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On the derived length of finite, graded Lie rings with prime-power order, and groups with prime-power order

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This work relates to a question of Burnside (1913) about groups with prime-power order: given a prime p and a positive integer k, what is the smallest order of a group with p-power order and derived length k? We denote this order by $p^{\beta_p(k)}$. For $k \in \{1,2,3\}$, the answer is known. For $k \geqslant 4$, the most recent lower bound for $\beta_p(k)$ is $2^{k-1} + 2k - 4$ (see [4]). A brief history of results is given in [3]. Also in that paper, examples of groups with derived length $k \geqslant 4$ and order p^{2^k-2} are exhibited. With these results, for derived length 4 we have $12 \leqslant \beta_p(4) \leqslant 14$.

Given any p-group, its lower central series can be used to associate a graded Lie ring of p-power order with it. Problems concerning p-groups can often be transformed into questions concerning the corresponding associated Lie ring, which are often easier to treat. In this context one must be cautions however, as the derived series of the graded Lie ring does not necessarily correspond to the derived series of the group. In the thesis, much more specific attention is given to graded Lie rings. We denote the smallest order of a graded Lie ring with p-power order and derived length k by $p^{\alpha_p(k)}$.

The main result shows that if a graded Lie ring has derived length $k \geqslant 4$, then $\alpha_p(4) \geqslant 14$, for $p \geqslant 5$. The link between p-groups and graded Lie rings gives as a corollary that if a group G has order p^{13}, then G has derived length at most 3; this confirms [1, Theorem 10]. Thus $\beta_p(4) = 14$ for $p \geqslant 5$. This leads to the improvement $\beta_p(k) \geqslant 2^{k-1} + 2k - 3$, for $k \geqslant 5$ and $p \geqslant 5$.

For the primes 2 and 3, the story is more complicated because the relationship between the groups and the rings is less straight-forward. There are graded Lie rings with order 2^{12} and derived length 4, and some with order 3^{13} and derived length 4. These are the smallest. However, they do not arise as associated with groups.

All groups of order 2^{12} and 3^{13} have derived length at most 3. The latter confirms a statement made (without proof) in [1]. Thus for odd primes, we now have that

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$\beta_p(4) = 14$ and $\beta_p(k) \geqslant 2^{k-1} + 2k - 3$, for $k \geqslant 5$. It is also confirmed, as stated in [1], that a group with order 2^{13} and derived length 4 must have nilpotency class 9 and lower central factor orders $(8, 4, 4, 2, 2, 2, 2, 2, 2)$. An example of such a 2-group produced by M.F. Newman using the computer system MAGMA (see [2]) is given in the thesis. Its derived factor orders are $[8, 16, 32, 2]$. Newman also used MAGMA to produce a group with order 3^{14}, lower central factor orders $(9, 3, 9, 3, 3, 3, 3, 3, 9, 3, 3)$, and derived factor orders $[9, 81, 2187, 3]$.

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