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A NOTE ON LACUNARY POWER SERIES WITH RATIONAL COEFFICIENTS

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Abstract

In this note, we prove that for any $\nu > 0$, there is no lacunary entire function $f(z) \in \mathbb{Q}[[z]]$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and $\operatorname{den} f(p/q) \ll q^{\nu}$, for all sufficiently large q.

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1. Introduction

A real number ξ is called a *Liouville number* if there exists a rational sequence $(p_k/q_k)_{k\geq 1}$, with $q_k > 1$, such that

$$0 < \left|\xi - \frac{p_k}{q_k}\right| < q_k^{-k}, \quad \text{for } k = 1, 2, \dots .$$

In his pioneering book, Maillet [3, Chapitre III] proved that the set of Liouville numbers is preserved under rational functions with rational coefficients. Based on this result, Mahler [2] posed the following question.

Question 1.1. Are there transcendental entire functions $f(z)$ such that if ξ is any Liouville number then so is $f(\xi)$?

Very recently, the first two authors and others (see [4–6]) constructed large classes of Liouville numbers which are mapped into Liouville numbers by transcendental entire functions. Marques and Moreira [4], for example, showed the existence of transcendental entire functions f such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and $\operatorname{den}(f(p/q)) < q^{8q^2}$, for all $p/q \in \mathbb{Q}$, with $q > 1$ (where $\operatorname{den}(z)$ denotes the denominator of the rational number z). The proof suggests that Mahler's question has an affirmative answer if the answer to the following question is also 'yes' (see also [6, Theorem 2.1]).

Question 1.2. Are there transcendental entire functions $f(z)$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and

$$\operatorname{den} f(p/q) \ll q^{\nu},$$

for some fixed $\nu > 0$ and for all sufficiently large q?

Throughout the paper, the implied constants in $\ll$ depend only on f.

Recall that a power series $\sum_{k\geq 0} a_k z^k$ is called *lacunary* if there exist two sequences of integers $\{s_1, s_2, \ldots\}$ and $\{t_0, t_1, t_2, \ldots\}$ such that $0 = t_0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots$, $a_{s_n} a_{t_n} \neq 0$, $a_k = 0$ if $s_n < k < t_n$ and $\lim_{n\to\infty}(t_n - s_n) = \infty$. Lacunary entire functions are transcendental (since they are not polynomials). In 1965, Mahler [1] studied some arithmetic properties of lacunary power series with integral coefficients.

In this note, we shall prove that the answer to the previous question is 'no' for lacunary power series with rational coefficients. More precisely, we have the following result.

Theorem 1.3. *For any positive real number ν, there is no lacunary entire function $f(z) \in \mathbb{Q}[[z]]$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and*

$$\operatorname{den} f(p/q) \ll q^{\nu},$$

for all sufficiently large q.

2. The proof

Suppose, towards a contradiction, that for some $\nu > 0$, there exists such a function, say $f(z) = \sum_{k\geq 0} a_k z^k \in \mathbb{Q}[[z]]$. Let $(t_n)_n$ and $(s_n)_n$ be the sequences defining the gaps in the lacunary power series (according to the definition given above). In particular, there exists a positive integer N such that $t_N - s_N > \nu + 1$.

By setting $f_N(z) = \sum_{k=0}^{s_N} a_k z^k$, we claim that $f_N(1/q) \neq f(1/q)$, for infinitely many integers $q \geq 2$. In fact, on the contrary,

$$0 = q^{t_N}(f(1/q) - f_N(1/q)) = a_{t_N} + \frac{a_{t_N+1}}{q} + \frac{a_{t_N+2}}{q^2} + \cdots,$$

for all sufficiently large q. However, this implies that $a_{t_N} = 0$ (by passing to the limit when q tends to infinity and using the uniform convergence), which is a contradiction (since $a_{s_N} a_{t_N} \neq 0$). Let q be such an integer. Note that $f_N(1/q) = A_{N,q}/cq^{s_N} \in \mathbb{Q}$, where $c = \operatorname{den}(a_0) \cdots \operatorname{den}(a_{s_N})$ and $A_{N,q} \in \mathbb{Z}$. Since $f(1/q)$ is a rational number,

$$|f(1/q) - f_N(1/q)| \geq \frac{1}{c \operatorname{den}(f(1/q)) q^{s_N}}. \tag{2.1}$$

On the other hand, since $q \geq 2$,

$$\begin{aligned} |f(1/q) - f_N(1/q)| &\leq \frac{1}{q^{t_N}}\Big(|a_{t_N}| + \frac{|a_{t_N+1}|}{q} + \frac{|a_{t_N+2}|}{q^2} + \cdots\Big) \\ &\ll \frac{1}{q^{t_N}}. \end{aligned} \tag{2.2}$$

Here we observe that $\sum_{k=0}^{\infty} a_{t_N+k} z^k = (f(z) - f_N(z))/z^{t_N}$ is also an entire function.

By combining (2.1), (2.2) and using the assumption about f,

$$q^{\nu+1} < q^{t_N - s_N} \ll \operatorname{den} f(1/q) \ll q^{\nu},$$

for infinitely many integers $q \geq 2$, leading to a contradiction. The proof is then complete. □

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REMARK 2.1. We finish by pointing out that with a very similar proof we can prove that there is no transcendental entire function $f(z) \in \mathbb{Q}[[z]]$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and

$$\operatorname{den} f(p/q) = o(q).$$

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