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OSCILLATION CONDITIONS IN SCALAR LINEAR DELAY DIFFERENTIAL EQUATIONS

ISTVÁN GYŐRI

Sufficient conditions are obtained for all solutions of a general scalar linear functional differential equation to be oscillatory. Our main theorem concerns some particular cases of a conjecture of Hunt and Yorke.

Hunt and Yorke in [2] investigated the question of when do all non-zero solutions (for $t \in R_+ = (0,\infty)$) of the scalar differential delay equation

$$\dot{x}(t) = -\sum_{i=1}^{n} q_i(t)\, x(t-T_i(t)), \quad t \geq 0 , \tag{1}$$

oscillate about 0, which means that they have infinitely many zeros on R_+, where $T_i, q_i : R_+ \longrightarrow R_+$, $(i = \overline{1,n})$, are continuous functions. In their paper an example is given from which they made the following

CONJECTURE. If there are constants q_0 and r for which $0 \leq q_i(t) \leq q_0$ and $0 \leq T_i(t) \leq r$ for $i = \overline{1,n}$ and $t \geq 0$, then

$$\inf_{\lambda, t \in (0,\infty)} \frac{1}{\lambda} \sum_{i=1}^{n} q_i(t) e^{\lambda T_i(t)} > 1 \tag{2}$$

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implies that all solutions to (1) oscillate on R_+ .

In this paper we will prove this conjecture in particular cases starting from the investigation of the more general scalar linear functional differential equation

$$\dot{x}(t) = -\ell(t,x_t) , \quad t \geq 0 , \tag{3}$$

using a technique in the proofs which proved effective in examination similar questions in [1, 3, 4].

Here

$$\ell : R_+ \times C([-r,o], R) \longrightarrow R, \quad (R=(-\infty,\infty), \quad o < r < \infty) ,$$

is a given function and for any $x \in C([-r,\infty), R)$,

$$x_t(s) = x(t+s), \quad -r \leq s \leq 0, \quad t \geq 0 .$$

In our investigations we use the following hypotheses:

(H_1) $\ell : R_+ \times C([-r,0] , R) \longrightarrow R$ is continuous and for any fixed $t \in R_+, \ell(t,.)$ is a linear and bounded functional on $C([-r,0] , R)$;

(H_2) there exists $0 < r_o < r$ and a continuous function $a : R_+ \longrightarrow R_+$ such that for any large enough t ,

$$\ell(t,\phi) \geq a(t) \min_{-r \leq s \leq -r_o} \phi(s), \ (t,\phi) \in C([-r,0], R_+) \tag{4}$$

and

$$\liminf_{t \to +\infty} a(t) > 0. \tag{5}$$

LEMMA. *Suppose that* (H_1) *and* (H_2) *hold and the continuous function* $\alpha : [T-r,\infty) \longrightarrow R_+$, $(0 \leq T < \infty)$, *satisfies the following:*

$$\alpha(t) = \ell(t, \exp(\int_{t+\cdot}^{t} \alpha(u)du)), \quad t \geq T. \tag{6}$$

Then

$$\liminf_{t \longrightarrow +\infty} \alpha(t) < \infty . \tag{7}$$

Proof. From (4) and (6), we have

$$\alpha(t) \geq a(t) \min_{-r \leq s \leq -r_o} \exp\left(\int_{t+s}^{t} \alpha(u)du\right), \quad t \geq T ,$$

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that is,

$$\alpha(t) \geq a(t) \exp\left(\int_{t-r_o}^{t} \alpha(u)du\right), \quad t \geq T .$$

But, from (5), we have

$$\liminf_{t \longrightarrow +\infty} \int_{t-r_o}^{t} a(u)du > 0 ,$$

and thus we can apply Lemma 2.1. of [4] , which implies

$$\liminf_{t \longrightarrow +\infty} \int_{t-r_o}^{t} \alpha(u)du < \infty .$$

From this (7) immediately follows, thus the proof of the lemma is finished.

Our main result is the following

THEOREM. *If* (H_1) *and* (H_2) *hold and*

$$\inf_{\lambda, t \in (o,\infty)} \frac{1}{\lambda} \ell(t, exp(-\lambda.)) > 1 \tag{8}$$

then all nonzero solutions to (3) oscillate on R_+.

Proof. We prove our statement indirectly, that is, we suppose that under the condition (8) there exists a solution x of (3) which does not oscillate, for instance, $x(t) > 0$ on some interval $[T-r,\infty) \subset [-r,\infty)$. Then x can be written in the form

$$x(t) = x_o \exp\left(-\int_{T-r}^{t} \alpha(u)du\right), \quad t \geq T,$$

where $x_o = x(T-r) > 0$ and

$$\alpha(t) = - \frac{\dot{x}(t)}{x(t)} = \frac{\ell(t,x_t)}{x(t)}$$

is a continuous and nonnegative function, since x satisfies (3) and (4) holds.

Again, with the aid of equation (3) we obtain that $\alpha(t)$ is a solution of

$$\alpha(t) \exp\left(-\int_{T-r}^{t} \alpha(u)du\right) = \ell\left(t, \exp\left(-\int_{T-r}^{t+\cdot} \alpha(u)du\right)\right), \quad t \geq T ,$$

that is, the nonnegative and continuous function $\alpha(t)$ satisfies

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equation (6).

On the other hand our Lemma implies that (7) is satisfied by this α. Thus we can distinguish between the following cases:

$$\text{(i)} \quad 0 < \lambda_0 = \liminf_{t \to +\infty} \alpha(t) < \infty .$$

Then for any ε such that $0 < \varepsilon < \lambda_0$ there exists a $t_1 = t_1(\varepsilon) \geq T$ such that

$$\alpha(t) \geq \lambda_0 - \varepsilon \qquad t \geq t_1 - r .$$

Thus, from (7) and (8), we have

$$\alpha(t) \geq \ell(t, \exp(-(\lambda_0 - \varepsilon).)) > \lambda_0 - \varepsilon \quad t > t_1 ,$$

which yields the following contradiction

$$\lambda_0 = \liminf_{t \to +\infty} \alpha(t) > \lambda_0 - \varepsilon .$$

$$\text{(ii)} \quad \lambda_0 = \liminf_{t \to +\infty} \alpha(t) = 0 .$$

Then (7) implies that

$$\alpha(t) \geq \ell(t,1), \qquad t \geq T ,$$

and thus

$$0 = \liminf_{t \to +\infty} \alpha(t) \geq \liminf_{t \to +\infty} \ell(t,1) .$$

But this is also a contradiction to (8) since from (4) and (5) we have that with a suitable constant $c > 0$ the inequality

$$\exp(\lambda r)\ \ell(t,1) \geq \ell(t, \exp(-\lambda.)) > (1+c)\lambda ,$$

holds, for any $t > 0$ and $\lambda > 0$, therefore

$$\liminf_{t \to +\infty} \ell(t,1) > 0 .$$

Summarizing our observations we see that from the existence of a solution of (3) which is not oscillatory on R_+, we have a contradiction, which means that all nontrivial solutions of (3) oscillate on R_+.

COROLLARY 1. *Suppose that* $q_i : R_+ \longrightarrow R_+$, $(i = \overline{1,n})$, *and* $T_i : R_+ \longrightarrow [0,r]$, $(i = \overline{1,n})$, *are continuous functions, moreover there are some indices* $i_\ell \in \{1,2,\ldots,n\}$, $1 \leq \ell \leq m$, *such that*

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$$\liminf_{t \to +\infty} T_{i_\ell}(t) \geq r_o > 0 \,, \tag{9}$$

and

$$\liminf_{t \to +\infty} \sum_{j=1}^{m} q_{i_j}(t) > 0 \,. \tag{10}$$

Then from the condition (2) it follows that all solutions of (1) oscillate on R_+ .

Proof. Let us define $\ell : R_+ \times C([-r,0],R) \to R$ by

$$\ell(t,\phi) = \sum_{i=1}^{n} q_i(t)\phi(-T_i(t)) \,,$$

for any $(t,\phi) \in R_+ \times C([-r,0],R)$.

Then it is self-evident that ℓ satisfies the hypothesis (H_1) and for any $\phi \in C([-r,0], R_+)$,

$$\ell(t,\phi) = \sum_{i=1}^{n} q_i(t)\phi(-T_i(t) \geq \sum_{j=1}^{m} q_{i_j}(t)) \min_{-r \leq s \leq -r_o} \phi(s) \,,$$

where the function

$$a(t) = \sum_{j=1}^{m} q_{i_j}(t)$$

satisfies (10) and its equivalent (5).

Thus, the hypothesis (H_2) is also true for this ℓ .

From condition (2), we have

$$\inf_{\lambda, t \in (o,\infty)} \ell(t, e^{-\lambda \cdot}) = \inf_{\lambda, t \in (o,\infty)} \sum_{i=1}^{n} q_i(t)\, e^{\lambda T_i(t)} > 1 \,,$$

that is, for the function ℓ condition (8) is satisfied. Thus, with this definition of ℓ , all conditions of the Theorem are satisfied which implies the statement of Corollary 1.

From Corollary 1, we get the following statements proving that the conjecture of Hunt and Yorke is true in some particular cases. But these results do not answer the question as to whether this conjecture is valid in its original form.

PROPOSITION 1. *Suppose that* $q_i : R_+ \longrightarrow R_+$, $(i = \overline{1,n})$, *are continuous functions,* $0 < T_i < r$, $(i = \overline{1,n})$, *are real numbers and*

$$\inf_{(t,\lambda)\in(0,\infty)} \frac{1}{\lambda} \sum_{i=1}^{n} q_i(t) e^{\lambda T_i} > 1 . \tag{11}$$

Then all solutions of

$$\dot{x}(t) = - \sum_{i=1}^{n} q_i(t) \;\; x(t-T_i), \qquad t \ge 0 , \tag{12}$$

must oscillate on R_+ .

Proof. From (11) we have that there exists a constant $c > 0$ and a $t_1 \ge 0$ such that

$$\sum_{i=1}^{n} q_i(t) . e^{\lambda T_i} > (1+c)\lambda , \qquad t \ge t_1 ,$$

that is

$$e^{\lambda r} \sum_{i=1}^{n} q_i(t) > (1+c)\lambda , \qquad t \ge t_1 ,$$

which implies

$$\liminf_{t \to +\infty} \sum_{i=1}^{n} q_i(t) > 0 .$$

Thus, choosing for the r_0 and $\{i_j\}_{j=1}^{m}$ in Corollary 1.

$$r_0 = \min_{1\le i\le n} T_i > 0 ,$$

$i_j = j$, $j = \overline{1,n}$, and $m = n$, we have that all conditions of Corollary 1 are satisfied. Thus the statement of Proposition 1 is a simple consequence of Corollary 1.

Remark 1. In the case that the delays T_i are constants, Proposition 1 is more general then the conjecture, since we need not use the boundary conditions for the coefficients $q_i(t)$.

PROPOSITION 2. *Suppose that* $q_i : R_+ \to R_+$, $i = 1,2$, *and* $T_2 : R_+ \to [0,r]$ *are continuous functions,*

$$m_1 = \sup_{t\geq 0} q_1(t) < \infty \tag{13}$$

and

$$\inf_{\lambda, t\in(0,\infty)} \frac{1}{\lambda}[q_1(t)+q_2(t)\, \exp(\lambda T_2(t))] > 1 . \tag{14}$$

Then all solutions of

$$\dot{x}(t) = -q_1(t)\, x(t)-q_2(t)\, x(t-T_2(t)), \quad t \geq 0 , \tag{15}$$

must oscillate on R_+.

Proof. This proposition is a simple consequence of Proposition 1 if we show that under the condition (14) the following are satisfied:

$$r_0 = \liminf_{t\to+\infty} T_2(t) > 0 , \tag{16}$$

and

$$\liminf_{t\to+\infty} q_2(t) > 0 . \tag{17}$$

From (14), we have the existence of a constant $c > 0$ such that

$$q_1(t) + q_2(t)\, \exp(\lambda T_2(t)) > (1+c)\lambda , \tag{18}$$

for any $0 < \lambda , t < \infty$

Then, from (13), we have

$$m_1 + q_2(t)\, \exp(\lambda r) > (1+c)\lambda ,$$

that is,

$$q_2(t) > [(1+c)\lambda-m_1]\, \exp(-\lambda r) > 0 , \quad t \geq 0 ,$$

for any λ which is large enough. This means that (17) is valid for q_2.

Let us denote

$$m_2 = \inf_{t\geq 0} q_2(t) \quad (>0) ,$$

and

$$k = (1+d+\frac{m_1}{m_2})\, \frac{1}{1+c} ;$$

where $d > 0$ is an arbitrary real number.

Then for the function

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$$\lambda(t) = k\cdot q_2(t)\ , \qquad t \ge 0\ ,$$

we have from (18) that

$$q_1(t)+q_2(t)\exp(k\, q_2(t)\ T_2(t)) > (1+c)k\ q_2(t), \quad t \ge 0\ .$$

Thus,

$$\exp(k\cdot q_2(t)\ T_2(t)) \ge (1+c)k - \frac{q_1(t)}{q_2(t)} \ge (1+c)k - \frac{m_1}{m_2}\ .$$

that is, from the definition of k, we have

$$k\cdot q_2(t)\ T_2(t) \ge \ln(1+d)\ .$$

But, using the definition of m_2 we have

$$T_2(t) \ge \ln(1+d)\frac{1}{k\ m_2} > 0\ ,$$

that is, (17) is valid. Therefore the proof of this proposition is complete.

Remark 2. Proposition 2 is more general than the conjecture when $n = 2$, since we need not assume that $q_2(t)$ is a bounded function. The importance of the condition $q_1(t)$ bounded on $(0,\infty)$ was suggested by example of Hunt and Yorke in [2]. But the following statement shows that it is not necessary to suppose the boundedness of $q_1(t)$ either, provided we assume that

$$m_2 = \liminf_{t\to+\infty} q_2(t) > 0\ . \tag{19}$$

PROPOSITION 3. *If $q_i:R_+ \longrightarrow R_+$, $i = 1,2$, and $T_2:R_+ \longrightarrow [0,r]$ are continuous functions and (14) and (19) are satisfied, then all solutions of (15) oscillate on R_+.*

Proof. Using Corollary 1 the statement will be proved by showing that (16) holds. But using the same technique as in the proof of Proposition 2, (18) implies (16).

Remark 3. Now we consider the example of Hunt and Yorke, that is, the case when

$$q_1(t) = a(g(t)-1)$$

and

$$q_2(t) = a \exp(-g(t)) ,$$

where

$$g(t) = e^{bt}(1-b)$$

and $b > 0$ is a constant such that $a = b/(1-\exp(-b)) > 1$. Then the equation

$$\dot{x}(t) = -q_1(t)\, x(t) - q_2(t)\, x(t-1), \quad t \geq 0 , \tag{20}$$

has the solution

$$x(t) = \exp(-\exp(bt) ,$$

which is not oscillatory. Thus, comparing this example with Proposition 2 and Proposition 3, we recognize that the reason for the existence of a nonoscillatory solution of (20) is that in this case both of the conditions

$$\liminf_{t \to +\infty} q_2(t) = a \lim_{t \to +\infty} \exp(-g(t)) = 0$$

and

$$\lim_{t \to +\infty} q_1(t) = a \lim_{t \to +\infty} (g(t)-1) = +\infty$$

are satisfied.

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Computing Centre of the Szeged
University of Medicine
6720 Szeged
Pecsi u. 4/a,
Hungary.