THE STANDARD ERROR OF CHAIN LADDER RESERVE ESTIMATES: RECURSIVE CALCULATION AND INCLUSION OF A TAIL FACTOR

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ABSTRACT

In Mack (1993), a formula for the standard error of chain ladder reserve estimates has been derived. In the present communication, a very intuitive and easily programmable recursive way of calculating the formula is given. Moreover, this recursive way shows how a tail factor can be implemented in the calculation of the standard error.

KEYWORDS

Chain Ladder, Standard Error, Recursive Calculation, Tail Factor

INTRODUCTION

Let $C_{ik}$ denote the cumulative loss amount of accident year $i = 1, ..., n$ at the end of development year (age) $k = 1, ..., n$. The amounts $C_{ik}$ have been observed for $k \leq n + 1 - i$ whereas the other amounts have to be predicted. The chain ladder algorithm consists of the stepwise prediction rule

$$\hat{C}_{i,k+1} = \hat{C}_{ik}f_k$$

starting with $\hat{C}_{i,n+1-i} = C_{i,n+1-i}$. Here, the age-to-age factor $f_k$ is defined by

$$f_k = \frac{\sum_{i=1}^{n-k} w_{ik} C_{ik}^\alpha F_{ik}}{\sum_{i=1}^{n-k} w_{ik} C_{ik}^\alpha}, \quad \alpha \in \{0; 1; 2\},$$

where

$$F_{ik} = C_{i,k+1}/C_{ik}, \quad 1 \leq i \leq n, \quad 1 \leq k \leq n - 1,$$

are the individual development factors and where

$$w_{ik} \in [0; 1]$$
are arbitrary weights which can be used by the actuary to downweight any outlying $F_{ik}$. Normally, $w_{ik} = 1$ for all $i, k$. Then, $\alpha = 1$ gives the historical chain ladder age-to-age factors, $\alpha = 0$ gives the straight average of the observed individual development factors and $\alpha = 2$ is the result of an ordinary regression of $C_{i,k+1}$ against $C_{ik}$ with intercept 0. Note that in case $C_{ik} = 0$, the corresponding two summands should be omitted when calculating $\hat{f}_k$.

The above stepwise rule finally leads to the prediction

$$\hat{C}_{in} = C_{i,n+1} \cdot \hat{f}_{n+1-i} \cdot \cdots \cdot \hat{f}_{n-1}$$

of $C_{in}$ but – because of limited data – the loss development of accident year $i$ does not need to be finished at age $n$. Therefore, the actuary often uses a tail factor $\hat{f}_{ult} > 1$ in order to estimate the ultimate loss amount $C_{i,ult}$ by

$$\hat{C}_{i,ult} = \hat{C}_{in} \hat{f}_{ult}.$$ 

A possible way to arrive at an estimate for the tail factor is a linear extrapolation of $\ln(\hat{f}_k - 1)$ by a straight line $a \cdot k + b$, $a < 0$, together with

$$\hat{f}_{ult} = \prod_{k=n}^{\infty} \hat{f}_k.$$ 

However, the tail factor used must be plausible and, therefore, the final tail factor is the result of the personal assessment of the future development by the actuary.

In Mack (1993), a formula for the standard error of the predictor $\hat{C}_{in}$ was derived for $\alpha = 1$ and all $w_{ik} = 1$. In the next section, this formula is generalized for the cases $\alpha = 0$ or $\alpha = 2$ and $w_{ik} < 1$. Furthermore, a recursive way of calculating the standard error is given. In the last section it is shown how a tail factor can be implemented in the calculation of the standard error.

**Recursive Calculation of the Standard Error**

In order to calculate the standard error of the prediction $\hat{C}_{in}$ as compared to the true loss amount $C_{in}$, Mack (1993) introduced an underlying stochastic model (for $\alpha = 1$ and $w_{ik} = 1$) which is given here in its more general form without the restriction on $\alpha$ and $w_{ik}$:

(C1) \[ E(F_{ik}|C_{i1}, \ldots, C_{ik}) = f_k, \quad 1 \leq i \leq n, \quad 1 \leq k \leq n - 1, \]

(C2) \[ \text{Var}(F_{ik}|C_{i1}, \ldots, C_{ik}) = \frac{\sigma^2}{w_{ik}C_{ik}}, \quad 1 \leq i \leq n, \quad 1 \leq k \leq n - 1, \]

(C3) \[ \text{The accident years} \ (C_{i1}, \ldots, C_{in}), \ 1 \leq i \leq n, \text{are independent.} \]
Within this model, the following statements hold (see Mack (1993)):

\[ E(C_{i,k+1}|C_{i1}, \ldots, C_{ik}) = C_{ik} f_k, \]
\[ E(C_{in}|C_{i1}, \ldots, C_{i,n+1-i}) = C_{i,n+1-i} f_{n+1-i} \cdot \ldots \cdot f_{n-1}, \]
\[ \hat{f}_k \] is the minimum variance unbiased linear estimator of \( f_k \) (for \( w_{ik} \) and \( \alpha \) given),
\[ \hat{f}_{n+1-i} \cdot \ldots \cdot \hat{f}_{n-1} \] is an unbiased estimator of \( f_{n+1-i} \cdot \ldots \cdot f_{n-1}. \)

Therefore, the model CL1-3 can be called underlying the chain ladder algorithm. Furthermore,

\[ \sigma_k^2 = \frac{1}{n-k-1} \sum_{i=1}^{n-k} w_{ik} C_{ik}^\alpha (F_{ik} - \hat{f}_k)^2, \quad 1 \leq k \leq n-2, \]

is an unbiased estimator for \( \sigma_k^2 \) which can be supplemented by

\[ \sigma_{n-1}^2 = \min(\sigma_{n-2}^4/\sigma_{n-3}^2, \min(\sigma_{n-3}^2, \sigma_{n-2}^2)) . \]

Based on this model for \( \alpha = 1 \) and all \( w_{ik} = 1 \), Mack (1993) derived the following formula for the standard error of \( \hat{C}_{in} \), which at the same time is the standard error of the estimate \( \hat{R}_i = \hat{C}_{in} - C_{i,n+1-i} \) for the claims reserve \( R_i = C_{in} - C_{i,n+1-i} \):

\[ (\text{s.e.}(\hat{C}_{in}))^2 = \hat{C}_{in}^2 \sum_{k=n+1-i}^{n-1} \frac{\sigma_k^2}{\hat{f}_k^2} \left( \frac{1}{C_{ik}} + \frac{1}{\sum_{j=1}^{n-k} C_{jk}} \right). \]

This formula can be rewritten as

\[ (*) \quad (\text{s.e.}(\hat{C}_{in}))^2 = \hat{C}_{in}^2 \sum_{k=n+1-i}^{n-1} \left( (\text{s.e.}(F_{ik}))^2 + (\text{s.e.}(\hat{f}_k))^2 \right) / \hat{f}_k^2 \]

where \( (\text{s.e.}(F_{ik}))^2 \) is an estimate of \( \text{Var}(F_{ik}|C_{i1}, \ldots, C_{ik}) \) and \( (\text{s.e.}(\hat{f}_k))^2 \) is an estimate of

\[ \text{Var}(\hat{f}_k) = \sigma_k^2 / \sum_{j=1}^{n-k} w_{jk} C_{jk}^\alpha . \]

In this last form, formula \((*)\) also holds for \( \alpha = 0 \) and \( \alpha = 2 \) and any \( w_{ik} \in [0; 1] \) as can be seen by applying the proof for \( \alpha = 0 \) and \( w_{ik} = 1 \) analogously. Moreover, from this proof the following easily programmable recursion can be gathered:

\[ (\text{s.e.}(\hat{C}_{i,k+1}))^2 = \hat{C}_{ik}^2 \left( (\text{s.e.}(F_{ik}))^2 + (\text{s.e.}(\hat{f}_k))^2 \right) + (\text{s.e.}(\hat{C}_{ik}))^2 \hat{f}_k^2 \]
with starting value \( \text{s.e.}(\hat{C}_{i,n+1-i}) = 0 \). This recursion, which leads to formula (*), is very intuitive: \( (\text{s.e.}(F_{ik}))^2 \) estimates the (squared) random error \( \text{Var}(F_{ik}) = E(F_{ik} - \hat{f}_k)^2 \), i.e. the mean squared deviation of an individual \( F_{ik} \) from its true mean \( \hat{f}_k \), and \( (\text{s.e.}(\hat{f}_k))^2 \) estimates the (squared) estimation error \( \text{Var}(\hat{f}_k) = E(\hat{f}_k - f_k)^2 \), i.e. the mean squared deviation of the estimated average \( \hat{f}_k \) of the \( F_{ik} \), \( 1 \leq i \leq n \), from the true \( f_k \). From this interpretation it is clear that we have \( \text{Var}(f_k) < \text{Var}(F_{ik}) \) if \( f_k \) is unbiased and accident year \( i \) belongs to those years over which \( f_k \) is the average.

**Inclusion of a Tail Factor**

The recursion can immediately be extended to include a tail factor \( \hat{f}_{ult} \):

\[
(\text{s.e.}(\hat{C}_{i,ult}))^2 = \hat{C}_{in}^2 \left( (\text{s.e.}(F_{i,ult}))^2 + (\text{s.e.}(\hat{f}_{ult}))^2 \right) + (\text{s.e.}(\hat{C}_{in}))^2 \hat{f}_{ult}^2
\]

and an actuary who develops an estimate for \( f_{ult} \) should also be able to develop an estimate \( \text{s.e.}(\hat{f}_{ult}) \) for its estimation error \( \sqrt{\text{Var}(\hat{f}_{ult})} \) (How far will \( \hat{f}_{ult} \) deviate from \( f_{ult} \)?) and an estimate \( \text{s.e.}(F_{i,ult}) \) for the corresponding random error \( \sqrt{\text{Var}(F_{i,ult})} \) (How far will any individual \( F_{i,ult} \) deviate from \( f_{ult} \) on average?). Note that at \( F_{ik}, f_k \) and \( \sigma_k \), index \( k = ult \) is the same as \( k = n \) whereas at \( C_{ik} \) we have \( ult = n + 1 \).

As a plausibility consideration, we will usually be able to find an index \( k < n \) with

\[
\hat{f}_{k-1} > \hat{f}_{ult} > \hat{f}_k.
\]

Then we can check whether it is reasonable to assume that the inequalities

\[
\text{s.e.}(\hat{f}_{k-1}) > \text{s.e.}(\hat{f}_{ult}) > \text{s.e.}(\hat{f}_k)
\]

and

\[
\text{s.e.}(F_{i,k-1}) > \text{s.e.}(F_{i,ult}) > \text{s.e.}(F_{i,k})
\]

hold, too, or whether there are reasons to fix \( \text{s.e.}(\hat{f}_{ult}) \) and/or \( \text{s.e.}(F_{i,ult}) \) outside these inequalities.

As an example, we take the data of Table 4 from Mack (1993). From these (using \( \alpha = 1 \) and all \( w_{ik} = 1 \)), we get the results given in Table 1 for \( k = 1, ..., 8 \):
The parameter estimates \( \hat{f}_k \) and \( \hat{\sigma}_k \) for \( 1 \leq k \leq 8 \) are the same as in Mack (1993). From these, the estimates \( \text{s.e.}(\hat{f}_k) = \hat{\sigma}_k \sqrt{\sum_{j=1}^{n-k} C_{jk}} \) and s.e. \( (F_{ik}) = \hat{\sigma}_k / \sqrt{C_{ik}} \) for \( k \leq n + 1 - i \) or s.e. \( (F_{ik}) = \hat{\sigma}_k / \sqrt{C_{ik}} \) for \( k > n + 1 - i \) are calculated which give the estimation error and the random error, respectively. Note that the random error s.e.\( (F_{ik}) \) varies also over the accident years because model assumption CL2 states that for \( \alpha = 1 \) the variance of the individual development factor \( F_{ik} \) is the smaller the greater the previous claims amount (volume) \( C_{ik} \) is. Therefore, only the values of s.e.\( (F_{ik}) \) for accident year \( i = 3 \) of average volume are given. The last column of Table 1 shows a possible tail estimation by the actuary: He expects a tail factor of 1.05 with an estimation error of ±0.02 and a random error of ±0.03 for accident year \( i = 3 \). From this, the estimate \( \hat{\sigma}_{ult} = \text{s.e.}(F_{3,ult}) \sqrt{C_{3,n}} = 71.0 \) has been deduced and is used to calculate s.e.\( (F_{i,ult}) \) for the other accident years. These tail estimates fit well between the columns \( k = 6 \) and \( k = 7 \). (Note that the extrapolated estimate for \( \sigma_8 \) leads to a rather small s.e.\( (F_{3,ult}) \) as compared to s.e.\( (F_8) \). This is due to the fact that \( f_8 \) does not follow a loglinear decay as it was assumed for the calculation of \( \sigma_8 \). Therefore, an estimate \( \hat{\sigma}_8 \approx 30 \) would have been more reasonable.)

Table 2 shows the resulting estimates for the ultimate claims amounts. The rows \( \hat{C}_{i,9} \) and s.e.\( (\hat{C}_{i,9}) \) are identical to the results given in Mack (1993). Row \( \hat{C}_{i,ult} \) is 5% higher than row \( \hat{C}_{i,9} \) and the last row s.e.\( (\hat{C}_{i,ult}) \) shows the standard errors which result from the formula given above.
Finally, we give a recursive formula for the total reserve of all accident years together:

\[
\left( \text{s.e.} \left( \sum_{i=n+1-k}^{n} \hat{C}_{i,k+1} \right) \right)^2 = \left( \text{s.e.} \left( \sum_{i=n+1-k}^{n} \hat{C}_{ik} \right) \right)^2 \cdot \hat{f}_k^2 + \\
+ \sum_{i=n+1-k}^{n} \hat{C}_{ik} \cdot \left( \text{s.e.}(F_{ik}) \right)^2 + \left( \sum_{i=n+1-k}^{n} \hat{C}_{ik} \right)^2 \cdot \left( \text{s.e.}(\hat{f}_k) \right)^2
\]

starting at \( k = 1 \). This formula can also be gathered from the proof of the corollary to Theorem 3 in Mack (1993). In the above example, this formula yields

\[
\text{s.e.} \left( \sum_{i=1}^{9} \hat{C}_{i,\text{ult}} \right) = 4054
\]

as standard error of the ultimate total claims amount \( \sum_{i=1}^{9} \hat{C}_{i,\text{ult}} = 48906 \) (amounts in 1000s).

**REFERENCE**


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