NOTE ON THE STRONG SUMMABILITY OF SERIES

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1. *Definitions and Preliminary Remarks.* Given the series $\sum_{n=0}^{\infty} a_n$, the n-th Cesàro sum of order k is defined by the relation

$$A_n^{(k)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(k)} a_\nu,$$

where $E_n^{(k)}$ is the binomial coefficient $\binom{k+n}{n}$. Let $C_n^{(k)} = A_n^{(k)}/E_n^{(k)}$. Then Σa_n is said to be summable $(C;\ k)$ to the sum s if, as $n \to \infty$, $C_n^{(k)} \to s$. The series is said to be absolutely summable $(C;\ k)$, or summable $|C;\ k|$, if $\sum_{n=0}^{\infty} |C_n^{(k)} - C_{n-1}^{(k)}|$ is convergent. The series is said to be strongly summable $(C;\ k)$ with index p, or summable $[C;\ k, p]$, to the sum s if

$$\sum_{\nu=0}^{n} |C_\nu^{(k-1)} - s|^p = o(n).$$

It is assumed that k and p are positive.

In this note a consistency theorem and necessary and sufficient conditions for summability $[C;\ k, p]$ are obtained. It is also shown that $[C;\ k, p]$, $p \geqslant 1$ implies $(C;\ \lambda)$, for some λ, and that, whereas $|C;\ k|$ implies $[C;\ k, p]$, $p \leqslant 1$, this is not true for $p > 1$. Properties of strong summability have already been obtained by various writers, for example by Kuttner* in the case $k=1$ and by Winn† in the case $p=1$, but $[C;\ k, p]$, for general k and p does not seem to have been considered hitherto in detail.

In the proofs of the theorems the following relations, all of which are well known, will be required.

$$A_n^{(k+\delta)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(\delta-1)} A_\nu^{(k)}, \quad \delta > 0, \qquad \text{(1)}$$

$$E_n^{(k)} = O(n^k), \qquad \text{(2)}$$

$$-n\{C_n^{(k)} - C_{n-1}^{(k)}\} = k\{C_n^{(k)} - C_n^{(k-1)}\}, \qquad \text{(3)}$$

$$nE_n^{(k+\delta)} a_n^{(k+\delta)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(\delta-1)} \nu E_\nu^{(k)} a_\nu^{(k)}, \quad \delta > 0, \qquad \text{(4)}$$

where $a_n^{(k)} = C_n^{(k)} - C_{n-1}^{(k)}$.

2. *Summability* $[C;\ k, p]$. That $[C;\ 1, p]$ implies $[C;\ 1, p-\delta]$, $0 \leqslant \delta < p$, is well known, and Theorem 1 below is merely a formal extension of this result.

THEOREM 1. *A series which is summable* $[C;\ k, p]$ *is also summable* $[C;\ k, p-\delta]$ *for every* δ *such that* $0 \leqslant \delta < p$.

By Hölder's inequality

$$\sum_{\nu=0}^{n} |C_\nu^{(k-1)} - s|^{p-\delta} \leqslant \left\{\sum_{\nu=0}^{n} |C_\nu^{(k-1)} - s|^p\right\}^{(p-\delta)/p} \left\{\sum_{\nu=0}^{n} 1\right\}^{1/\mu},$$

* B. Kuttner, *Journal London Math. Soc.*, **21** (1946), 118–122.

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$ and $\lambda=p/(p-\delta)$. Thus

$$\sum_{\nu=0}^{n} | C_\nu^{(k-1)} - s |^{p-\delta} = o\{n^{(p-\delta)/p}\} . O(n^{1/\mu}) = o(n).$$

It has been shown by Winn* that $[C; k, 1]$ implies $(C; k)$, whence, by Theorem 1, it follows that, for $p \geqslant 1$, $[C; k, p]$ implies $(C; k)$. This result is not true† when $p<1$. The consistency theorem for Cesàro summability shows that $[C; 1, 1]$ implies $(C; 1+\delta)$, for $\delta \geqslant 0$. On the other hand, Kuttner‡ has shown that $[C; 1, p]$ implies $\left(C; \frac{1}{p}+\delta\right)$, for $\delta>0$, and that δ cannot be replaced by zero. For $[C; k, p]$ we have the following theorem.

THEOREM 2. *If $kp>1$, $p>1$, and if Σa_n is summable $[C; k, p]$ then Σa_n is summable $\left(C; k+\frac{1}{p}+\delta-1\right)$, for any positive δ.*

We may suppose, without loss of generality that the sum of the series Σa_n is zero. If $\lambda>0$ we have

$$C_\nu^{(k+\lambda-1)} = \frac{A_\nu^{(k+\lambda-1)}}{E_\nu^{(k+\lambda-1)}} = \frac{1}{E_\nu^{(k+\lambda-1)}} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(\lambda-1)} A_\mu^{(k-1)}$$

$$= \frac{1}{E_\nu^{(k+\lambda-1)}} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(\lambda-1)} E_\mu^{(k-1)} C_\mu^{(k-1)}.$$

By Hölder's inequality,

$$| C_\nu^{(k+\lambda-1)} | \leqslant \frac{1}{E_\nu^{(k+\lambda-1)}} \left\{ \sum_{\mu=0}^{\nu} | C_\mu^{(k-1)} |^p \right\}^{1/p} \left[\sum_{\mu=0}^{\nu} \{E_{\nu-\mu}^{(\lambda-1)} E_\mu^{(k-1)}\}^{p'} \right]^{1/p'},$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $p>1$. Thus

$$C_\nu^{(k+\lambda-1)} = o\left[\nu^{\frac{1}{p}+1-k-\lambda} \left\{ \sum_{\mu=0}^{\nu} (\nu-\mu+1)^{p'(\lambda-1)} (\mu+1)^{p'(k-1)} \right\}^{1/p'} \right]$$

$$= o\left\{ \nu^{\frac{1}{p}+1-k-\lambda}\, \nu^{\lambda+k-2+\frac{1}{p'}} \right\},$$

if $p'(\lambda-1)>-1$, $p'(k-1)>-1$. Hence if $\lambda>1-\frac{1}{p'}=\frac{1}{p}$, $k>\frac{1}{p}$, $p>1$, we have

$$C_\nu^{(k+\lambda-1)} = o(1),$$

which proves the theorem.

We now obtain necessary and sufficient conditions for strong summability.

THEOREM 3. *Necessary and sufficient conditions for a series to be summable $[C; k, p]$, $p \geqslant 1$ are that it be summable $(C; k)$ and that*

$$\sum_{\nu=0}^{n} \nu^p | C_\nu^{(k)} - C_{\nu-1}^{(k)} |^p = o(n).$$

Suppose that the sum of the given series is zero.

* C. E. Winn, *Math. Zeitschrift*, **37** (1933), 481–492.

† It has been shown that, given any T-matrix, there is a series summable $[C; 1, p]$, $p<1$, but not summable (T). See B. Kuttner, *loc. cit.*

‡ B. Kuttner, *loc. cit.*

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If the series is summable $[C;\ k, p]$, $p \geqslant 1$, it is summable (C, k). From relation (3) and Minkowski's inequality, we have, if $p > 1$,

$$\left[\sum_{\nu=0}^{n} \nu^p \,|\, C_\nu^{(k)} - C_{\nu-1}^{(k)} \,|^p\right]^{1/p} \leqslant \left[k^p \sum_{\nu=0}^{n} |\, C_\nu^{(k)} \,|^p\right]^{1/p} + \left[k^p \sum_{\nu=0}^{n} |\, C_\nu^{(k-1)} \,|^p\right]^{1/p}.$$

By hypothesis, the second term on the right is $o(n^{1/p})$. Also $C_\nu^{(k)} = o(1)$, since the series is summable $(C;\ k)$ to the sum zero. Thus the first term on the right is also $o(n^{1/p})$. Hence, when $p > 1$,

$$\sum_{\nu=0}^{n} \nu^p \,|\, C_\nu^{(k)} - C_{\nu-1}^{(k)} \,|^p = o(n).$$

If $p = 1$,

$$\sum_{\nu=0}^{n} \nu \,|\, C_\nu^{(k)} - C_{\nu-1}^{(k)} \,| \leqslant k \sum_{\nu=0}^{n} |\, C_\nu^{(k)} \,| + k \sum_{\nu=0}^{n} |\, C_\nu^{(k-1)} \,| = o(n).$$

The two conditions are therefore necessary. To prove sufficiency write (3) in the form

$$kC_n^{(k-1)} = kC_n^{(k)} + n\,\{C_n^{(k)} - C_{n-1}^{(k)}\}.$$

When $p > 1$, Minkowski's inequality gives

$$\left\{\sum_{\nu=0}^{n} |\, kC_\nu^{(k-1)} \,|^p\right\}^{1/p} \leqslant \left\{\sum_{\nu=0}^{n} |\, kC_\nu^{(k)} \,|^p\right\}^{1/p} + \left[\sum_{\nu=0}^{n} \nu^p \,|\, C_\nu^{(k)} - C_{\nu-1}^{(k)} \,|^p\right]^{1/p}.$$

The second term is $o(n^{1/p})$ by hypothesis, and so also is the first, since $C_\nu^{(k)} = o(1)$. When $p = 1$, the proof, as in the case of necessity, is obvious.

This theorem at once suggests a definition corresponding to summability $[C;\ 0, p]$. The series Σa_n may be said to be strongly convergent with index p, if it is convergent and if $\sum_{\nu=0}^{n} \nu^p \,|\, a_\nu \,|^p = o(n)$. Strong convergence with index unity may conveniently be called strong convergence. Examples of strongly convergent series are easy to construct. All convergent series whose n-th terms are $o\left(\frac{1}{n}\right)$ are clearly strongly convergent. It will be noted that the condition $\sum_{\nu=0}^{n} \nu \,|\, a_\nu \,| = o(n)$ does not itself imply convergence, since it is satisfied in the case $a_\nu = (\nu \log \nu)^{-1}$, $\nu \geqslant 2$. It is obvious that absolute convergence implies strong convergence.

We shall now show that strong convergence with index p, $p \geqslant 1$, implies summability $[C;\ k, p]$ for any positive k. This is included in a wider consistency theorem (Theorem 4 below), which is based on certain lemmas. The first of these is very general in scope.

LEMMA 1. *If* * $p > 1$, $f(x) \geqslant 0$, $K(x, y) \geqslant 0$ and $K(x, y)$ *is homogeneous of degree* -1, *and if*

$$\int_0^\infty K(x, 1)\, x^{-1/p}\, dx = \lambda,$$

then

$$\int_0^\infty dy \left\{\int_0^\infty K(x, y) f(x)\, dx\right\}^p \leqslant \lambda^p \int_0^\infty \{f(x)\}^p\, dx.$$

LEMMA 2. *If* $f(x) \geqslant 0$, $f(x) = 0$ *for* $x > n$, *if* $k \geqslant 0$, $\delta > 0$, $p > 1$, *then*

$$\int_0^n dy \left\{\int_0^y \frac{(y-x)^{\delta-1} x^k}{y^{k+\delta}} f(x)\, dx\right\}^p \leqslant K \int_0^n \{f(x)\}^p\, dx,$$

where K *is independent of* n.

* See Hardy, Littlewood and Pólya, *Inequalities* (Cambridge University Press, 1934), 229.

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In Lemma 1, take

$$K(x, y) = \frac{(y-x)^{\delta-1} x^k}{y^{k+\delta}}, \quad (x \leqslant y),$$
$$= 0, \qquad (x > y).$$

Then $K(x, y)$ is homogeneous of degree -1, and

$$\int_0^\infty K(x, 1) x^{-1/p}\, dx = \int_0^1 (1-x)^{\delta-1} x^{k-1/p}\, dx = \frac{\Gamma(\delta)\, \Gamma\left(k+1-\frac{1}{p}\right)}{\Gamma\left(k+1+\delta-\frac{1}{p}\right)} = \lambda,$$

say. Thus

$$\int_0^\infty dy \left\{\int_0^y \frac{(y-x)^{\delta-1} x^k}{y^{k+\delta}} f(x)\, dx\right\}^p \leqslant \lambda^p \int_0^\infty \{f(x)\}^p\, dx$$
$$= \lambda^p \int_0^n \{f(x)\}^p\, dx,$$

from which the result follows.

Lemma 3. *If* $\alpha_\mu \geqslant 0$, $k \geqslant 0$, $0 < \delta < 1$, $p > 1$,

$$\sum_{\nu=0}^{n-1} \left\{\sum_{\mu=0}^{\nu-1} \frac{(\nu+1-\mu)^{\delta-1} \mu^k}{(\nu+1)^{k+\delta}} \alpha_\mu\right\}^p \leqslant K \sum_{\mu=0}^{n} \alpha_\mu^p,$$

where K is independent of N.

In Lemma 2 let

$$f(x) = \alpha_\mu, \; \mu \leqslant x < \mu+1, \; \mu = 0, 1, \dots n-1.$$

Then
$$\int_0^n \{f(x)\}^p\, dx = \sum_{\mu=0}^{n-1} \int_\mu^{\mu+1} \{f(x)\}^p\, dx \leqslant \sum_{\mu=0}^{n} \alpha_\mu^p,$$

and
$$\int_0^n dy \left\{\int_0^y \frac{(y-x)^{\delta-1} x^k}{y^{k+\delta}} f(x)\, dx\right\}^p = \sum_{\nu=0}^{n-1} \int_\nu^{\nu+1} dy \left\{\int_0^y \frac{(y-x)^{\delta-1} x^k}{y^{k+\delta}} f(x)\, dx\right\}^p$$
$$\geqslant \sum_{\nu=0}^{n-1} \frac{1}{(\nu+1)^{p(k+\delta)}} \left\{\int_0^\nu (\nu+1-x)^{\delta-1} x^k f(x)\, dx\right\}^p$$
$$= \sum_{\nu=0}^{n-1} \frac{1}{(\nu+1)^{p(k+\delta)}} \left\{\sum_{\mu=0}^{\nu-1} \alpha_\mu \int_\mu^{\mu+1} (\nu+1-x)^{\delta-1} x\; dx\right\}^p$$
$$\geqslant \sum_{\nu=0}^{n-1} \left\{\sum_{\mu=0}^{\nu-1} \frac{(\nu+1-\mu)^{\delta-1} \mu^k}{(\nu+1)^{k+\delta}} \alpha_\mu\right\}^p.$$

Lemma 4. *If* * $\alpha_\nu \geqslant 0$, $\sum_{\nu=0}^{n} \alpha_\nu = o(n)$ *then, for* $\lambda > -1$,

$$\sum_{\nu=0}^{n} \nu^\lambda \alpha_\nu = o(n^{\lambda+1}).$$

The main consistency theorem may be stated as follows :

Theorem 4. *If* Σa_n *is summable* $[C;\ k, p]$, $k \geqslant 0$, $p \geqslant 1$, *then it is summable* $[C;\ k+\delta, q]$ *for any* $\delta > 0$ *and any* $q \leqslant p$.

The case $k > 0$, $p = 1$ has been proved by Winn.*

By Theorem 1 it is sufficient to show that the hypothesis implies summability $[C;\ k+\delta, p]$, and there is no loss in generality in assuming that $0 < \delta < 1$.

By hypothesis and Theorem 3 the series is summable $(C;\ k)$ and therefore summable $(C;\ k+\delta)$. Hence to prove the theorem it is sufficient to show that

$$\sum_{\nu=0}^{n} \{\nu \mid a_\nu^{(k+\delta)} \mid\}^p = o(n).$$

* C. E. Winn, *loc. cit.*

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When $k=0$, $p=1$ we have, from (4),

$$\sum_{\nu=0}^{n} \nu \mid a_\nu^{(\delta)} \mid \leqslant \sum_{\nu=0}^{n} \frac{1}{E_\nu^{(\delta)}} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(\delta-1)} \mu \mid a_\mu \mid$$
$$= \sum_{\mu=0}^{n} \mu \mid a_\mu \mid \sum_{\nu=\mu}^{n} \frac{E_{\nu-\mu}^{(\delta-1)}}{E_\nu^{(\delta)}}$$
$$= O\left\{ \sum_{\mu=0}^{n} \mu^{1-\delta} \mid a_\mu \mid \sum_{\nu=\mu}^{n} E_{\nu-\mu}^{(\delta-1)} \right\}$$
$$= O\left\{ n^\delta \sum_{\mu=0}^{n} \mu^{1-\delta} \mid a_\mu \mid \right\} = o(n),$$

by Lemma 4.

Assuming now that $k \geqslant 0$, $p > 1$ we have, from (4),

$$\nu E_\nu^{(k+\delta)} \mid a_\nu^{(k+\delta)} \mid \leqslant \sum_{\mu=0}^{\nu-1} E_{\nu-\mu}^{(\delta-1)} E_\mu^{(k)} \mu \mid a_\mu^{(k)} \mid + E_\nu^{(k)} \nu \mid a_\nu^{(k)} \mid,$$

and, since $(a+b)^p \leqslant 2^p(a^p+b^p)$, $a \geqslant 0$, $b \geqslant 0$, we have

$$\{\nu \mid a_\nu^{(k+\delta)} \mid\}^p \leqslant 2^p \left\{ \frac{1}{E_\nu^{(k+\delta)}} \sum_{\mu=0}^{\nu-1} E_{\nu-\mu}^{(\delta-1)} E_\mu^{(k)} \mu \mid a_\mu^{(k)} \mid \right\}^p + 2^p \left\{ \frac{E_\nu^{(k)} \nu \mid a_\nu^{(k)} \mid}{E_\nu^{(k+\delta)}} \right\}^p.$$

Thus $$\sum_{\nu=0}^{n-1} \{\nu \mid a_\nu^{(k+\delta)} \mid\}^p = O\left[\sum_{\nu=0}^{n-1} \left\{ \sum_{\mu=0}^{\nu-1} \frac{(\nu+1-\mu)^{\delta-1} \mu^k}{(\nu+1)^{k+\delta}} \mu \mid a_\mu^{(k)} \mid \right\}^p \right] + O\left[\sum_{\nu=0}^{n-1} \{\nu \mid a_\nu^{(k)} \mid\}^p \right].$$

The second term is $o(n)$ and, by Lemma 3, the first is

$$O\left[\sum_{\nu=0}^{n} \{\nu \mid a_\nu^{(k)} \mid\}^p \right] = o(n).$$

The theorem is therefore proved.

3. *Relationship between* $| C; k |$ *and* $[C; k, p]$. It is easy to see that $| C; k |$ implies $[C; k, 1]$ and therefore, by Theorem 1, that it implies $[C; k, p]$ for $0 < p \leqslant 1$. Any series which is summable $| C; k |$ is summable $(C; k)$, and, if it is also to be summable $[C; k, p]$, for $p > 1$, we must have, by Theorem 3,

$$\sum_{\nu=0}^{n} \nu^p \mid a_\nu^{(k)} \mid^p = o(n).$$

Write $\alpha_\nu = | a_\nu^{(k)} |$. Now it is possible to find a convergent series of positive terms $\Sigma \alpha_\nu$ which is such that, for $p > 1$,

$$\sum_{\nu=0}^{n} (\nu \alpha_\nu)^p \neq o(n).$$

For example, if

$$\alpha_\nu = e^{-\mu}, \text{ when } \nu \text{ is of the form } [e^{e^\mu}],$$
$$= 0, \text{ all other } \nu,$$

it is not difficult to show that, for any $\eta > 0$,

$$\frac{1}{n} \sum_{\nu=0}^{n} (\nu \alpha_\nu)^{1+\eta} \to \infty,$$

as $n \to \infty$ through values of the form $[e^{e^m}]$. It follows that, when $p > 1$, $| C; k |$ does not imply $[C; k, p]$.

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