & & \downarrow Q^* \\
H^{n+p}(\Gamma U, P_\infty) & \xrightarrow{\pi} & H^{n+p}(U^2, *) & \xrightarrow{\theta_3} & H^{n+p}(\Gamma U, P_\infty) & \xrightarrow{m\,\cdot} & H^{n+p+1}(\Gamma U, P_\infty),
\end{array}
\tag{2}
$$

and the exact sequence:

$$
H^{n+p}(\Lambda^2 U, \pi B) \xrightarrow{j^*} H^{n+p}(\Lambda^2 U, \Delta U) \xrightarrow{k^*} H^{n+p}(\pi B, \Delta U) \tag{3}
$$

where always j and k are appropriate inclusions of pairs. The remainder of the proof is simple diagram chasing, using the previous lemmas. $H^{n+p}(U^2, B)$ has dimension 3 over Z_2, with generators $uv \otimes 1 + 1 \otimes uv$, $u \otimes v + v \otimes u$, and $u \otimes v + uv \otimes 1$. Since $T^* \circ \pi^* = \pi^*$ and $\theta \circ T^* = \theta$, where $T : U^2 \to U^2$ exchanges coordinates, we can easily see that $H^{n+p}(\Lambda^2 U, \pi B)$ has dimension 2, with independent generators α and β, where $\pi^*\alpha = uv \otimes 1 + 1 \otimes uv$ and $\pi^*\beta = u \otimes v + v \otimes u$. Now $\pi B/\Delta U \cong (\Lambda^2 K_n/\Delta K_n) \vee (\Lambda^2 K_p/\Delta K_p)$, hence (in diagram (3)) $k^*(\Lambda u \Lambda v) = 0$. Using the exactness of (3) and the commutativity of the upper left square of (2), we have that $j^*\alpha = \Lambda(uv)$ and $j^*\beta = \Lambda(uv) + \Lambda u \Lambda v$. Now

$$P^* \circ j^* : H^{n+p}(U^2, B) \to H^{n+p}(U^2, *)$$

is obviously mono, by commutativity of the two left squares and by the zero in the upper left corner. By Lemmas 1 and 4,

$$\theta_3(uv \otimes 1 + u \otimes v) = \Gamma v \theta_3(u \otimes 1) = \Gamma v \Gamma u \in H^{n+p}(\Gamma U, P_\infty),$$

hence

$$\theta_1(uv \otimes 1 + u \otimes v) = \alpha + \beta \in H^{n+p}(\Lambda^2 U, \pi B),$$

hence

$$\theta_2(uv \otimes 1 + u \otimes v) = \Lambda u \Lambda v \in H^{n+p}(\Lambda^2 U, \Delta U);$$

thus (i) is proved. Part (ii) is an immediate corollary.

LEMMA 8. *If* $x \in H^n(X)$, $y \in H^p(X)$, *and* $z \in H^q(X)$, *then* $\Lambda x \Lambda y \Lambda z = \Lambda x \Lambda(yz) = \Lambda y \Lambda(xz) + \Lambda z \Lambda(xy)$.

Proof. It is sufficient to consider the universal example, namely $U = K_n \times K_p \times K_q$, $u = \iota_n \otimes 1 \otimes 1$, $v = 1 \otimes \iota_p \otimes 1$, and $w = 1 \otimes 1 \otimes \iota_q$. Let

$$B = \Delta U \cup (K_n \times K_p \times *)^2 \cup (K_n \times * \times K_q)^2 \cup (* \times K_p \times K_q)^2.$$

We have a commutative diagram with the row and column exact:

$$\begin{array}{ccccccc} 0 \longrightarrow & H^{n+p+q}(\Lambda^2 U, \pi B) & \xrightarrow{\pi^*} & H^{n+p+q}(U^2, B) & \xrightarrow{\theta} & H^{n+p+q}(\Lambda^2 U, \pi B) \\ & \downarrow j^* & & \downarrow j^* & & \\ & H^{n+p+q}(\Lambda^2 U, \Delta U) & \xrightarrow{\pi^*} & H^{n+p+q}(U^2, \Delta U) & & \\ & \downarrow k^* & & & & \\ & H^{n+p+q}(\pi B, \Delta U) \cong H^{n+p+q}((\Lambda^2 K_n/\Delta K_n) \vee (\Lambda^2 K_p/\Delta K_p) \vee (\Lambda^2 K_q/\Delta K_q)) & & & & \end{array}$$

where j and k are appropriate inclusions of pairs. Now $H^{n+p+q}(U^2, B)$ has dimension 7 over Z_2; its symmetric part, Ker θ, has four independent generators. It is a simple matter of diagram chasing to verify that $j^* : H^{n+p+q}(\Lambda^2 U, \pi B) \to H^{n+p+q}(\Lambda^2 U, \Delta U)$ is mono, and its image has dimension 4, generated by $\Lambda(uvw)$, $\Lambda u\Lambda(vw)$, $\Lambda v\Lambda(uw)$, and $\Lambda w\Lambda(uv)$. Since $k^*\Lambda u\Lambda v\Lambda w = 0$, $\Lambda u\Lambda v\Lambda w$ must be a linear combination of those four generators. The stated result is the only possibility which agrees with Lemma 4.

Let

$$\mu : \Sigma(X^2/\Delta X) \to (\Sigma X)^2/\Delta\Sigma X$$

be defined as follows: for every $x, y \in X$ and $t \in I$, let $\mu[[x, t], t] = [[x, t], [y, t]]$. Let $\tau : \Sigma(\Lambda^2 X/\Delta X) \to \Lambda^2\Sigma X/\Delta\Sigma X$ be the corresponding map on the quotient spaces.

LEMMA 9. *For any $x \in H^n(X)$ and any $i \geq 0$, $\tau^* m^i\Lambda(sx) = sm^i\Lambda x$* ($s$ = suspension isomorphism).

Proof. Consider first the case where $i = 0$. Let $X = K_n$, $x = \iota_n$; the universal example. We have a commutative diagram:

$$\begin{array}{ccc} Z_2 \cong H^{n+1}(\Lambda^2\Sigma K_n/\Delta\Sigma K_n) & \xrightarrow{\pi^*} & H^{n+1}((\Sigma K_n)^2/\Delta\Sigma K_n) \\ \downarrow \tau^* & & \downarrow \mu^* \\ Z_2 \cong H^{n+1}(\Sigma(\Lambda^2 X/\Delta X)) & \xrightarrow{(\Sigma\pi)^*} & H^{n+1}(\Sigma(X^2/\Delta X)). \end{array}$$

Now $\mu^*\pi^*\Lambda(s\iota_n) = \mu^*(s\iota_n \otimes 1 + 1 \otimes s\iota_n) = s(\iota_n \otimes 1 + 1 \otimes \iota_n) = (\Sigma\pi)^* s\Lambda\iota_n$. Since both groups on the left of the diagram have one generator each, $\Lambda(s\iota_n)$ and $s\Lambda\iota_n$, respectively, $\tau^*\Lambda(s\iota_n) = s\Lambda\iota_n$. Now τ comes from μ; thus multiplication by m commutes with τ^*, and we are done.

LEMMA 10. *If $x \in H^n(X)$, $Sq^i\Lambda x = \sum_{j=0}^{i} m^{i-j}\Lambda Sq^j x$ for all $i \geq 0$.*

Proof. We use a 2-step induction process. The formula obviously holds if $i = 0$. We first show that the formula holds for (n, i) if it holds for $(n, i - 1)$, provided $n > i$; secondly, we show that it holds for (n, i) if it holds for

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$(n + 1, i)$. In each case, we look at the universal example: $X = K_n$, $x = \iota_n$.

Step I: Since $i < n$, $H^{n+i}(K_n{}^2, \Delta K_n)$ is generated by elements of the form $Sq^I\iota_n \otimes 1 + 1 \otimes Sq^I\iota_n = \pi^*\Lambda Sq^I\iota_n$, where I is an admissable monomial and Sq^I is the Steenrod square; hence multiplication by m is mono on $H^{j+i}(\Lambda^2 K_n, \Delta K_n)$. By the Cartan formula and Lemma 6, we have

$$m\,\Lambda Sq^i\iota_n = \delta Sq^i\Lambda\iota_n = Sq^i\delta\iota_n = Sq^i(m\,\Lambda\iota_n) = mSq^i\Lambda\iota_n + m^2\sum_{j=0}^{i-1} m^{i-j-1}\Lambda Sq^j\iota_n.$$

Cancelling m, we are done.

Step 2: Consider the map $\tau : \Sigma(\Lambda^2K_n/\Delta K_n) \to \Lambda^2\Sigma K_n/\Delta\Sigma K_n$. By Lemma 9 we have

$$sSq^i\Lambda\iota_n = \tau^*Sq^i\Lambda s\iota_n = \tau^*\sum_{j=0}^{i} m^{i-j}\Lambda sSq^j\iota_n = s\sum_{j=0}^{i} m^{i-j}\Lambda Sq^j\iota_n$$

Since s is an isomorphism, we are done.

Let $Z_2[m]$ be the algebra of finite polynomials in m with coefficients in Z_2. $H^*(\Lambda^2X, \Delta X)$ is a commutative associative graded algebra over $Z_2[m]$. The cohomology of the pair $(\Lambda^2X, \Delta X)$ is then fully described by the following theorem:

THEOREM 11 (structure theorem). *As an algebra over $Z_2[m]$, $H^*(\Lambda^2X, \Delta X)$ is generated by all Λx for $x \in \tilde{H}^*(X)$, subject only to the following relations* ($\tilde{H}^*$ = reduced cohomology):

(i) $\Lambda(x + y) = \Lambda x + \Lambda y$ *for all* $x, y \in \tilde{H}^*(X)$.

(ii) $m\Lambda x\Lambda y = 0$ *for all* $x, y \in \tilde{H}^*(X)$.

(iii) $\Lambda x\Lambda y\Lambda z = \Lambda x\Lambda(yz) + \Lambda y\Lambda(xz) + \Lambda z\Lambda(xy)$ *for all* $x, y, z \in \tilde{H}^*(X)$.

(iv) $(\Lambda x)^2 = \sum_{j=0}^{n} m^{n-j}\Lambda Sq^j x$ *for all* $x \in \tilde{H}^n(X)$, $n > 0$.

Proof. Let $\mathbf{H}^*$ be the commutative associative graded algebra over $Z_2[m]$ generated by $\{\Lambda x : x \in \tilde{H}^*(X)\}$ subject to the relations (i) through (iv) above; i.e., $\mathbf{H}^*$ is what the theorem claims $H^*(\Lambda^2X, \Delta X)$ to be. Let $\iota^* : \mathbf{H}^* \to H^*(\Lambda^2X, \Delta X)$ be the graded $Z_2[m]$-homomorphism which takes Λx to Λx for all x; ι^* is well-defined by Lemmas 5, 7, 8, and 10. Consider the diagram of groups and homomorphisms:

$$\begin{array}{ccccccccc}
H^k(X^2, \Delta X) & \xrightarrow{\beta} & \mathbf{H}^k & \xrightarrow{m\,\cdot} & \mathbf{H}^{k+1} & \xrightarrow{\alpha} & H^{k+1}(X^2, \Delta X) & \xrightarrow{\beta} & \mathbf{H}^{k+1} \\
\downarrow = & & \downarrow \iota^k & & \downarrow \iota^{k+1} & & \downarrow = & & \downarrow \iota^{k+1} \\
H^k(X^2, \Delta X) & \xrightarrow{\theta} & H^k(\Lambda^2X, \Delta X) & \xrightarrow{m\,\cdot} & H^{k+1}(\Lambda^2X, \Delta X) & \xrightarrow{\pi^*} & H^{k+1}(X^2, \Delta X) & \xrightarrow{\theta} & H^{k+1}(\Lambda^2X, \Delta X)
\end{array}$$

where

$$\alpha(\Lambda x) = x \otimes 1 + 1 \otimes x,\ \alpha(\Lambda x\Lambda y) = xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x,$$

and $\alpha(m^i\Lambda x) = 0$ for all $x, y \in \tilde{H}^*(X)$ and all $i \geqq 1$; and where $\beta(x \otimes 1 + 1 \otimes x) = 0$ and $\beta(x \otimes y + xy \otimes 1) = \Lambda x\Lambda y$ for all $x, y \in \tilde{H}^*(X)$.

By the definition of Λ, and also by Lemma 7, the diagram is commutative. The bottom row is the Thom-Gysin sequence, and it is a routine algebraic exercise that the top row is also exact. We prove that ι^k is an isomorphism by induction on k. Clearly it is if $k \leqq 0$. Suppose ι^k is an isomorphism. By the 5-lemma, ι^{k+1} is one-to-one. Using that fact and the 5-lemma again, we have that ι^{k+1} is onto. Thus $\iota^* : \mathbf{H}^* \to H^*(\Lambda^2 X, \Delta X)$ is an isomorphism, as claimed.

3. The cohomology of $(\Lambda^2 X, \Delta X)$ **with other coefficients.** Let G be a cyclic group, and let $G[m]$ be the sheaf of coefficients over $\Lambda^2 X - \Delta X$, locally isomorphic to G, twisted by m. Let $M \in H^1(\Lambda^2 X - \Delta X; Z[m])$ be the twisted integer class representing m; $M = (\delta)^T 1$, where $1 \in H^0(\Lambda^2 X - \Delta X; Z_2)$ is the unit and $(\delta)^T$ is the Bokstein of the sequence

$$Z[m] \xrightarrow{\times 2} Z[m] \longrightarrow Z_2,$$

hence $2M = 0$. We have two long exact sequences, where θ and θ^T are the transfer maps:

$$\cdots \longrightarrow H^{n-1}(\Lambda^2 X, \Delta X; G[m]) \xrightarrow{\cup M} H^n(\Lambda^2 X, \Delta X; G) \xrightarrow{\pi^*} H^n(X^2, \Delta X; G) \xrightarrow{\theta^T} H^n(\Lambda^2 X, \Delta X; G[m]) \xrightarrow{\cup M} \cdots$$

$$\cdots \longrightarrow H^{n-1}(\Lambda^2 X, \Delta X; G) \xrightarrow{\cup M} H^n(\Lambda^2 X, \Delta X; G[m]) \xrightarrow{(\pi^*)^T} H^n(X^2, \Delta X; G) \xrightarrow{\theta} H^n(\Lambda^2 X, \Delta X; G) \xrightarrow{\cup M} \cdots .$$

The compositions $\theta \circ \pi^*$ and $\theta^T \circ (\pi^*)^T$ are both multiplication by 2.

Let $p \geqq 1$ be an integer, and let $K = K(G, p)$. We define $\Lambda\iota_p \in H^p(\Lambda^2 K, \Delta K; G[m])$ by the equation $(\pi^*)^T \Lambda\iota_p = \iota_p \otimes 1 - 1 \otimes \iota_p$, where ι_p is the fundamental class of K. If $x \in H^p(X; G)$, let $\Lambda x = (\Lambda^2 f)^* \Lambda\iota_p$, where $f : X \to K$ classifies x.

Let $p \geqq 1$ and $q \geqq 1$ be integers, and let $K_p = K(G, p)$ and $K_q = K(G, q)$. Let $\alpha = \iota_p \otimes 1$ and $\beta = 1 \otimes \iota_q$, elements of $H^*(K_p \times K_q; G)$. Let $U = K_p \times K_q$, and $B = (K_p \vee K_q)^2 \cup \Delta U \subset U^2$. Now $\pi : (U^2, B) \to (\Lambda^2 U, \pi B)$ is a relative 2-1 covering, and U^2/B is $(p + q - 1)$-connected, hence, using the Thom-Gysin sequence of that covering, we can verify that

$$(\pi^*)^T : H^{p+q}(\Lambda^2 U, \pi B; G[m]) \to H^{p+q}(U^2, B)$$

is a monomorphism. It is also not difficult to show that $\sigma = \alpha \otimes \beta - (-1)^{pq} \beta \otimes \alpha$ must lie in the image of $(\pi^*)^T$. We then define $\Delta(\alpha, \beta)$ to be $(\Lambda^2 j)^*((\pi^*)^T)^{-1}\sigma \in H^{p+q}(\Lambda^2 U, \Delta U; G[m])$, where $j : (\Lambda^2 U, \Delta U) \to (\Lambda^2 U, \pi B)$ is the inclusion. If $x \in H^p(X; G)$ and $y \in H^q(X; G)$, let $f : X \to K_p$ and $g : X \to K_q$ be maps which classify x and y, respectively: we then define $\Delta(x, y)$ to be $(\Lambda^2(f \times g))^* \Delta(\alpha, \beta) \in H^{p+q}(\Lambda^2 X, \Delta X; G[m])$. We immediately have:

Remark 12. $\Delta(y, x) = (-1)^{pq+1} \Delta(x, y)$.

Now, if G has odd order, both Gysin sequences split, since M has order 2. Hence:

Remark 13. If G has odd order, π^* and $(\pi^*)^T$ are both mono; in fact, $H^*(\Lambda^2 X, \Delta X; G)$ and $H^*(\Lambda^2 X, \Delta X; G[m])$ are isomorphic to the symmetric and antisymmetric parts of $H^*(X^2, \Delta X; G)$, respectively.

Let k be an odd prime. Then

$$A^* = H^*(\Lambda^2 X, \Delta K; Z_k) \oplus H^*(\Lambda^2 X, \Delta X; Z_k[m]) \cong H^*(X^2, \Delta X; Z_k)$$

is a commutative graded algebra over Z_k in the obvious way. Similar to Theorem 11, we have:

THEOREM 14. *If k is an odd prime and X is a connected C.W. complex, then $H^*(\Lambda^2 X, \Delta X; Z_k) \oplus H^*(\Lambda^2 X, \Delta X; Z_k[m])$, considered as a commutative graded algebra over Z_k, is generated only by elements of the form:*

(i) $\Lambda x \in H^p(\Lambda^2 X, \Delta X; Z_k[m])$ *for all* $x \in H^p(X; Z_k)$, $p \geqq 1$

(ii) $\Delta(x, y) \in H^{p+q}(\Lambda^2 X, \Delta X; Z_k[m])$ *for all* $x \in H^p(X; Z_k)$, $y \in H^q(X; Z_k)$, $p, q \geqq 1$,

subject only to the following relations (*where in each case*, $\dim x = \dim x' = p$, $\dim y = q$, $\dim z = r$, *and* $\dim w = s$):

(i) $\Lambda(x + x') = \Lambda x + \Lambda x'$,

(ii) $\Lambda x \Lambda y \Lambda z = \Lambda xyz + \Delta(x, yz) - \Delta(xy, z) + (-1)^{qr}\Delta(xz, y)$,

(iii) $\Delta(y, x) = (-1)^{pq+1}\Delta(x, y)$,

(iv) $\Delta(x + x', y) = \Delta(x, y) + \Delta(x', y)$,

(v) $\Lambda x \Delta(y, z) = (-1)^{pq}\Lambda y \Lambda xz - \Lambda xy \Lambda z$,

(vi) $\Delta(x, y)\Delta(z, w) = (-1)^{(r+q)s}\Lambda xw \Lambda yz + (-1)^{qr+1}\Lambda xz \Lambda yw$.

We omit the proof, which is trivial given Remark 13.

Henceforth, for any integers r and s, we let $\beta_r{}^s$ be the Bokstein of the coefficient sequence $Z_s \to Z_{rs} \to Z_r$; we also let ρ_r denote reduction mod r from any coefficient group whose order is infinite or a multiple of r. Let $(\beta_r{}^s)^T$ and $(\rho_r)^T$ denote the twisted versions of these. Directly from the definitions of Λ and Δ and from Remark 13, we conclude:

Remark 15. For any $x \in H^*(X; Z_t)$, t a power of an odd prime, $(\beta_t{}^t)^T \Lambda x = \Lambda \beta_t{}^t x$.

Remark 16. For any $x, y \in H^*(X; Z_t)$, t a power of an odd prime, $\beta_t{}^t(\Lambda x \Lambda y) = \Lambda \beta_t{}^t x \Lambda y + (-1)^p \Lambda x \Lambda \beta_t{}^t y$, and $(\beta_t{}^t)^T \Delta(x, y) = \Delta(\beta_t{}^t x, y) + (-1)^p \Delta(x, \beta_t{}^t y)$ where $p = \dim x$.

Now let $K_n = K(Z_2, n)$, for $n \geqq 1$, and let ι_n be the fundamental class of K_n. First, consider even n. By Theorem 11, $(\Lambda\iota_n)^2$ and $(\Lambda\iota_n)Sq^1\Lambda\iota_n + Sq^n Sq^1 \Lambda\iota_n$ are both non-zero; thus $\mathfrak{p}\Lambda\iota_n$ has order 4, where $\mathfrak{p}$ is the Pontrjagin square. We define $\Upsilon\iota_n \in H^{2n}(\Lambda^2 K_n, \Delta K_n; Z_4)$ to satisfy the equations:

$$\rho_2 \Upsilon \iota_n = \sum_{i=0}^{\frac{1}{2}n} m^{2i} \Lambda Sq^{n-2i} \iota_n,$$

and

$$\beta_4{}^2\Upsilon\iota_n = \Lambda\iota_n\Lambda Sq^1\iota_n + \sum_{i=0}^{\frac{1}{2}n} m^{2i}\Lambda Sq^{n-2i}Sq^1\iota_n.$$

Note that $\Upsilon\iota_n$ differs from $\mathfrak{p}\Lambda\iota_n$ by an element of order 2, hence $\Upsilon\iota_n$ also has order 4. Suppose, secondly, that n is odd. Let $y = \sum_{i=1}^{\frac{1}{2}(n-1)} m^{n-2i}\Lambda Sq^{2i}\iota_n$. By Lemma 10 and Theorem 11, $Sq^1y + my = 0$, but $y \notin \operatorname{Im}(Sq^1 + m)$. Thus y is the reduction of a twisted Z_4 class of order 4, which we call $\Upsilon\iota_n$. A quick check of the Thom-Gysin sequence shows that y cannot be the reduction of a twisted Z_8 class, hence $(\beta_4{}^2)^T\Upsilon\iota_n \neq 0$. Now $(\beta_4{}^2)^T\Upsilon\iota_n$ must lie in the kernel, but not the image, of $Sq^1 + m$; the only candidate, up to indeterminacy, is

$$z = \Lambda\iota_n\Lambda Sq^1\iota_n + \Lambda\iota_n Sq^1\iota_n + \sum_{i=0}^{\frac{1}{2}(n-1)} m^{n-2i}\Lambda Sq^{2i}Sq^1\iota_n.$$

We can thus insist that $(\beta_4{}^2)^T\Upsilon\iota_n = z$. Using K_n as a universal example, we can define Υx for any $x \in H^n(X; Z_2)$; $\Upsilon x \in H^{2n}(\Lambda^2X, \Delta X; S)$, where $S = Z_4$ if n is even, $S = Z_4[m]$ if n is odd. Υx has order 4 if and only if $Sq^1x \neq 0$.

If $x \in H^n(X; Z_r)$, where $n \geqq 1$ and $r \geqq 4$ is a power of 2, we can, using the appropriate universal example, define $\Upsilon x \in H^{2n}(\Lambda^2X, \Delta X; S)$, where $S = Z_{2r}$ if n is even, $S = Z_{2r}[m]$ if n is odd; such that, if n is even,

$$\rho_2\Upsilon x = \sum_{i=0}^{\frac{1}{2}n} m^{2i}\Lambda Sq^{n-2i}\rho_2 x \quad \text{and} \quad \beta_r{}^2\Upsilon x = \Lambda\rho_2 x\Lambda\beta_r{}^2 x;$$

and if n is odd,

$$(\rho_2)^T\Upsilon x = \sum_{i=0}^{\frac{1}{2}(n-1)} m^{n-2i}\Lambda Sq^{2i}\rho_2 x, \quad \text{and} \quad (\beta_r{}^2)^T\Upsilon x = \Lambda\rho_2 x\Lambda\beta_r{}^2 x.$$

In either case, Υx has order $2r$ if and only if $\beta_r{}^2 x \neq 0$. We leave the details to the reader.

3. The category of coefficients. We define a category C, which we call the category of coefficients, as follows. The objects of C are all Abelian groups. If A and B are objects, we let

$$\operatorname{Hom}_C(A, B) = \operatorname{Hom}_C{}^0(A, B) \oplus \operatorname{Hom}_C{}^1(A, B),$$

a graded Abelian group, where $\operatorname{Hom}_C{}^0(A, B) = \operatorname{Hom}(A, B)$ and $\operatorname{Hom}_C{}^1(A, B) = \operatorname{Ext}(A, B)$, both Hom and Ext being over the integers. Let $\alpha \in \operatorname{Hom}_C(B, C)$ and $\beta \in \operatorname{Hom}_C(A, B)$, for any three objects A, B, and C. If $\deg\alpha = \deg\beta = 0$, let $\alpha \circ \beta$ be the ordinary composition. If $\deg\alpha = \deg\beta = 1$, let $\alpha \circ \beta = 0$. If $\deg\alpha = 0$ and $\deg\beta = 1$, $\alpha \circ \beta$ is defined by the pushout diagram:

$$\begin{array}{rccccccccc}
\beta: & 0 & \to & B & \to & E & \to & A & \to & 0 \\
 & & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle =} & & \\
\alpha\circ\beta: & 0 & \to & C & \to & E' & \to & A & \to & 0
\end{array}$$

while if $\deg \alpha = 1$ and $\deg \beta = 0$, $\alpha \circ \beta$ is given by the pullback:

$$\begin{array}{ccccccccc} \alpha\circ\beta : 0 & \rightarrow & C & \rightarrow & E' & \rightarrow & A & \rightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow \beta & & \\ \alpha : 0 & \rightarrow & C & \rightarrow & E & \rightarrow & B & \rightarrow & 0. \end{array}$$

Composition in C is then a homomorphism of degree 0:

$$\mathrm{Hom}_C(B, C) \otimes \mathrm{Hom}_C(A, B) \rightarrow \mathrm{Hom}_C(A, C)$$

Let A^* be the category of all graded Abelian groups and homomorphisms. We define a C-module to be a functor $H^* : C \rightarrow A^*$ such that $H^* : \mathrm{Hom}_C(A, B) \rightarrow \mathrm{Hom}_{A^*}(H^*(A), H^*(B))$ is a homomorphism of degree 0 for any A, $B \in C$. If H^* and H''^* are C-modules, and k is an integer, we say $\gamma : H^* \rightarrow H'^*$ is a C-module map of degree k if, for every $A \in C$, $\gamma(A) \in \mathrm{Hom}_{A^*}(H^*(A), H'^*(A))$, and, for every $A, B \in C$ and every $\alpha \in \mathrm{Hom}_C(A, B)$, the following diagram is commutative:

$$\begin{array}{ccc} H^*(A) & \xrightarrow{H^*(\alpha)} & H^*(B) \\ \downarrow \gamma(A) & & \downarrow \gamma(B) \\ H'^*(A) & \xrightarrow{H'^*(\alpha)} & H'^*(B). \end{array}$$

We define composition of C-module maps $\gamma : H^* \rightarrow H'^*$ and $\delta : H'^* \rightarrow H''^*$ by the equation $(\delta \circ \gamma)(A) = \delta(A) \circ \gamma(A)$ for all $A \in C$. Direct sum of C-modules is also defined in the obvious way.

If $H^* : C \rightarrow A^*$ is a C-module, and if $x \in H^k(A)$ for some integer k and some $A \in C$, we say that x is an element of H^* of degree k with coefficients in A.

The obvious example of a C-module is, of course, the cohomology of a C.W. pair, (K, L), $H^*(K, L) = H^* : C \rightarrow A^*$ defined by: $H^*(A) = H^*(K, L; A)$ for any Abelian group A; $H^*(\alpha) = \alpha^*$ if $\alpha \in \mathrm{Hom}(A, B)$, and $H^*(\alpha) = \beta$, the Bokstein of $\alpha : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$, if $\alpha \in \mathrm{Ext}(A, B)$.

Let p be a prime. A C-module

$$H^* : C \rightarrow A^*$$

we call p-adic if $H^*(A)$ has no torsion other than p-torsion for any $A \in C$. The p-component of the cohomology of (K, L) is a p-adic C-module; we leave the details to the reader.

Pick any Abelian group G. Let $1_G \in \mathrm{Hom}(G, G)$ be the identity map. We define a C-module $F_G^* : C \rightarrow A^*$ as follows: for any $A \in C$, $F_G^*(A) = \mathrm{Hom}_C(G, A)$; for any $A, B \in C$ and $\alpha \in \mathrm{Hom}_C(A, B)$, $F_G^*(\alpha) : \mathrm{Hom}_C(G, A) \rightarrow \mathrm{Hom}_C(G, B)$ is composition by α. We call F_G^* the free C-module generated by 1_G. Let p be a prime. Let $[F_G^*]_p$ be the p-adic C-module where $[F_G^*]_p(A)$ and $[F_G^*]_p(\alpha)$ are the p-components of $F_G^*(A)$ and $F_G^*(\alpha)$, respectively, for any object A and map α. We call $[F_G{}^*]_p$ the free p-adic C-module generated by 1_G.

If $H^* : C \rightarrow A^*$ is any C-module, and if $\{x_i\}$ are elements of H^*, we say that H^* is freely generated by $\{x_i\}$ (freely generated by $\{x_i\}$ as a p-adic C-module) if H^* is isomorphic to a direct sum of free C-modules (free p-adic C-modules) such that each x_i corresponds to the generator of one summand.

Note that the cohomology of any Moore space is a free C-module on one generator, its fundamental class.

THEOREM 17. *Let (K, L) be a C.W. pair of finite type. Then*

(i) *$H^*(K, L)$ is freely generated as a C-module by a countable set of cohomology elements, and*

(ii) *the p-component of $H^*(K, L)$ is freely generated as a p-adic C-module by a countable set.*

Furthermore, in both cases, we may specify that each generator have coefficients in a cyclic group of infinite or prime power order.

Proof. To prove (i), write $H^*(K, L; Z)$ as a countable direct sum, $\sum_{i=1}^{N} S_i$ for some $N \geqq \infty$, where each S_i has pure degree and is cyclic of infinite or prime power order. For each i, let y_i be a generator of S_i. If y_i has infinite order, let $x_i = y_i$. If y_i has order $m = q^k$, q a prime, pick $x_i \in H^*(K, L; Z_m)$ such that $\beta_m x_i = y_i$, where β_m is the Bokstein of the coefficient sequence $Z \rightarrow Z \rightarrow Z_m$. A simple cohomology argument, which we leave to the reader, shows that $\{x_i\}$ is the desired set of generators. From this set, omit all x_i of order a power of a prime other than p; the set of remaining generators will freely generate the p-component of $H^*(K, L)$, as a p-adic C-module; (ii) is proved.

For example, let P_n be real projective n-space. Then $H^*(P_n)$ is freely generated, as a C-module, by $\{1, u, u^3, u^5, \ldots u^{n-1}\}$ if n is even, and by $\{1, u, u^3, u^5, \ldots u^{n-2}, t\}$ if n is odd, where $1 \in H^0(P_n; Z)$; $u \in H^1(P_n; Z_2)$ is the fundamental class; and $t \in H^n(P_n; Z)$ is the top class, of infinite order, if n is odd.

Another important example of a C-module is twisted cohomology. Again, suppose (K, L) is a C.W. pair, and pick $a \in H^1(K; Z_2)$. For any Abelian group A, let $A[a] = A \otimes Z[a]$, where $Z[a]$ is the twisted integer sheaf over K, twisted by a. Let $H^*(K, L; [a]) : C \rightarrow A^*$ be the C-module where $H^*(K, L; [a])(A) = H^*(K, L; A[a])$ for all A. If $\alpha : A \rightarrow B$ is a map, $H^*(K, L; [a])(\alpha) = (\alpha \otimes 1)_*$; while if $\alpha \in \text{Ext}(A, B)$, $H^*(K, L; [a])[\alpha)$ is the Bokstein of

$$\alpha \otimes 1 : 0 \rightarrow A[a] \rightarrow E[a] \rightarrow B[a] \rightarrow 0.$$

Analogous to Theorem 12, we have:

THEOREM 18. *If (K, L) is a C.W. pair of finite type, and if $a \in H^1(K; Z_2)$, then:*

(i) $H^*(K, L; [a])$ *is freely generated as a C-module by a countable set, and*

(ii) *the p-component of* $H^*(K, L; [a])$ *is freely generated, as a p-adic C-module, by a countable set.*

Furthermore, in both cases, we may assume that all generators have coefficients in cyclic groups of infinite or prime power order.

We leave the proof, analogous to that of Theorem 12, to the reader. As an example, consider twisted cohomology of real projective space, $H^*(P_n; [u])$, which is generated by $\{1, u^2, u^4, \ldots u^{n-1}\}$ if n is odd, and by $\{1, u^2, u^4, \ldots u^{n-2}, t\}$ if n is even; where $1 \in H^0(P_n; Z_2)$, and $t \in H^n(P_n; Z[u])$ is the top class, of infinite order, if n is even.

The generators of the cohomology, or twisted cohomology, of a pair, which are given by Theorem 17, or 18, we call Moore generators.

5. The structure of $H^*(\Lambda^2 X, \Delta X)$ and $H^*(\Lambda^2 X, \Delta X; [m])$ as C-modules.

We shall need a lemma:

LEMMA 19. *If* G^* *and* H^* *are free C-modules and if* $\gamma : G^* \to H^*$ *is a C-module map such that* $\gamma(Z_p) : G^*(Z_p) \to H^*(Z_p)$ *is an isomorphism for each prime* p, *then* γ *is an isomorphism.*

We leave the proof to the reader. The central idea is that if β is the Bokstein of the coefficient sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\delta} C,$$

and if F^* is a free C-module, the following triangle is exact:

$$\begin{array}{ccc} F^*(A) & \xrightarrow{\alpha^*} & F^*(B) \\ & {}_{\beta}\nwarrow \quad \swarrow_{\delta_*} & \\ & F^*(C) & \end{array} .$$

Consider now, as before, X to be any connected C.W. complex of finite type. Let $* \in X$ be a base-point. Now $H^*(X, *)$ is a free C-module; let $\mathscr{X} = \{x_1, x_2, \ldots\}$ be a set of Moore generators, as given by Theorem 17. Using $\mathscr{X}$, one may obtain a complete list of Moore generators for $H^*(\Lambda^2 X, \Delta X)$ and also for $H^*(\Lambda^2 X, \Delta X; [m])$:

THEOREM 20. (I) $H^*(\Lambda^2 X, \Delta X)$ *is freely generated, as a C-module, by all elements of the following types:*

(i) $m^{2i}\Lambda\rho_2 x$ *for all* $x \in \mathscr{X}$ *of order infinite or a power of* 2, $0 \leqq 2i < \dim x$,

(ii) $m^{2i}\Lambda\beta_r{}^2 x$ *for all* $x \in \mathscr{X}$ *of order r, a power of* 2, $0 \leqq 2i \leqq \dim x$,

(iii) Υx *for all* $x \in \mathscr{X}$ *of order a power of* 2 *and of even dimension*,

(iv) $\Lambda x \Lambda\beta_r{}^r x$ *for all* $x \in \mathscr{X}$ *of finite order r, provided r is a power of* 2 *or* $\dim x$ *is odd*,

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(v) $(\Lambda x)^2$ *for all* $x \in \mathscr{X}$ *of odd or infinite order and even dimension*,

(vi) $\Lambda x_i \Lambda x_j$ *if* $x_i, x_j \in \mathscr{X}$ *both have infinite order and* $i < j$,

(vii) $\Lambda x_i \Lambda \rho_r x_j$ *for all* $x_i, x_j \in \mathscr{X}$; *if* x_i *has order* $r < \infty$, x_j, *has order* $s \leqq \infty$, *a multiple of* r; *and* $i < j$ *if* $r = s$,

(viii) $\Lambda x_i \Lambda \beta_s{}^r x_j$ *for all* $x_i, x_j \in \mathscr{X}$ *such that* x_i *has order* $r < \infty$, x_j *has order* $s < \infty$, *a multiple of* r; *and* $i < j$ *if* $r = s$.

(II) $H^*(\Lambda^2 X, \Delta X; [m])$ *is freely generated, as a* C*-module, by all elements of the following types:*

(i) Λx *for all* $x \in \mathscr{X}$,

(ii) $m^{2i+1}\Lambda\rho_2 x$ *for all* $x \in \mathscr{X}$ *of order infinite or a power of* 2, $0 < 2i + 1 <$ dim x,

(iii) $m^{2i+1}\Lambda\beta_r{}^2 x$ *for all* $x \in \mathscr{X}$ *of order* r, *a power of* 2, $0 < 2i + 1 \leqq \dim x$,

(iv) Υx *for all* $x \in \mathscr{X}$ *of order a power of* 2 *and of odd dimension*,

(v) $\Delta(x, \beta_r{}^r x)$ *for all* $x \in \mathscr{X}$ *of finite order* r, *provided* r *is a power of* 2 *or* dim x *is even*,

(vi) $\Delta(x, x)$ *for all* $x \in \mathscr{X}$ *of odd or infinite order and odd dimension*,

(vii) $\Delta(x_i, x_j)$ *for all* $x_i, x_j \in \mathscr{X}$, *both of infinite order, if* $i < j$,

(viii) $\Delta(x_i, \rho_r x_j)$ *for all* $x_i, x_j \in \mathscr{X}$, *if* x_i *has order* $r < \infty$, x_j *has order* $s \leqq \infty$, *a multiple of* r, *and* $i < j$ *if* $r = s$,

(ix) $\Delta(x_i, \beta_s{}^r x_j)$ *for all* $x_i, x_j \in \mathscr{X}$ *such that* x_i *has order* $r < \infty$, x_j *has order* $s \leqq \infty$, *a multiple of* r, *and* $i < j$ *if* $r = s$.

Proof. Let G^* be the formal free C-module generated by the elements specified in the statement of (I), and let $\gamma : G^* \to H^*(\Lambda^2 X, \Delta X)$ be the C-module map which sends each element to itself. We may routinely check, using Theorems 11 and 14, that $\gamma(Z_p)$ is an isomorphism for each prime p. By Lemma 19, (I) is proved. (II) is proved similarly.

6. Projective spaces. For any $n \geqq 1$, let P_n be real projective n-space. Let $* \in P_n$ be a basepoint. Now if n is even, $H^*(P_n, *)$, as a C-module, has only generators of order 2, namely $u, u^3, \ldots u^{n-1}$, i.e., odd powers of the fundamental class $u \in H^1(P_n, *; Z_2)$. Thus, as a C-module, $H^*(\Lambda^2 P_n, \Delta P_n)$ has only Moore generators of order 2, namely $m^{2i}\Lambda u^k$ for all $1 \leqq k \leqq n, 0 \leqq 2i \leqq k$, and $u^{2i+1}\Lambda u^k$ for all $1 \leqq 2i + 1 < k \leqq n$; while $H^*(\Lambda^2 P_n, \Delta P_n; [m])$ has Moore generators Υu^{2i+1} of order 4 for all $1 \leqq 2i + 1 < n$; as well as Λu^{2i+1} for all $1 \leqq 2i + 1 < n$, and $m^{2i+1}\Lambda u^k$ and $\Delta(u^{2i+1}, u^k)$ for all $2 \leqq k \leqq n$, $1 \leqq 2i + 1 < n$, f order 2. Thus,

$$H^*(\Lambda^2 P_n, \Delta P_n; Z) \cong \bigoplus_s Z_2$$

$$H^*(\Lambda^2 P_n, \Delta P_n; Z[m]) \cong \bigoplus_{s-\delta} Z_2 \oplus \bigoplus_{\delta} Z_4$$

where $s = \begin{cases} [\frac{1}{2}k], \text{ if } k \leqq n \\ [n + 1 - \frac{1}{2}k], \text{ if } k > n \end{cases}$ and $\delta = \begin{cases} 1, \text{ if } k \equiv 3 \text{ modulo } 4 \\ 0, \text{ otherwise.} \end{cases}$

Suppose now that n is odd. $H^*(P_n, *)$ has Moore generators $u, u^3, \ldots u^{n-2}$ of

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order 2, and $\tau \in H^n(P_n, *; Z)$, the top class, of infinite order. Thus, by Theorem 20, $H^*(\Lambda^2 P_n, \Delta P_n)$ has only Moore generators of order 2, while $H^*(\Lambda^2 P_n, \Delta P_n; [m])$ has Moore generators $\Lambda\tau$ and $\Delta(\tau, \tau)$ of infinite order, and γu^{2i+1} for all $0 < 2i + 1 < n$, of order 4; the others all have order 2. Hence, for all $1 \leqq k \leqq 2n$, we have:

$$H^k(\Lambda^2 P_n, \Delta P_n; Z) \cong \bigoplus_s Z_2$$

$$H^k(\Lambda^2 P_n, \Delta P_n; Z[m]) \cong \bigoplus_t Z_2 \oplus G$$

where $s = [\frac{1}{2}k]$ if $k \leqq n$, $s = [n + 1 - \frac{1}{2}k]$ if $k > n$, $\delta = 1$ if $k \equiv 3 \pmod 4$, 0 otherwise, and $G = Z_4$ if $k \equiv 3 \pmod 4$, $G = Z$ if $k = n$ or $2n$, and $G = 0$ otherwise; and $t = s - \delta - (-1)^k$ if $k > n$, $t = s - \delta$ if $k \leqq n$.

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