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SYMMETRICALLY COMPLETELY BOUNDED LINEAR MAPS BETWEEN C^*-ALGEBRAS

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ABSTRACT. We study the properties of a new class $\mathrm{SCB}(\mathcal{L}, \mathcal{B})$ of bounded linear maps, called symmetrically completely bounded maps, from a linear subspace $\mathcal{L}$ of a C^*-algebra to another C^*-algebra $\mathcal{B}$. This class contains the class of all completely bounded linear maps from $\mathcal{L}$ to $\mathcal{B}$. In particular, we obtain a representation theorem for maps in $\mathrm{SCB}(\mathcal{L}, \mathcal{B})$ when $\mathcal{B}$ is the algebra of all bounded linear operators on a Hilbert space.

1. **Introduction.** Completely bounded linear maps between C^*-algebras play an important role in recent development of the theory of operator algebras. In this paper, we study the properties of a larger class of linear maps, called *symmetrically completely bounded linear maps* between C^*-algebras.

Throughout this paper, C^*-algebras are usually written in scripts $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. Elements in C^*-algebras are written in capital roman type A, B, etc. A C^*-algebra is *unital* if it has an identity for multiplication and this is denoted by I. $B(H)$ denotes the C^*-algebra of all bounded linear operators on a Hilbert space H. M_n denotes the algebra of all $n \times n$ complex matrices. $M_n(\mathcal{A})$ is the algebra of all $n \times n$ matrices over $\mathcal{A}$. We often identify the tensor product $M_n \otimes \mathcal{A}$ of M_n and $\mathcal{A}$ with the C^*-algebra $M_n(\mathcal{A})$. The set of all positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}^+$. $B(\mathcal{A}, \mathcal{B})$ denotes the set of all bounded linear maps from $\mathcal{A}$ to $\mathcal{B}$. We write id_n for the identity map on M_n. The transpose map on M_n is denoted by tr_n.

Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and let $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Φ is said to be *positive* if $\Phi(\mathcal{A}^+) \subset \mathcal{B}^+$. For every positive integer n, let $\mathrm{id}_n \otimes \Phi\colon M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ be the map defined by

$$\mathrm{id}_n \otimes \Phi([A_{ij}]_{i,j=1}^n) = [\Phi(A_{ij})]_{i,j=1}^n, \quad [A_{ij}]_{i,j=1}^n \in M_n(\mathcal{A}),$$

and let $\mathrm{tr}_n \otimes \Phi\colon M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ be defined by

$$\mathrm{tr}_n \otimes \Phi([A_{ij}]_{i,j=1}^n) = [\Phi(A_{ji})]_{i,j=1}^n, \quad [A_{ij}]_{i,j=1}^n \in M_n(\mathcal{A}).$$

Φ is said to be *completely positive* (respectively *completely copositive*) if $\mathrm{id}_n \otimes \Phi$ (respectively $\mathrm{tr}_n \otimes \Phi$) is positive for every positive integer n [1, 9, 15]. Φ is said to be *decomposable* if it is the sum of a completely positive linear map and a completely copositive linear map [11, 15]. If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, Φ is said to be *unital* if $\Phi(I) = I$.

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Let $\mathcal{L}$ be a linear subspace of a C^*-algebra, let $\mathcal{B}$ be a C^*-algebra and let $\Phi: \mathcal{L} \longrightarrow \mathcal{B}$ be a linear map. Let $\mathrm{id}_n \otimes \Phi: M_n(\mathcal{L}) \longrightarrow M_n(\mathcal{B})$ and $\mathrm{tr}_n \otimes \Phi: M_n(\mathcal{L}) \longrightarrow M_n(\mathcal{B})$ be defined as above. Φ is said to be *completely bounded* (respectively *completely cobounded*) if $\sup_n \| \mathrm{id}_n \otimes \Phi\|$ (respectively $\sup_n \| \mathrm{tr}_n \otimes \Phi\|$) is finite. We denote by $\mathrm{CB}(\mathcal{L}, \mathcal{B})$ $\big($respectively $\mathrm{CCB}(\mathcal{L}, \mathcal{B})\big)$ the linear space of all completely bounded (respectively all completely cobounded) linear maps from $\mathcal{L}$ to $\mathcal{B}$. If Φ is in $\mathrm{CB}(\mathcal{L}, \mathcal{B})$, the quantity

$$\|\Phi\|_{\mathrm{cb}} = \sup_n \| \mathrm{id}_n \otimes \Phi\|$$

is called the *completely bounded norm* of Φ. Φ is said to be *completely contractive* if $\|\Phi\|_{\mathrm{cb}} \leq 1$. The corresponding definitions for maps in $\mathrm{CCB}(\mathcal{L}, \mathcal{B})$ are obvious. In particular, Φ is *completely cocontractive* if $\|\Phi\|_{\mathrm{ccb}} = \sup_n \| \mathrm{tr}_n \otimes \Phi\| \leq 1$. Completely bounded linear maps between C^*-algebras have been studied in various papers and in a book [7] by Paulsen. (See the references listed in [7]).

In this paper, we study a new class $\mathrm{SCB}(\mathcal{L}, \mathcal{B})$ of linear maps, called *symmetrically completely bounded linear maps* (Definition 1), from a linear subspace $\mathcal{L}$ of a C^*-algebra to a C^*-algebra $\mathcal{B}$. This class contains the linear span of completely bounded linear maps and completely cobounded linear maps from $\mathcal{L}$ to $\mathcal{B}$; we show that equality holds if the range algebra is $B(H)$ (Theorem 4), and prove an extension theorem and a representation theorem for maps in $\mathrm{SCB}\big(\mathcal{L}, B(H)\big)$ (Theorem 4).

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2. **Symmetrically completely bounded linear maps.** Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\mathcal{L}$ be a linear subspace of $\mathcal{A}$. For every positive integer n, let

$$M_n(\mathcal{L})^s = \big\{ [A_{ij}]_{i,j=1}^n \in M_n(\mathcal{L}) : A_{ij} = A_{ji}, 1 \leq i,j \leq n \big\}.$$

Let $\Phi: \mathcal{L} \longrightarrow \mathcal{B}$ be a bounded linear map. Then

$$\mathrm{id}_n \otimes \Phi | M_n(\mathcal{L})^s = \mathrm{tr}_n \otimes \Phi | M_n(\mathcal{L})^s.$$

Hence for every n,

$$\begin{aligned}\|\Phi\| &\leq \| \mathrm{id}_n \otimes \Phi | M_n(\mathcal{L})^s\| \\ &\leq \min(\| \mathrm{id}_n \otimes \Phi\|, \| \mathrm{tr}_n \otimes \Phi\|).\end{aligned}$$

It is obvious that the sequence $\big\{ \| \mathrm{id}_n \otimes \Phi | M_n(\mathcal{L})^s\| \big\}_{n=1}^{\infty}$ is monotone increasing.

DEFINITION 1. Let

$$\mathrm{SCB}(\mathcal{L}, \mathcal{B}) = \big\{ \Phi \in B(\mathcal{L}, \mathcal{B}) \mid \sup_n \| \mathrm{id}_n \otimes \Phi | M_n(\mathcal{L})^s\| < \infty \big\}.$$

A linear map $\Phi\colon \mathcal{L} \longrightarrow \mathcal{B}$ is said to be *symmetrically completely bounded* if Φ is in $\mathrm{SCB}(\mathcal{L}, \mathcal{B})$ and the quantity

$$\|\Phi\|_{\mathrm{scb}} = \sup_n \|\, \mathrm{id}_n \otimes \Phi | M_n(\mathcal{L})^s \|$$

is called the *symmetrically completely bounded norm* of Φ. Φ is said to be *symmetrically completely contractive* if $\|\Phi\|_{\mathrm{scb}} \leq 1$. Note that these terms have been used for different objects in [3].

The following characterization of maps in $\mathrm{SCB}(\mathcal{L}, \mathcal{B})$ is sometimes useful.

LEMMA 2. *Let $\Phi\colon \mathcal{L} \longrightarrow \mathcal{B}$ be a linear map. The following conditions are equivalent:*
(i) Φ is symmetrically completely bounded.
(ii) There exists a positive constant c such that

$$\|\, [\Phi(A_{ij})]_{i,j=1}^n \| \leq c \cdot \max(\|\, [A_{ij}]_{i,j=1}^n \|, \|\, [A_{ij}]_{i,j=1}^n \|)$$

for every positive integer n and every $[A_{ij}]_{i,j=1}^n$ in $M_n(\mathcal{L})$.
Furthermore, if Φ satisfies (i) or (ii), then

$$(*) \qquad \begin{aligned} \|\Phi\|_{\mathrm{scb}} = \sup\{ \|\, [\Phi(A_{ij})]_{i,j=1}^n \| : [A_{ij}]_{i,j=1}^n \in M_n(\mathcal{L}), \\ n \text{ positive}, \ \max(\|\, [A_{ij}]_{i,j=1}^n \|, \|\, [A_{ji}]_{i,j=1}^n \|) \leq 1 \}. \end{aligned}$$

PROOF. (i) means there exists a positive constant c such that

$$\|\, [\Phi(A_{ij})]_{i,j=1}^n \| \leq c \|\, [A_{ij}]_{i,j=1}^n \|$$

for every positive integer n and every $[A_{ij}]_{i,j=1}^n \in M_n(\mathcal{L})^s$.
Hence (ii) $\Rightarrow$ (i) and we have the inequality "$\leq$" in $(*)$.

(i) $\Rightarrow$ (ii): Let $[A_{ij}]_{i,j=1}^n \in M_n(\mathcal{L})$. Then

$$\begin{bmatrix} 0 & [A_{ij}]_{i,j=1}^n \\ [A_{ji}]_{i,j=1}^n & 0 \end{bmatrix}$$

is in $M_{2n}(\mathcal{L})^s$. By (i),

$$\begin{aligned} \max(\|\, [\Phi(A_{ij})]_{i,j=1}^n \|, \|\, [\Phi(A_{ji})]_{i,j=1}^n \|) &= \left\| \begin{bmatrix} 0 & [\Phi(A_{ij})]_{i,j=1}^n \\ [\Phi(A_{ji})]_{i,j=1}^n & 0 \end{bmatrix} \right\| \\ &\leq \|\Phi\|_{\mathrm{scb}} \left\| \begin{bmatrix} 0 & [A_{ij}]_{i,j=1}^n \\ [A_{ji}]_{i,j=1}^n & 0 \end{bmatrix} \right\| \\ &= \|\Phi\|_{\mathrm{scb}} \cdot \max(\|\, [A_{ij}]_{i,j=1}^n \|, \|\, [A_{ji}]_{i,j=1}^n \|). \end{aligned}$$

Hence (ii) holds and we have the inequality "$\geq$" in $(*)$. ■

REMARK 3. Like $\big(\mathrm{CB}(\mathcal{L}, \mathcal{B}), \|\cdot\|_{\mathrm{cb}}\big)$, the space $(\mathrm{SCB}(\mathcal{L}, \mathcal{B}), \|\cdot\|_{\mathrm{scb}})$ is also complete. $\mathrm{SCB}(\mathcal{L}, \mathcal{B})$ contains the completely bounded linear maps and the completely cobounded linear maps. If Φ is completely bounded (respectively completely cobounded), then

$$\|\Phi\|_{\mathrm{scb}} \leq \|\Phi\|_{\mathrm{cb}} \text{ (respectively } \|\Phi\|_{\mathrm{scb}} \leq \|\Phi\|_{\mathrm{ccb}}).$$

We do not have equality for the norms in general. For example, since the transpose map tr_n on M_n is a $*$-antiisomorphism, $\| \mathrm{tr}_n \|_{\mathrm{scb}} = 1$ (each $\| \mathrm{id}_k \otimes \mathrm{tr}_n | M_k(M_n)^s \| = 1$). However by [13, Theorem 1.2], if $n \geq 2$

$$\| \mathrm{tr}_n \|_{\mathrm{cb}} = n > \| \mathrm{tr}_n \|_{\mathrm{scb}}.$$

A linear map Φ from a C^*-algebra $\mathcal{A}$ to a C^*-algebra $\mathcal{B}$ is a *Jordan homomorphism* if for every A, B in $\mathcal{A}$,

$$\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A).$$

It is easy to verify that if $\Pi_i: \mathcal{A} \longrightarrow B(K)$, $i = 1, 2$, Π_1 is a $*$-homomorphism, Π_2 is a $*$-antihomomorphism and Π_1 and Π_2 are orthogonal (i.e., $\Pi_1(A_1)\Pi_2(A_2) = \Pi_2(A_2)\Pi_1(A_1) = 0$ for every A_1, A_2 in $\mathcal{A}$), then the sum $\Pi = \Pi_1 + \Pi_2$ is a linear map is a $*$-Jordan homomorphism. Conversely, by a result of Størmer [10, Theorem 3.3], every $*$-Jordan homomorphism $\Pi: \mathcal{A} \longrightarrow B(K)$ is of this form. The next theorem shows that $*$-Jordan homomorphisms are closely related to symmetrically completely bounded linear maps.

THEOREM 4. *Let $\mathcal{L}$ be a linear subspace of a C^*-algebra $\mathcal{A}$ and let $\Phi: \mathcal{L} \longrightarrow B(H)$ be a linear map. The following conditions are equivalent:*

(i) Φ is symmetrically completely contractive (i.e. $\|\Phi\|_{\mathrm{scb}} \leq 1$).

(ii) For every positive integer n and every $[A_{ij}]_{i,j=1}^n \in M_n(\mathcal{L})$,

$$\| [\Phi(A_{ij})]_{i,j=1}^n \| \leq \max(\| [A_{ij}]_{i,j=1}^n \|, \| [A_{ji}]_{i,j=1}^n \|).$$

(iii) There exist a Hilbert space K, a $*$-Jordan homomorphism $\Pi: \mathcal{A} \longrightarrow B(K)$ and isometries V and W from H to K such that

$$\Phi(A) = V^*\Pi(A)W \text{ for every } A \in \mathcal{L}.$$

If $\mathcal{A}$ is unital and Φ satisfies (i) or (ii), then we can choose Π to be unital in (iii).

PROOF. (i)$\Longleftrightarrow$(ii) follows from Lemma 2.

(ii)$\Rightarrow$(iii): Let $\mathcal{A} \subset B(L)$ for some Hilbert space L. Let $X \longrightarrow X^{\mathrm{tr}}$ be the transpose map on $B(L)$ with respect to some fixed orthonormal basis of L. The map

$$\mathrm{tr}_n \otimes \mathrm{tr}: M_n\big(B(L)\big) \longrightarrow M_n\big(B(L)\big) \text{ given by}$$

$$\mathrm{tr}_n \otimes \mathrm{tr}([X_{ij}]_{i,j}) = [X_{ji}^{\mathrm{tr}}]_{i,j}$$

is a unital $*$-antiisomorphism for every n. If $[X_{ij}]_{i,j} \in M_n\big(B(L)\big)$, then

$$\| [X_{ij}^{\mathrm{tr}}]_{i,j} \| = \| \mathrm{tr}_n \otimes \mathrm{tr}([X_{ji}]_{i,j}) \| = \| [X_{ji}]_{i,j} \|.$$

Therefore

$$\begin{aligned} \| [X_{ij} \oplus X_{ij}^{\mathrm{tr}}]_{i,j=1}^n \| &= \max(\| [X_{ij}]_{i,j=1}^n \|, \| [X_{ij}^{\mathrm{tr}}]_{i,j=1}^n \|) \\ &= \max(\| [X_{ij}]_{i,j=1}^n \|, \| [X_{ji}]_{i,j=1}^n \|). \end{aligned}$$

Let

$$S = \{A \oplus A^{\mathrm{tr}} \in B(L) \oplus B(L) : A \in \mathcal{L}\}.$$

Then S is a linear subspace of $B(L) \oplus B(L)$. Define

$$\Psi: S \longrightarrow B(H) \text{ by } \Psi(A \oplus A^{\mathrm{tr}}) = \Phi(A).$$

By (ii),

$$\begin{aligned} \|[\Psi(A_{ij} \oplus A_{ij}^{\mathrm{tr}})]_{i,j=1}^{n}\| &= \|[\Phi(A_{ij})]_{i,j=1}^{n}\| \\ &\leq \max(\|[A_{ij}]_{i,j=1}^{n}\|, \|[A_{ji}]_{i,j=1}^{n}\|) \\ &= \|[A_{ij} \oplus A_{ij}^{\mathrm{tr}}]_{i,j=1}^{n}\| \end{aligned}$$

for every $[A_{ij}]_{i,j=1}^{n} \in M_n(\mathcal{L})$ and for every n. Hence Ψ is n-contractive for every positive integer n. By the extension theorem for completely bounded maps ([7, Theorem 7.2], [14, Theorem 3.1]), there exists a completely contractive extension

$$\tilde{\Psi}: B(L) \oplus B(L) \longrightarrow B(H)$$

of Ψ such that $\|\tilde{\Psi}\|_{\mathrm{cb}} = \|\Psi\|_{\mathrm{cb}}$. Furthermore, by the representation theorem for completely bounded maps ([6, Theorem 2.7], [7, Theorem 7.4]), there exist a Hilbert space K, a unital $*$-homomorphism $\tilde{\Pi}: B(L) \oplus B(L) \longrightarrow B(K)$, and isometries V and W from H to K such that

$$\tilde{\Psi}(\cdot) = V^*\tilde{\Pi}(\cdot)W.$$

For every $A \in \mathcal{L}$,

$$\begin{aligned} \Phi(A) &= \tilde{\Psi}(A \oplus A^{tr}) \\ &= V^*\tilde{\Pi}(A \oplus A^{\mathrm{tr}})W \\ &= V^*\Pi(A)W, \end{aligned}$$

where $\Pi: \mathcal{A} \longrightarrow B(K)$ given by $\Pi(A) = \tilde{\Pi}(A \oplus A^{\mathrm{tr}})$ is a $*$-Jordan homomorphism. Thus (iii) holds. If $\mathcal{A}$ is unital, then

$$\Pi(I) = \tilde{\Pi}(I \oplus I) = I.$$

We omit the routine proof of (iii)$\Rightarrow$(ii). ∎

If the range algebra of a linear map is injective (see [2]), we still have the following

COROLLARY 5. *Let $\mathcal{L}$ be a linear subspace of a C^*-algebra $\mathcal{A}$, let $\mathcal{B}$ be an injective C^*-algebra and let $\Phi: \mathcal{L} \longrightarrow \mathcal{B}$ be a symmetrically completely bounded linear map. Then there exists a symmetrically completely bounded linear map $\tilde{\Phi}: \mathcal{A} \longrightarrow \mathcal{B}$ which extends Φ and $\|\tilde{\Phi}\|_{\mathrm{scb}} = \|\Phi\|_{\mathrm{scb}}$. Moreover, there exist a completely bounded linear map $\tilde{\Phi}_1$ and a completely cobounded linear map $\tilde{\Phi}_2$ from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$\tilde{\Phi} = \tilde{\Phi}_1 + \tilde{\Phi}_2,$$
$$\|\tilde{\Phi}_1\|_{\mathrm{cb}} \leq \|\Phi\|_{\mathrm{scb}} \text{ and } \|\tilde{\Phi}_2\|_{\mathrm{ccb}} \leq \|\Phi\|_{\mathrm{scb}}.$$

COROLLARY 6. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\Phi\colon \mathcal{A} \longrightarrow B(H)$ be a unital linear map. The following conditions are equivalent:*

(i) Φ is symmetrically completely contractive.

(ii) There exist a Hilbert space K, a unital $*$-Jordan homomorphism $\Pi\colon \mathcal{A} \longrightarrow B(K)$ and an isometry V from H to K such that

$$\Phi(\cdot) = V^*\Pi(\cdot)V,$$

i.e., Φ is decomposable (in the sense of [11, 15]).

PROOF. (i)$\Rightarrow$(ii): Suppose that (i) holds. As in the proof of (ii)$\Rightarrow$(iii) in Theorem 4, the map Ψ is unital in this case. Hence the map

$$\tilde{\Psi}\colon B(L) \oplus B(L) \longrightarrow B(H)$$

is unital completely contractive and so it is completely positive (see [1, Theorem 1.2.9]). By Stinespring's theorem [9] instead of the representation theorem for completely bounded maps, we can represent Φ in the form in (ii).

(ii)$\Rightarrow$(i): This follows from Theorem 4 (iii)$\Rightarrow$(i). ∎

We should compare Theorem 4 and Corollary 6 with the following result. The main part, the equivalence of (i) and (ii), is due to Størmer [12].

THEOREM 7. *Let $\mathcal{A}$ be a C^*-algebra and Φ a linear map for $\mathcal{A}$ to $B(H)$. The following conditions are equivalent:*

(i) Φ is decomposable.

(ii) For every positive integer n, if both $[A_{ij}]_{i,j=1}^n$ and $[A_{ji}]_{i,j=1}^n$ are in $M_n(\mathcal{A})^+$, then $[\Phi(A_{ij})]_{i,j=1}^n$ is in $M_n\big(B(H)\big)^+$.

(iii) For every positive integer n, if $[A_{ij}]_{i,j=1}^n \in \big(M_n(\mathcal{A})^s\big)^+$, then $[\Phi(A_{ij})]_{i,j=1}^n \in M_n\big(B(H)\big)^+$.

PROOF. (i)$\Longleftrightarrow$(ii) is in [12]. It remains to prove (ii)$\Longleftrightarrow$(iii). (ii)$\Rightarrow$(iii) is obvious. The proof of (iii)$\Rightarrow$(ii) depends on the following observation: Let $\mathcal{A}$ be a C^*-algebra, let $X = [A_{ij}]_{i,j} \in M_n(\mathcal{A})$ and $\tau_n(X) = [A_{ji}]_{i,j}$. Then the element

$$\tilde{X} = \frac{1}{2}\begin{bmatrix} X + \tau_n(X) & -i\big(X - \tau_n(X)\big) \\ i\big(X - \tau_n(X)\big) & X + \tau_n(X) \end{bmatrix}$$

is in $M_{2n}(\mathcal{A})^s$, and $\tilde{X}$ is positive if and only if both X and $\tau_n(X)$ are positive. (By direct verification, $\tilde{X}$ is in $M_{2n}(\mathcal{A})^s$; the second assertion follows from the relation

$$\tilde{X} = U^*\begin{bmatrix} X & 0 \\ 0 & \tau_n(X) \end{bmatrix} U$$

where

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

is a unitary operator.)

Now suppose that (iii) holds. Let both $X = [A_{ij}]_{i,j=1}^{n}$ and $\tau_n(X) = [A_{ji}]_{i,j=1}^{n}$ be positive. Then $\tilde{X} \in \big(M_{2n}(\mathcal{A})^s\big)^+$ and by (iii),

$$\big(\mathrm{id}_n \otimes \Phi(X)\big)^{\sim} = \mathrm{id}_{2n} \otimes \Phi(\tilde{X}) \in M_{2n}\big(B(H)\big)^+.$$

Hence $\mathrm{id}_n \otimes \Phi(X) \in M_n\big(B(H)\big)^+$. Thus (ii) holds. ∎

REMARK 8. (i) Theorem 7 states that if $\Phi: \mathcal{A} \longrightarrow B(H)$ is a linear map from a C^*-algebra $\mathcal{A}$ to $B(H)$ and if for every n, $\mathrm{id}_n \otimes \Phi$ is positive on $\big(M_n(\mathcal{A})^s\big)^+$, then Φ is the sum of a completely positive linear map and a completely copositive linear map. Theorem 4 shows an analogous result: if for every n, $\mathrm{id}_n \otimes \Phi$ is contractive on $M_n(\mathcal{A})^s$, then Φ is the sum of a completely contractive linear map and a completely cocontractive linear map.
(ii) Combining Theorem 7 and Corollary 6, we have the following: Let $\mathcal{A}$ be a unital C^*-algebra and let Φ be a unital linear map from $\mathcal{A}$ to $B(H)$, then Φ is symmetrically completely contractive if and only if $\mathrm{id}_n \otimes \Phi$ is positive on $\big(M_n(\mathcal{A})^s\big)^+$ for every n (Φ is "symmetrically completely positive" in a sense). This is analogous to the result that for a unital linear map between two C^*-algebras, complete contractivity is equivalent to complete positivity. (See [1, p. 154]).

For a pair of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we have the obvious inclusions

$$\mathrm{CB}(\mathcal{A}, \mathcal{B}) \subseteq \mathrm{SCB}(\mathcal{A}, \mathcal{B}) \subseteq B(\mathcal{A}, \mathcal{B}).$$

Huruya and Tomiyama [5] and Smith [8] have considered the case $\mathrm{CB}(\mathcal{A}, \mathcal{B}) = B(\mathcal{A}, \mathcal{B})$. By modifying their techniques, we can determine when equality holds at each level of the above inclusions (Theorems 9 and 10). We omit the proofs.

THEOREM 9. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The following conditions are equivalent:*

(i) $\mathrm{CB}(\mathcal{A}, \mathcal{B}) = B(\mathcal{A}, \mathcal{B})$

(i′) $\mathrm{CCB}(\mathcal{A}, \mathcal{B}) = B(\mathcal{A}, \mathcal{B})$

(ii) $\mathrm{SCB}(\mathcal{A}, \mathcal{B}) = B(\mathcal{A}, \mathcal{B})$

(iii) Either $\mathcal{A}$ is finite dimensional or $\mathcal{B}$ is a C^*-subalgebra of some matrix C^*-algebra $M_n(\mathcal{C})$ for some commutative C^*-algebra $\mathcal{C}$ and some n.

THEOREM 10. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The following conditions are equivalent:*

(i) $\mathrm{CB}(\mathcal{A}, \mathcal{B}) = \mathrm{SCB}(\mathcal{A}, \mathcal{B})$.

(i′) $\mathrm{CCB}(\mathcal{A}, \mathcal{B}) = \mathrm{SCB}(\mathcal{A}, \mathcal{B})$.

(ii) $\mathrm{CB}(\mathcal{A}, \mathcal{B})$ *is $\|\cdot\|_{\mathrm{scb}}$-closed in* $\mathrm{SCB}(\mathcal{A}, \mathcal{B})$.

(ii′) $\mathrm{CCB}(\mathcal{A}, \mathcal{B})$ *is $\|\cdot\|_{\mathrm{scb}}$-closed in* $\mathrm{SCB}(\mathcal{A}, \mathcal{B})$.

(iii) Either $\mathcal{A}$ or $\mathcal{B}$ is a C^*-subalgebra of a matrix C^*-algebra $M_n(\mathcal{C})$ for some commutative C^*-algebra $\mathcal{C}$ and some n.

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3. Some conjectures.

In this section, we propose some unsolved problems concerning symmetrically completely bounded maps.

Let $\mathcal{L}$ be a linear subspace of a C^*-algebra and let $\mathcal{B}$ be another C^*-algebra.

QUESTION 11. Is $\mathrm{SCB}(\mathcal{L}, \mathcal{B}) = \mathrm{CB}(\mathcal{L}, \mathcal{B}) + \mathrm{CCB}(\mathcal{L}, \mathcal{B})$?

By Corollary 5, this is true if $\mathcal{B}$ is injective.

In view of Theorem 10, the following conjecture is plausible.

CONJECTURE 12. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then $\mathrm{CB}(\mathcal{A}, \mathcal{B}) = \mathrm{CCB}(\mathcal{A}, \mathcal{B})$ if and only if either $\mathcal{A}$ or $\mathcal{B}$ is a C^*-subalgebra of $M_n(C)$ for some commutative C^*-algebra C and some n.

In [4], Haagerup has proved the celebrated result that a bounded homomorphism $\Pi: \mathcal{A} \longrightarrow B(H)$ from a C^*-algebra $\mathcal{A}$ to $B(H)$ is similar to a $*$-homomorphism (i.e., there exists an invertible operator T in $B(H)$ such that $T\Pi(\cdot)T^{-1}$ is a $*$-homomorphism) if and only if Π is completely bounded. M.-D. Choi has made the following conjecture, which is a natural symmetrization of Haagerup's result.

CONJECTURE 13. Let $\mathcal{A}$ be a C^*-algebra and let $\Pi: \mathcal{A} \longrightarrow B(H)$ be a bounded Jordan homomorphism. Then Π is similar to a $*$-Jordan homorphism if and only if Π is symmetrically completely bounded.

Observe that the "only if" part of this conjecture follows easily from Theorem 4. The difficulty of the "if" part seems to arise from the phenomenon that for a Jordan homomorphism Π, $\mathrm{id}_n \otimes \Pi$ may not be a Jordan homomorphism for $n \geq 2$.

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