On a Class of Integral Equations.

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(*Read and Received 12th February 1915*).

1. *Introductory.*

In a paper which appeared in the *Proceedings of the Edinburgh Mathematical Society*, Vol. XXXII., Session 1913–14,* I showed how the application of Laplace's transformation to certain linear differential equations enables us to solve some homogeneous integral equations of the first and second kinds, and I obtained, by an extension of this method, the solution of integral equations whose nucleus is of the form $f(zt)$ or $e^{f(z)f(t)}$.

I propose now to show how this method furnishes the solution of an infinite number of homogeneous integral equations of the first and second kinds where the nucleus $f(z, t)$ satisfies a partial differential equation with respect to z and t, of a rather general form; and I shall then make some remarks on these integral equations.

2. *Exposition of the method.*

Z and S being functions of z, and h an arbitrary constant, let us consider the linear differential equation of the first order

$$Z\frac{dy}{dz}+(S+h)\,y=0 \quad\text{..............................(1)}$$

Its solution is

$$y(z)=Ce^{-\int\frac{S+h}{Z}\,dz}.$$

* References to this first paper will be made in the present one under the symbol I, followed by the indication of the paragraph.

Let us now make, in equation (1), the following change of variable

$$y(z) = \int_a^b f(z, t)\, v(t)\, dt$$

where a and b are two constants, conveniently chosen, and where $f(z, t)$ is a function of z aud t, which satisfies the partial differential equation of the first order

$$Z\frac{\partial f}{\partial z} + T\frac{\partial f}{\partial z} + (S + \Theta) f = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In this equation Z and S are the functions of z already spoken of, and T and Θ are functions of t. It is evident that neither Z nor T can vanish. All our reasoning supposes that essential condition.

We have

$$\frac{dy}{dz} = \int_a^b \frac{\partial f}{\partial z} v\, dt.$$

And equation (1) becomes

$$\int_a^b v(t)\, dt \left[Z\frac{\partial f}{\partial z} + Sf + hf \right] = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

But, using equation (2), this becomes

$$\int_a^b v(t)\, dt \left[-T\frac{\partial f}{\partial t} - \Theta f + hf \right] = 0.$$

Now we can write, by integrating by parts,

$$\int_a^b \frac{\partial f}{\partial t} Tv(t)\, dt = \Big[fTv \Big]_a^b - \int_a^b f(Tv' + T'v)\, dt.$$

If then we have chosen the constants a and b such that

$$\Big[fTv \Big]_a^b = 0,$$

our equation becomes

$$\int_a^b f\, dt \left[Tv' + v(T' - \Theta + h) \right] = 0,$$

and the suitable value of v is given by the linear differential equation, where t is the independent variable,

$$Tv' + v(T' - \Theta + h) = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The solution is

$$v(t) = \frac{C}{T} e^{\int \frac{\Theta - h}{T} dt}$$

and we find then the homogeneous integral equation of the first kind

$$e^{-\int \frac{S+h}{Z} dz} = \lambda_h \int_a^b f(z, t)\, e^{\int \frac{\Theta - h}{T} dt} \frac{dt}{T} \quad \text{.................(5)}$$

In this equation a and b must be determined as we have explained above; we can take two constants, independent of z, such that

$$f(z, a) = f(z, b) = 0.$$

The value of λ, which is also to be determined, will be easily obtained by giving to z a known value, such as 0 or 1. Its value changes with the constant h.

It may be understood that we are thus led to a very general class of integral equations. Depending upon a partial differential equation, the nuclei will always contain an arbitrary function; but, generally, they will be unsymmetrical.

The two nuclei studied in the former paper, $f(zt)$ and $e^{f(z)f(t)}$ (I., § 3–5), are mere particular cases of those we are now speaking of.

3. *Case in which the integral equation is of second kind.*

As was pointed out in § 2 of our precedent paper, the integral equation will be of second kind when the equation (4) turns out to be precisely the same as (1); this occurs if we have between the four functions Z, S, T, Θ (in which we suppose a single variable, say u, has been written instead of z and t), and h:

$$\frac{S+h}{Z} = \frac{h + T' - \Theta}{T} \quad \text{..............................(6)}$$

Very often, for a given function $f(z, t)$, this condition will be fulfilled only for a particular value of h. For instance, with the nucleus $f(zt)$, it is fulfilled only for $h = \frac{1}{2}$; if we give to h any other value, we shall obtain an integral equation of the first kind.

4. *Examples and applications.*

A. Let us consider, as a nucleus, the function

$$f(z, t) = \frac{1}{z+t}\, \phi\left(\frac{z}{t}\right),$$

ϕ being an arbitrary function.

It satisfies the partial differential equation

$$z\frac{\partial f}{\partial z}+t\frac{\partial f}{\partial t}+f=0,$$

which is of the form in question; and in that case, as we have

$$Z=z,\ \ T=t,\ \ S=1,\ \ \Theta=0,$$

the corresponding integral equation will be of the second kind.

It is easily seen that the solution is z^k, k being a constant.

So we have the integral equation of the second kind

$$z^k=\lambda_k\int_a^b\phi\left(\frac{z}{t}\right)\frac{1}{z+t}t^k\,dt.$$

If, for $t=0$, we have $\phi\left(\frac{z}{t}\right)=0$; and if, for $t=+\infty$, $\phi\left(\frac{z}{t}\right)\neq 0$, we can take 0 and $+\infty$ for the limits a and b. This will be the case, for instance, if we take $\phi\left(\frac{z}{t}\right)=e^{-\frac{z}{t}}$.

The value of λ_k will be obtained by making $z=1$, and so

$$\frac{1}{\lambda_k}=\int_0^\infty\phi\left(\frac{1}{t}\right)\frac{t^k}{t+1}\,dt.$$

We are now able to solve the equation of the first kind

$$F(z)=\int_0^\infty\phi\left(\frac{z}{t}\right)\frac{1}{z+t}\,\Phi(t)\,dt$$

where F is known and Φ unknown: for, let us expand F in a Laurent series for points z within an annulus whose centre is at the origin:

$$F(z)=z^h\left[a_0+a_1z+a_2z^2+\ldots+b_1z^{-1}+b_2z^{-2}+\ldots\right],$$

we have for the unknown function $\Phi(t)$

$$\Phi(t)=z^h\left[\lambda_h a_0+\lambda_{h+1}a_1z+\lambda_{h+2}a_2z^2+\ldots+\lambda_{h-1}b_1z^{-1}+\lambda_{h-2}b_2z^{-2}+\ldots\right],$$

the λ's being given by the above formula.

B. The function

$$f_1(z,t)=\phi\left(\frac{z}{t}\right)\frac{1}{z-t}$$

satisfying the same partial differential equation, the above results remain unchanged for integral equations with this function f_1 as nucleus.

https://doi.org/10.1017/S0013091500002339 Published online by Cambridge University Press

C. Again, the function

$$f(z, t) = \phi\left(\frac{z}{t}\right) \frac{1}{z^n \pm t^n},$$

which satisfies the partial differential equation

$$z\frac{\partial f}{\partial z} + t\frac{\partial f}{\partial t} + nf = 0,$$

leads us to the integral equation of the first kind,

$$z^k = \lambda_k \int_0^\infty \phi\left(\frac{z}{t}\right) \frac{1}{z^n \pm t^n} t^{n+k-1} \, dt.$$

As above, we can deduce from this equation the solution of the integral equation of the first kind,

$$F(z) = \int \phi\left(\frac{z}{t}\right) \frac{1}{z^n \pm t^n} \Phi(t) \, dt,$$

by the process of developing $F(z)$ in a Laurent's series of powers of z.

D. Another example of integral equations of the first kind obtained by our method is the following:

$$e^{-\frac{1}{2}z^2} = \lambda \int_{-\infty}^{+\infty} \phi(z+t) \, e^{zt} e^{\frac{1}{2}t^2} \, dt,$$

the functions ϕ being such as

$$\lim_{u = \pm\infty} \phi(u) \, e^u = 0.$$

5. *Change of variable.*

A great number of new integral equations (generally of the first kind) can be deduced from known ones by change of variable. For, if we know the function $v(t)$ which satisfies the integral equation

$$y(z) = \int f(z, t) \, v(t) \, dt,$$

we can at once write the solution of the integral equation

$$y_1(z) = y(\zeta) = \int f(\zeta, \theta) \, v_1(t) \, dt$$

where ζ is a function of z, θ a function of t: it appears clearly that

$$v_1(t) = v(\theta) \frac{d\theta}{dt}.$$

It is by using the change of variable that it is possible to deduce an integral equation of the second kind from one of the first kind (see an example of this in I., § 7).

It may be seen that the equation obtained above in example C can be deduced from the equations in examples A and B by writing

$$\zeta^n = z \quad \theta^n = t.$$

And also the integral equation

$$z^k = \lambda_k \int \phi \left(\frac{z}{t}\right) t^{k-1}\, dt,$$

which is a particular case of example C, with $n = 0$, can be deduced from the following (I., § 4)

$$z^k = \lambda_k \int \phi\,(zt)\, t^{-1-k}\, dt$$

by changing t into $\frac{1}{t}$.

Starting from this last formula, let us put, instead of t, a function $g(t)$. We have

$$z^k = \lambda_k \int \phi\,[z\,g(t)]\,[g(t)]^{-1-k} g'(t)\, dt,$$

and this gives us the solution of the integral equation of the first kind

$$F(z) = \int \phi\,[z\,g(t)]\,\Phi\,(t)\, dt,$$

which solution will be obtained by the process of developing $F(z)$ in series, as we did in the preceding paragraph.

For instance, let us take

$$g(t) = \log t\,;$$

we have

$$z^k = \lambda_k \int \phi\,(z \log t)\,(\log t)^{-1-k}\, \frac{dt}{t},$$

which gives us the solution of integral equations of first kind with the nucleus $\phi\,(z \log t)$, or, what is the samé, $\phi\,(t^z)$. This result can also be obtained by our direct method, the function

$$f(z, t) = \phi\,(t^z)$$

satisfying the partial differential equation

$$z\frac{\partial f}{\partial z} - t \log t\, \frac{\partial f}{\partial t} = 0.$$

A similar change of variables gives us the integral equation

$$[f_1(z)]^k = \lambda_k \int \phi\,[f_1(z) f_2(t)]\,[f_2(t)]^{-1-k} f_2'(t)\,dt,$$

and gives us a method of solving an integral equation of the first kind, with this nucleus, when the known function of z can be developed in a series of powers of $f_1(z)$.

6. *Extension.*

It will be easily understood that our method can be extended to nuclei $f(z, t)$ satisfying a partial differential equation of the second order, of suitable form. The differential equations for y and v will in this case be of the second order; and we shall then obtain a relation such as

$$c_1 y_1 + c_2 y_2 = \lambda \int f(z, t)\,[c_1' v_1 + c_2' v_2]\,dt,$$

the y's and v's being particular solutions of their respective differential equations. This is an integral equation of the first kind, and one has to determine the suitable values of the c's. This case is much more complicated than the case of the first order, and, naturally, so also would be those obtained by extension to the third, fourth, ..., n^{th} order.