Appendix B
Basic Ideas from Topology

We are assuming that the reader is familiar with the basic concepts of topology, and we review some definitions and results used for the reader's convenience. All of the concepts in this appendix are covered in standard textbooks on topology, for example, Engelking (1989). We mention again that we write $U \Subset X$ if U is an open subset of a topological space X.

B.1 Initial and Final Topologies

Let X be a set and $(f_i)_{i \in I}$ be a family of mappings $f_i : X \to Y_i$ to topological spaces Y_i. The coarsest topology on X making each f_i continuous is called the *initial topology on X with respect to the mappings* $(f_i)_{i \in I}$. Dually, if $\{g_i : Y_i \to X\}_{i \in I}$ is a family of mappings from topological spaces to X there is a finest topology making all g_i continuous. This topology is called the *final topology* with respect to the g_i.

B.1 Remark Note that the sets $\bigcap_{i \in F} f_i^{-1}(U_i)$, where $F \subseteq I$ is finite and $U_i \Subset Y_i$ form a basis of the initial topology (even if we restrict our choice of U_i to a basis of the topology of Y_i).

B.2 Lemma *The initial topology on X with respect to a family* $(f_i)_{i \in I}$ *is the unique topology which satisfies* $g : Z \to X$ *is continuous if and only if* $f_i \circ g : Z \to Y_i$ *is continuous for every* $i \in I$.

Proof Assume first that X carries the initial topology $\mathcal{I}$. Then clearly if g is continuous $f_i \circ g$ is continuous for every $i \in I$. Conversely, since $\mathcal{S} := \{f_i^{-1}(V_i) \mid V_i \Subset Y_i\}$ is a subbase for $\mathcal{I}$ by Remark B.1, we see that $f_i \circ g$ continuous implies that $g^{-1}(W)$ is open in Z for each $W \in \mathcal{S}$. Thus g is continuous. We conclude that $\mathcal{I}$ has the claimed property.

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Now let $\mathcal{T}$ be another topology which satisfies the above property. Then the identity maps $\mathrm{id}\colon (X,\mathcal{I}) \to (X,\mathcal{T})$ and $\mathrm{id}\colon (X,\mathcal{T}) \to (X,\mathcal{I})$ are continuous, whence both topologies coincide. □

B.3 Example If X is a subset of a topological space Y, the *induced*, or *subspace, topology* is the initial topology with respect to the inclusion $\iota\colon X \to Y$, $x \mapsto x$.

B.4 Example We always endow the cartesian product $X := \prod_{i\in I} X_i$ of a family $(X_i)_{i\in I}$ of topological spaces with the *product topology*, which is the initial topology with respect to the family $(\mathrm{pr}_i)_{i\in I}$ of projections $\mathrm{pr}_j((x_i)_I) := x_j$.

Exercises

B.1.1 Let X be a topological space endowed with the final topology with respect to a family of mappings $f_i\colon X_i \to X$. Show that $g\colon X \to Y$ is continuous if and only if $g \circ f_i$ is continuous for all $i \in I$.

B.1.2 Let $(E_i)_{i\in I}$ be locally convex spaces and $E := \{(x_i)_{i\in I} \in \prod_{i\in I} E_i \mid$ almost all $x_i = 0\}$. Endow E with the final topology with respect to the family of inclusions $\iota_j\colon E_j \to E$, $x \mapsto (x_i)_{i\in I}$, with $x_j = x$ and $x_i = 0$ if $i \neq j$. Show that:

(a) The resulting topology is the box topology, that is, the topology generated by the base of sets $E \cap \prod_{i\in I} U_i$, where for every $i \in I$, U_i runs through a topological base of E_i.

(b) If I is a *countable* set and $f_i\colon E_i \to F$ is a continuous linear map to a locally convex space F, then there exists a unique continuous linear map $f\colon E \to F$ with $f \circ \iota_j = f_j$. This proves, in particular, that the box topology turns E into the direct locally convex sum of the spaces E_i.
Hint: For continuity of f consider the preimage of a 0-neighbourhood V in F. Construct inductively a sequence $(V_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods with $V_n + V_n \subseteq V_{n+1}$.

(c) Let I be uncountable and $E_i = \mathbb{R}$ for all $i \in I$. Show that the summation map $s\colon \{(x_i)_i \in \mathbb{R}^I \mid x_i = 0 \text{ for almost all } i\} \to \mathbb{R}$, $(x_i) \mapsto \sum_{i\in I} x_i$ is discontinuous in the box topology. Thus the box topology is properly coarser than the direct sum topology in this case.

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B.2 The Compact Open Topology

Let X,Y be (Hausdorff) topological spaces and $C(X,Y)$ the set of all continuous mappings from X to Y. We define a topology on $C(X,Y)$ by declaring a subbase consisting of the following sets:

$$\lfloor K,U \rfloor := \{f \in C(X,Y) \mid f(K) \subseteq U\}, \qquad K \subseteq X \text{ compact}, U \subseteq_{\circ} Y.$$

The resulting topology is called the *compact open topology* and we write $C(X,Y)_{\text{c.o.}}$ for the set of continuous mappings with this topology.[1]

B.5 Remark As singletons are compact, we see that the inclusion map

$$C(X,Y)_{\text{c.o.}} \to Y^X := \prod_{x \in X} Y$$

is continuous if we equip the right-hand side with the product topology. Thus $C(X,Y)_{\text{c.o.}}$ will again be Hausdorff and the evaluations $\text{ev}_x \colon C(X,Y)_{\text{c.o.}} \to Y, \text{ev}_x(f) := f(x)$ are continuous for every $x \in X$.

We will now show that if the target is a locally convex space the compact open topology coincides with the topology of compact convergence.

B.6 Let $(E, \{p_i\}_{i \in I})$ be a locally convex space and $K \subseteq X$ be a compact subset of a topological space. Then we define a seminorm on $C(X,E)_{\text{c.o.}}$ via

$$\|f\|_{p_i,K}(f) := \sup_{x \in K} p_i(f(x)).$$

Note that these seminorms are separating, since for $\gamma \in C(X,E)$ with $\gamma \neq 0$ we find $x \in X$ with $\gamma(x) \neq 0$ and thus $\|\gamma\|_{p_i,\{x\}} \neq 0$ for some $i \in I$. The locally convex topology generated by all seminorms $(\|\cdot\|_{p_i,K})_{i \in I,\ K \subseteq X \text{ compact}}$ is called *topology of compact convergence*.

B.7 Lemma *Let X be a topological space and E a locally convex space. Then the compact open topology coincides with the topology of compact convergence on $C(X,E)$. As a consequence, $C(X,E)_{\text{c.o.}}$ is again a locally convex space if E is locally convex.*

Proof First note that for every seminorm $\|\cdot\|_{K,p}$ for $K \subseteq X$ compact and p a continuous seminorm on E, we have

$$\{f \in C(X,E) \mid \|f\|_{K,p} < r\} = \lfloor K, \{x \in E \mid p(x) < r\} \rfloor. \tag{B.1}$$

To see that the topology of compact convergence is finer than the compact open topology, it suffices to prove that all sets $\lfloor K,U \rfloor$ with $K \subseteq X$ compact and

[1] If you prefer a video walkthrough covering (parts of) the material in this appendix, then take a look at www.youtube.com/watch?v=vGs-C9eEdJ0.

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$U \Subset E$ are open in that topology. Now let $\gamma \in \lfloor K, U \rfloor$. Then $\gamma(K) \subseteq U \Subset E$ is compact and by Lemma A.2(g) there is an open 0-neighbourhood $W \Subset E$ such that $\gamma(K) + W \subseteq U$. We choose a seminorm p and $r > 0$ such that $V := \{x \in E \mid p(x) < r\} \subseteq W$. Then $\gamma + \lfloor K, V \rfloor$ is an open neighbourhood of γ in the topology of compact convergence by (B.1). Moreover, $\gamma + \lfloor K, V \rfloor$ is contained in $\lfloor K, U \rfloor$ since $\gamma(x) + \eta(x) \in \gamma(K) + V \subseteq U$ for all $\eta \in \lfloor K, V \rfloor$ and $x \in K$. This shows that $\lfloor K, U \rfloor$ is a γ-neighbourhood in the topology of compact convergence, hence open.

Conversely, let $\gamma \in C(X, E)$. Thanks to Lemma A.2(e) and (B.1), we see that the sets $\gamma + \lfloor K, \{y \in E \mid p(y) < r\} \rfloor$ form a basis of open γ-neighbourhoods in the topology of compact convergence. Thus it suffices to prove that these sets are open in the compact-open topology. For this choose $0 \in V \Subset E$ such that $V - V \subseteq \{y \in E \mid p(y) < r\}$. As $\gamma|_K$ is continuous, we find for every $x \in K$ an x-neighbourhood $K_x \Subset K$ with $\gamma(\overline{K_x}) \subseteq \gamma(x) + V$. Using compactness, we choose a finite set $F \subseteq K$ with $K = \bigcup_{x \in F} K_x$. We will now show that

$$\Omega_\gamma := \bigcap_{x \in F} \lfloor \overline{K}_x, \gamma(x) + V \rfloor \subseteq \gamma + \lfloor K, \{y \in E \mid p(y) < r\} \rfloor.$$

If $\eta \in \Omega_\gamma$, then we find for every $y \in K$ a $x \in F$ with $y \in K_x$. Thus $(\gamma - \eta)(y) \in \gamma(x) + V - (\gamma(x) + V) \subseteq V - V \subseteq U$ and thus $\gamma - \eta \in \lfloor K, \{y \in E \mid p(y) < r\} \rfloor$. Now Ω_γ is open in the compact-open topology and contains γ by choice of the K_x. Thus both topologies coincide. □

B.8 Lemma *Let A, X, Y, Z be topological spaces and $h \colon A \to X, f \colon Y \to Z$ be continuous maps. Then the* pushforward *and the* pullback *map*

$$f_* \colon C(X, Y)_{\text{c.o.}} \to C(X, Z)_{\text{c.o.}}, \quad g \mapsto f \circ g,$$
$$h^* \colon C(X, Y)_{\text{c.o.}} \to C(A, Y)_{\text{c.o.}}, \quad g \mapsto g \circ h$$

are continuous.

Proof We begin with the pushforward f_* and show that $f_*^{-1}(\lfloor K, U \rfloor)$ is open for every $K \subseteq X$ compact and $U \Subset Y$. For $\gamma \in C(X, Y)$ we see that

$$f_*(\gamma) = f \circ \gamma \in \lfloor K, U \rfloor \Leftrightarrow f(\gamma(K)) \subseteq U \Leftrightarrow \gamma(K) \subseteq f^{-1}(U)$$
$$\Leftrightarrow \gamma \in \lfloor K, f^{-1}(U) \rfloor.$$

Thus $f_*^{-1}(\lfloor K, U \rfloor) = \lfloor K, f^{-1}(U) \rfloor \Subset C(X, Y)_{\text{c.o.}}$ by continuity of f.

Now for the continuity of the pullback h^*, pick $L \subseteq A$ compact and $V \Subset X$. Then for $\gamma \in C(X, Y)$, we have $h^*(\gamma) = \gamma \circ h \in \lfloor L, V \rfloor$ if and only if $\gamma(h(L)) \subseteq V$. In other words, if and only if $\gamma \in \lfloor h(L), V \rfloor$. By continuity of h, $h(L)$ is compact and thus $(f^*)^{-1}(\lfloor L, V \rfloor) = \lfloor h(L), V \rfloor \Subset C(X, Y)_{\text{c.o.}}$ and the pullback is continuous. □

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B.9 Lemma *If X, Y, Z are topological spaces and $\mathrm{pr}_Y : Y \times Z \to Y$ and $\mathrm{pr}_Z : Y \times Z \to Z$ the canonical projections, then the map*

$$\Theta\colon C(X, Y \times Z)_{\text{c.o.}} \to C(X,Y)_{\text{c.o.}} \times C(X,Z)_{\text{c.o.}}, \quad \gamma \mapsto ((\mathrm{pr}_Y)_*(\gamma), (\mathrm{pr}_Z)_*(\gamma))$$

is a homeomorphism, whence $C(X, Y \times Z)_{\text{c.o.}} \cong C(X,Y)_{\text{c.o.}} \times C(X,Z)_{\text{c.o.}}$.

Proof By Lemma B.8 the map Θ is continuous. Clearly Θ is a bijection and thus we only need to show that it takes open sets in a subbase to open sets to see that Θ is a homeomorphism. Now open rectangles $U \times V$, $U \Subset Y$, $V \Subset Z$ form a subbase of the product topology, whence Exercise B.2.2 shows that the sets $\lfloor K, U \times V \rfloor$, with $K \subseteq X$ compact, form a subbase of the topology on $C(X, Y \times Z)_{\text{c.o.}}$. Now $\Theta(\lfloor K, U \times V \rfloor) = \lfloor K, U \rfloor \times \lfloor K, V \rfloor$ is open in $C(X,Y)_{\text{c.o.}} \times C(X,Z)_{\text{c.o.}}$, and so Θ is open. □

B.10 Lemma *Let X, Y be topological spaces. If X is locally compact, the evaluation map*

$$\mathrm{ev}\colon C(X,Y)_{\text{c.o.}} \times X \to Y, \quad (f,x) \mapsto f(x)$$

is continuous.

Proof For $U \Subset Y$ we will show that $\mathrm{ev}^{-1}(U)$ is open in $C(X,Y)_{\text{c.o.}} \times X$. To this end, let $(\gamma, x) \in \mathrm{ev}^{-1}(U)$, that is, $\gamma(x) \in U$. By continuity of γ and local compactness of X there is a compact x-neighbourhood K such that $\gamma(K) \subseteq U$. Thus $\lfloor K, U \rfloor \times K$ is a (γ, x)-neighbourhood such that $\mathrm{ev}(\lfloor K, U \rfloor \times K) \subseteq U$. We see that $\mathrm{ev}^{-1}(U)$ is open and the evaluation is continuous. □

B.11 Proposition *Let X, Y, Z be topological spaces such that Y is locally compact. Then the composition map*

$$\mathrm{Comp}\colon C(X,Y)_{\text{c.o.}} \times C(Y,Z)_{\text{c.o.}} \to C(X,Z)_{\text{c.o.}}, \quad (f,g) \mapsto g \circ f$$

is continuous.

Proof Let $K \subseteq X$ be compact and $U \Subset Z$. Pick (γ, η) with $\mathrm{Comp}(\gamma, \eta) \in \lfloor K, U \rfloor$. We have $\eta(\gamma(K)) \subseteq U$. By Exercise B.2.2 we can pick a compact neighbourhood L of $\gamma(K)$ in $\eta^{-1}(U)$ and set $W := L^\circ$ (interior). Then $\lfloor K, V \rfloor \times \lfloor L, U \rfloor$ is a neigbourhood of (γ, η) which is contained in $\mathrm{Comp}^{-1}(\lfloor K, U \rfloor)$ by construction. Thus the composition is continuous. □

We will now consider continuous mappings on cartesian products. If $f\colon X \times Y \to Z$ is continuous we can form for every $x \in X$ a mapping $f(x, \cdot)\colon Y \to Z$, $y \mapsto f(x,y)$. Since f is continuous every partial map $f(x, \cdot)$ is continuous. It

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turns out that the mapping which assigns to each $x \in X$ the partial map $f(x,\cdot)$ is continuous as a mapping into the space of continuous functions.

B.12 Proposition *Let X, Y, Z be topological spaces and $f: X \times Y \to Z$ be continuous. Then $f^\vee: X \to C(Y,Z)_{\text{c.o.}}$, $f^\vee(x) := f(x,\cdot)$ is continuous.*

Proof Consider $\lfloor K, U \rfloor \subseteq C(Y,Z)_{\text{c.o.}}$. We then compute

$$\begin{aligned}(f^\vee)^{-1}(\lfloor K,U\rfloor) &= \{x \in X \mid f(x,y) \in U \text{ for all } y \in K\}\\ &= \{x \in X \mid f(\{x\} \times K) \subseteq U\}\\ &= \{x \in X \mid \{x\} \times K \subseteq f^{-1}(U)\}.\end{aligned}$$

Now $f^{-1}(U)$ is an open neighbourhood of the compact set $\{x\} \times K$ in the cartesian product $X \times Y$. From Engelking (1989, Lemma 3.1.15) we deduce that there are $A \Subset X$ and $B \Subset Y$ such that $\{x\} \times K \subseteq A \times B \subseteq f^{-1}(U)$. This shows that $(f^\vee)^{-1}(\lfloor K,U\rfloor)$ is a neighbourhood for every x contained in it and thus an open set. Since sets of the form $\lfloor K,U\rfloor$ form a subbase of the compact open topology, $f^\vee$ is continuous. □

B.13 Proposition (Exponential law for the compact open topology) *Let X, Y, Z be topological spaces and Y be locally compact. Then a mapping $f: X \to C(Y,Z)_{\text{c.o.}}$ is continuous if and only if the map*

$$f^\wedge: X \times Y \to Z, \quad (x,y) \mapsto f(x)(y)$$

is continuous.

Proof If f is continuous, then $f^\wedge(x,y) = \mathrm{ev}(f(x),y)$ is continuous by Lemma B.10. Conversely, assume that $f^\wedge$ is continuous. Then a quick calculation shows that $f = (f^\wedge)^\vee$, whence f is continuous by Proposition B.12. □

It is worth noting that local compactness is a crucial ingredient to obtain the exponential law. Indeed one can prove that under some requirement to the topological spaces involved, the exponential law can only hold if the topological space Y is locally compact. See Engelking (1989, Exercise 3.4.A) for details.

Exercises

B.2.1 Prove that if X carries the final topology with respect to a family $g_i: Y_i \to X, i \in I$, then $f: X \to Z$ is continuous if and only if $f \circ g_i$ is continuous for every $i \in I$.

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B.2.2 Let X, Y be topological spaces and $\mathcal{S}$ be a subbase for the topology on Y. Show that the sets $\lfloor K, V \rfloor$ with $K \subseteq X$ compact and $V \in \mathcal{S}$ form a subbase for the compact open topology on $C(X, Y)$.

B.2.3 Let X be a locally compact topological space and $K \subseteq X$ compact with $K \subseteq U \Subset X$. Show that there is a compact set $L \subseteq U$ whose interior contains K.

B.2.4 Let K be a compact topological space and $\Omega \Subset K \times Y$. Prove that the set

$$\Omega' := \{f \in C(K, Y) \mid \text{graph}(f) \subseteq \Omega\}$$

is open in $C(K, Y)_{\text{c.o.}}$. Here $\text{graph}(f) = \{(x, f(x)) \mid x \in K\} \subseteq K \times Y$. *Hint:* If $f \in \Omega'$, then for every $x \in K$ there are open $U_x \Subset K, V_x \Subset Y$ with $(x, f(x)) \in U_x \times V_x \Subset \Omega$.

B.2.5 Use Proposition B.13 to give an alternative proof of the continuity of the composition map Comp from Proposition B.11.
Hint: What is $\text{Comp}^\wedge$?

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