Bull. Aust. Math. Soc. **89** (2014), 494–499
doi:10.1017/S0004972713000889

ON A QUESTION OF J. M. RASSIAS

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(Received 1 August 2013; accepted 6 August 2013; first published online 27 September 2013)

Abstract

Answering a problem posed by John Michael Rassias, we study the functional inequality

$$f(x + y + xy) \le f(x) + f(y) + f(xy),$$

with real unknown mapping f.

2010 *Mathematics subject classification*: primary 39B62; secondary 26A51, 26Dxx.
Keywords and phrases: functional inequality, subadditive mappings, Dini derivatives.

1. Introduction

In a recent paper [1] the authors studied the functional equation

$$f(x + y + xy) = f(x) + f(y) + f(xy),$$

together with its Hyers–Ulam stability. In connection with this research, John Michael Rassias in a personal communication asked about real solutions of the related functional inequality

$$f(x + y + xy) \le f(x) + f(y) + f(xy). \tag{1.1}$$

The purpose of the present note is to answer his question.

2. Main results

We begin with some elementary observations. A map $f\colon \mathbb{R} \to \mathbb{R}$ is called subadditive if it satisfies the functional inequality

$$f(x + y) \le f(x) + f(y),$$

for all $x, y \in \mathbb{R}$. For a detailed discussion of the notion of subadditive mappings the reader is referred to Kuczma [4, Ch. 16]. It is straightforward to notice that every subadditive mapping is a solution of (1.1). However, the example below shows that the converse is not true.

The research of the author was supported by the Polish Ministry of Science and Higher Education in the years 2013–2014, under Project No. IP2012 011072.

EXAMPLE 2.1. Define $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 3 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

It is easy to note that, for every $x, y \in \mathbb{R}$,

$$f(x + y + xy) \leq 3 = 1 + 1 + 1 \leq f(x) + f(y) + f(xy).$$

Therefore, f is a solution of (1.1). On the other hand, to see that f is not a subadditive function, take $x = \sqrt{2}$ and $y = -\sqrt{2}$ and observe that

$$f(\sqrt{2} + (-\sqrt{2})) = f(0) = 3 > 2 = 1 + 1 = f(\sqrt{2}) + f(-\sqrt{2}).$$

However, it is easy to observe that, for every solution f of (1.1),

$$f(0) = f(0 + 0 + 0) \leq 3f(0);$$

therefore $f(0) \geq 0$.

In view of the foregoing example, in our subsequent studies we will assume additionally that f possesses some smoothness around zero and $f(0) = 0$.

We will employ the Dini derivatives of f. Assume that $I \subseteq \mathbb{R}$ is an open interval. For an arbitrary mapping $f : I \to \mathbb{R}$, the Dini derivatives are defined as follows:

$$D^{\pm} f(x) = \limsup_{h \to 0\pm} \frac{f(x + h) - f(x)}{h}$$

and

$$D_{\pm} f(x) = \liminf_{h \to 0\pm} \frac{f(x + h) - f(x)}{h},$$

for every $x \in I$. It is clear that the Dini derivatives can attain infinite values. Therefore, in each inequality which involves Dini derivatives it is to be understood that it is valid provided that both sides are meaningful (that is, no indefinite expression of the form $\infty - \infty$ or ∞/∞ appears).

We will prove the following lemma.

LEMMA 2.2. *Assume that I is a nonvoid open interval containing zero and $f : I \to \mathbb{R}$ is differentiable at zero, satisfies $f(0) = 0$ and solves (1.1) for all $x, y \in I$ such that $x + y + xy \in I$. Then the following estimate holds true:*

$$x \in I \setminus \{-1\} \implies D^{+} f(x) \leq f'(0) \leq D_{-} f(x). \tag{2.1}$$

PROOF. Fix $x, y \in I$ such that $x + y + xy \in I$ and $y > 0$ and rearrange (1.1) in the following way:

$$f(x + (1 + x)y) - f(x) \leq f(y) + f(xy)$$

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and

$$(1+x)\frac{f(x+(1+x)y)-f(x)}{(1+x)y} \le \frac{f(y)}{y} + x\frac{f(xy)}{xy}. \tag{2.2}$$

Let us assume that $x > -1$ and pick a sequence $(y_n)_{n\in\mathbb{N}}$ (possibly depending upon x) of positive elements of I tending to zero and for which the equality

$$\lim_{n\to+\infty} \frac{f(x+(1+x)y_n)-f(x)}{(1+x)y_n} = D^+ f(x)$$

holds true. Next, replace y by y_n in (2.2) and let $n \to +\infty$. Note that, since $y_n \to 0$, the condition $x + (1+x)y_n \in I$ will be fulfilled for every n large enough. Using the assumptions that $f(0) = 0$ and $f'(0)$ exists, we arrive at

$$(1+x)D^+ f(x) \le (1+x)f'(0),$$

that is, $D^+ f(x) \le f'(0)$ for all $x \in I$ such that $x > -1$.

Similarly, if $x < -1$, then we choose a sequence $(y_n)_{n\in\mathbb{N}}$ of positive real numbers tending to zero such that

$$\lim_{n\to+\infty} \frac{f(x+(1+x)y_n)-f(x)}{(1+x)y_n} = D_- f(x).$$

Taking the limit in (2.2),

$$(1+x)D_- f(x) \le (1+x)f'(0),$$

that is, $D_- f(x) \ge f'(0)$ for all $x \in I$ such that $x < -1$.

Next, for $x, y \in I$ such that $x + y + xy \in I$ and $y < 0$, we derive from (1.1) the inequality

$$(1+x)\frac{f(x+(1+x)y)-f(x)}{(1+x)y} \ge \frac{f(y)}{y} + x\frac{f(xy)}{xy}. \tag{2.3}$$

Assume that $x > -1$ and choose a sequence $(y_n)_{n\in\mathbb{N}}$ of negative elements of I tending to zero and for which the equality

$$\lim_{n\to+\infty} \frac{f(x+(1+x)y_n)-f(x)}{(1+x)y_n} = D_- f(x)$$

is satisfied. Taking the limit in (2.3) leads us to the inequality $D_- f(x) \ge f'(0)$ for all $x \in I$ such that $x > -1$.

For $x < -1$, we take a sequence $(y_n)_{n\in\mathbb{N}}$ of negative real numbers tending to zero such that

$$\lim_{n\to+\infty} \frac{f(x+(1+x)y_n)-f(x)}{(1+x)y_n} = D^+ f(x).$$

Taking the limit in (2.3), we obtain the inequality $D^+ f(x) \le f'(0)$ for all $x \in I$ such that $x < -1$. Therefore, the estimate (2.1) is proved. $\square$

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We have the following corollary.

COROLLARY 2.3. *Assume that I is a nonvoid open interval containing zero and $f : I \to \mathbb{R}$ is a mapping which is differentiable at zero, satisfies $f(0) = 0$ and solves (1.1) for all $x, y \in I$ such that $x + y + xy \in I$. Then, the following conditions are equivalent:*

(i) *f is convex;*
(ii) *f is differentiable on I;*
(iii) *f is of the form*
$$f(x) = f'(0) \cdot x \quad \text{for all } x \in I.$$

PROOF. We prove the implication (i) $\Rightarrow$ (ii). A well-known property of convex functions (see, for example, [4, Theorem 7.4.1]) guarantees that at every point $x \in I$ there exist the left derivative $f'_-(x)$ and the right derivative $f'_+(x)$ of f, and moreover, we have $f'_-(x) \le f'_+(x)$. This jointly with Lemma 2.2 implies that $f(x) = f'(0)x$ for every $x \in I \setminus \{-1\}$. Using the convexity of f once more we get that also $f(-1) = f'(0)(-1)$ in case $-1 \in I$.

Obviously (iii) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). Moreover, directly from Lemma 2.2 we obtain the implication (ii) $\Rightarrow$ (iii). □

The assumptions imposed upon f in the foregoing corollary seem to be fairly strong. However, the following example shows that one cannot drop them.

EXAMPLE 2.4. Let $a > -1$ be an arbitrary constant. Define a function $f : \mathbb{R} \to \mathbb{R}$ by the formula
$$f(x) = \begin{cases} x & \text{if } x \neq -1, \\ a & \text{if } x = -1. \end{cases}$$

One can check that f is a solution of (1.1) and f satisfies all the assumptions of Lemma 2.2. Indeed, note that the equality $x + y + xy = -1$ holds precisely if $x = -1$ or $y = -1$. Therefore, it is enough to verify (1.1) in the cases $x = -1$ and $y = -1$. And it is straightforward to see that this is equivalent to $f(x) + f(-x) \ge 0$ for all $x \in \mathbb{R}$, which is obviously true thanks to the assumption that $a > -1$.

On the other hand, note that f is not subadditive, since
$$f(-\tfrac{1}{2} + (-\tfrac{1}{2})) = f(-1) = a > -1 = f(-\tfrac{1}{2}) + f(-\tfrac{1}{2}).$$

Therefore, there exist nonconvex and nondifferentiable solutions of (1.1) which vanish at the origin and are differentiable except at one point different from zero.

In our next result, we prove some more about (1.1) in the case where its solutions are defined on the whole real line.

THEOREM 2.5. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a mapping which is differentiable at zero, satisfies $f(0) = 0$ and solves (1.1) for all $x, y \in \mathbb{R}$. Then*
$$f(x) = f'(0) \cdot x \quad \text{for all } x \in (-1, 0).$$

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Proof. Replace x in (1.1) by $x+y$ and simultaneously y by $-y$. After some rearrangements

$$f(x-(x+y)y)-f(x)\le f(x+y)-f(x)+f(-y)+f(-(x+y)y),$$

which is true for all $x, y\in\mathbb{R}$. If we assume that $y>0$, then we obtain the inequality

$$\begin{aligned} &-(x+y)\frac{f(x-(x+y)y)-f(x)}{-(x+y)y} \\ &\quad\le \frac{f(x+y)-f(x)}{y}-\frac{f(-y)}{-y}-(x+y)\frac{f(-(x+y)y)}{(x+y)y} \end{aligned} \tag{2.4}$$

and the converse one for $y<0$. Suppose that $x\in(-1,0)$. Pick a sequence $(y_n)_{n\in\mathbb{N}}$ of positive numbers which tend to zero such that

$$\lim_{n\to+\infty}\frac{f(x-(x+y_n)y_n)-f(x)}{(x+y_n)y_n}=D_+f(x).$$

Substitute $y\to y_n$ in (2.4). Letting $n\to+\infty$ we deduce the inequality

$$-xD_+f(x)\le D_+f(x)-(1+x)f'(0),$$

which gives us that $f'(0)\le D_+f(x)$ for all $x\in(-1,0)$.

Next, we can pick a sequence $(y_n)_{n\in\mathbb{N}}$ of negative numbers which tend to zero such that

$$\lim_{n\to+\infty}\frac{f(x-(x+y_n)y_n)-f(x)}{(x+y_n)y_n}=D^-f(x).$$

Letting $n\to+\infty$ in the inequality converse to (2.4) (for negative y),

$$-xD^-f(x)\le D^-f(x)-(1+x)f'(0),$$

which gives us that $D_+f(x)\le f'(0)$ for all $x\in(-1,0)$. Comparing our inequalities with Lemma 2.2 we eventually get that f is differentiable on the interval $(-1,0)$ and $f'(x)=f'(0)$ for all $x\in(-1,0)$. □

3. Remarks

Remark 3.1. Inequality (1.1) is similar to the functional inequality

$$f(x+y)+f(xy)\ge f(x)+f(y)+f(x)f(y),$$

postulated for all $x, y\in\mathbb{R}$, which was introduced by Hammer [3]; see also [2] for a generalisation. In fact, our method of solving (1.1) is an extension of the approach of Hammer.

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REMARK 3.2. Two other functional inequalities related to Hosszú's functional equation have already been studied. Hosszú's equation is

$$f(x + y - xy) = f(x) + f(y),$$

where $x, y \in (0, 1)$. Maksa and Páles [5] and later Pečarić [6] and Powązka [7] have dealt with the following two functional inequalities:

$$f(x + y - xy) \leq f(x) + f(y) \tag{3.1}$$

and

$$f(x + y - xy) + f(xy) \leq f(x) + f(y) \tag{3.2}$$

for a function f defined on the open interval $(0, 1)$. They established some connections of the solutions of (3.1) and (3.2) with Jensen-concave functions and Wright-concave functions.

References

[1] B. Bouikhalene, J. M. Rassias, A. Charifi and S. Kabbaj, 'On the approximate solution of the Hosszú's functional equation', *Int. J. Nonlinear Anal. Appl.* **3**(1) (2012), 40–44.

[2] W. Fechner, 'A note on alienation for functional inequalities', *J. Math. Anal. Appl.* **385**(1) (2012), 202–207.

[3] C. Hammer, 'Über die Funktionalungleichung $f(x + y) + f(xy) \geq f(x) + f(y) + f(x)f(y)$', *Aequationes Math.* **45**(2–3) (1993), 297–299.

[4] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, 2nd edn. (Birkhäuser, Basel, 2009), edited and with a preface by Attila Gilányi.

[5] G. Maksa and Z. Páles, 'On Hosszú's functional inequality', *Publ. Math. Debrecen* **36**(1–4) (1989), 187–189.

[6] J. E. Pečarić, 'Two remarks on Hosszú's functional inequality', *Publ. Math. Debrecen* **40**(3–4) (1992), 243–244.

[7] Z. Powązka, 'Über Hosszúfunktionalungleichung und die Jensenische Integralungleichung', *Rocznik Nauk.-Dydakt. Prace Mat.* **14** (1997), 121–128.

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