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REGULARITY OF RICHARDSON'S COMPACTIFICATION

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1. Introduction. In [**7**] Richardson constructed a Stone-Čech type compactification $R(E)$ of a Hausdorff convergence space E. Two questions arise in this regard. First, when is $R(E)$ homeomorphic to $\beta(E)$, $\beta(E)$ the topological Stone-Čech compactification of E, for a Tychonoff topological space E? Second, if E is a regular convergence space, when is $R(E)$ regular? The last question is motivated by the study of regular compactifications in [**6**]. In section 2 it will be shown that a necessary and sufficient condition in answer to both questions, is that $\alpha = \text{cl}(\alpha)$ for each nonconvergent ultrafilter α on E. In section 3 it is shown that if $R(E) = \beta(E)$, then $R(F) = \beta(F)$ for every extension $F = E \cup A$ in $\beta(E)$, A closed, $A \subset \beta(E) - E$. This provides a class of nondiscrete, noncompact topological spaces F for which $R(F) = \beta(F)$. Also it is shown that $R(E) = \beta(E)$ implies $\beta(E) = W(E)$, $W(E)$ the Wallman compactification of E, and that $\beta(E)$ is equivalent to the Fomin H-closed extension of E. From these results it follows that the class of E for which $R(E) = \beta(E)$ is neither finitely productive nor hereditary.

For use in the next section, we review the compression operator in a slightly modified setting. Let $\mathbf{\Phi}$ be a filter on $R(E)$. If $A \in \mathbf{\Phi}$ and f is any function on A for which $f(D) \in D$ for each $D \in A$, define $A(f) = \cup\ (f(D) : D \in A)$. Since $\mathbf{\Phi}$ is a filter base on the collection of all filters on E, the $A(f)$ constitute a filter base on E. Then $K(\mathbf{\Phi})$ is defined to be the filter, on E, generated by the $A(f)$.

Notation involving the compactification $R(E)$ is that of [**7**]. If E is a convergence space and $A \subset E$, cl (A, E) is the closure of A in E and, if α is a filter on E, cl (α, E) is the filter generated by the cl (A, E), $A \in \alpha$. The abbreviations cl (A), cl (α) will be used when there is no loss of clarity. The convergence space E is regular if cl $(\alpha) \to x$ whenever $\alpha \to x$ for each filter α on E. (See [**2**] for an equivalent definition of regularity which makes essential use of the compression operator K, and [**4**], for example, for a different concept of regularity.) In section 2 "space" means convergence space unless specified otherwise. In section 3 all spaces are topological Tychonoff.

Acknowledgement. In the original version of this paper, section 3 contained only an example of a Tychonoff E for which $R(E) \neq \beta(E)$. The referee pointed out that this was true for any non-normal Tychonoff space or any Tychonoff space containing an open set with noncompact boundary. Indeed, he suggested all the results of section 3. The author wishes to thank the referee for this and also for helpful suggestions concerning matters of exposition.

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2. Basic result.

LEMMA 1. *Suppose* $\mathbf{\Phi}$ *is a filter on* $R(E)$, $\alpha \in R(E)$ *and* $\mathbf{\Phi} \to \alpha$. *Then* $K(\mathbf{\Phi}) = \alpha$ *if* α *is nonconvergent and* $K(\mathbf{\Phi}) \to x$ *if* $\alpha = \dot{x}$.

Proof. First suppose α is nonconvergent. If $A \in \alpha$ then, by definition of convergence in $R(E)$, $H \subset \hat{A}$ for some $H \in \mathbf{\Phi}$. If f is defined on H by $f(h) = A$ for each $h \in H$, then

$$H(f) = \cup\ (f(h) : h \in H) = A \in K(\mathbf{\Phi}).$$

So $\alpha \leqq K(\mathbf{\Phi})$ and equality follows because α is an ultrafilter.

But if $\alpha = \dot{x}$ then $\mathbf{\Phi} \geqq \hat{\gamma}$ for some γ, $\gamma \to x$, and the same argument as above shows that $K(\mathbf{\Phi}) \geqq \gamma$.

LEMMA 2. *If* $A \subset E$, E *a Hausdorff space, then* $\text{cl}\ (\hat{A}, R(E)) \subset (\text{cl}\ (A, E))\hat{}$.

Proof. Let $\alpha \in \text{cl}\ (\hat{A}, R(E))$ and suppose first that α is nonconvergent. Then there is a filter $\mathbf{\Phi}$ with $\mathbf{\Phi} \to \alpha$ and $\hat{A} \in \mathbf{\Phi}$. By Lemma 1, $K(\mathbf{\Phi}) = \alpha$ so $\hat{A}(f) \in K(\mathbf{\Phi}) = \alpha$, where f is defined on $\hat{A}$ by $f(h) = A$ for $h \in \hat{A}$. Thus, $A = \hat{A}(f) \in \alpha$. It follows that $\alpha \in \hat{A}$ so $\alpha \in (\text{cl}\ (A, E))\hat{}$.

Now suppose that $\alpha = \dot{x} \in \text{cl}\ (\hat{A}, R(E))$. There is a filter $\mathbf{\Phi}$ with $\mathbf{\Phi} \to \dot{x}$ and $\hat{A} \in \mathbf{\Phi}$. By Lemma 1, $K(\mathbf{\Phi}) \to x$ and $\hat{A} \in \mathbf{\Phi}$. With f defined on $\hat{A}$ by $f(h) = A$ for $h \in \hat{A}$, we have $A = \hat{A}(f) \in K(\mathbf{\Phi})$, $K(\mathbf{\Phi}) \to x$. This means $x \in \text{cl}\ (A, E)$ and it follows that $\dot{x} \in (\text{cl}\ (A, E))\hat{}$.

THEOREM 1. *Let* E *be a regular Hausdorff space. Then* $R(E)$ *is regular if and only if* $\alpha = \text{cl}\ (\alpha)$ *for each nonconvergent ultrafilter* α *on* E.

Proof. Suppose that $R(E)$ is regular, that i is the embedding of E into $R(E)$ and that α is a nonconvergent ultrafilter on E. Then $i(\alpha) \to \alpha$ hence $\text{cl}\ (i(\alpha), R(E)) \to \alpha$. So if $A \in \alpha$, $\text{cl}\ (i(B), R(E)) \subset \hat{A}$ for some $B \in \alpha$. We claim that $\text{cl}\ (B, E) \subset A$. For, if $x \in \text{cl}\ (B, E)$, there is a filter γ with $\gamma \to x$ and $B \in \gamma$. Then $i(\gamma) \to i(x)$, $i(B) \in i(B) \in i(\gamma)$, so $\dot{x} = i(x) \in \text{cl}\ (i(B), R(E)) \subset \hat{A}$. It follows from this that $x \in A$ and our claim is true. Thus, $\text{cl}\ (\alpha, E) \geqq \alpha$ and $\text{cl}\ (\alpha, E) = \alpha$ follows.

Conversely, suppose that the closure of each nonconvergent ultrafilter on E is itself. Let $\mathbf{\Phi} \to \alpha$ in $R(E)$.

Case 1. α is nonconvergent. Then $\mathbf{\Phi} \geqq \hat{\alpha}$. By Lemma 2 $(\text{cl}\ (\alpha, E))\hat{} \leqq \text{cl}\ (\hat{\alpha}, R(E))$ and, by assumption, $\alpha = \text{cl}\ (\alpha)$ so $\text{cl}\ (\mathbf{\Phi}, R(E)) \geqq \text{cl}\ (\hat{\alpha}, \mathrm{R}(E)) \geqq (\text{cl}\ (\alpha, E))\hat{} = \hat{\alpha}$. From this $\text{cl}\ (\mathbf{\Phi}) \to \alpha$.

Case 2. $\alpha = \dot{x}$. Then $\mathbf{\Phi} \geqq \hat{\gamma}$ for some γ, $\gamma \to x$. Since E is regular, $\text{cl}\ (\gamma, E) \to x$ and, using Lemma 2 again, $\text{cl}\ (\mathbf{\Phi}, R(E)) \geqq \text{cl}\ (\hat{\gamma}, R(E)) \geqq (\text{cl}\ (\gamma, E))\hat{}$. Thus $\text{cl}\ (\mathbf{\Phi}, R(E)) \to \dot{x}$.

In each case it has been shown that $\text{cl}\ (\mathbf{\Phi}) \to \alpha$ whenever $\mathbf{\Phi} \to \alpha$ so $R(E)$ is regular.

COROLLARY 1. *Let E be a regular, principal, Hausdorff space in which $\alpha =$ cl (α) for each nonconvergent ultrafilter α on E. Then E is a completely regular topological space.*

Proof. From results of [7] and Theorem 1, $R(E)$ is Hausdorff, regular and compact. Clearly $R(E)$ is also principal so it is an immediate consequence of [**6**, Proposition 1] that $R(E)$ is a completely regular topological space. Since E is embedded in $R(E)$, E is also completely regular.

COROLLARY 2. *Let E be a Tychonoff topological space. Then, a necessary and sufficient condition that $R(E) = \beta(E)$ is that $\alpha =$ cl (α) for each nonconvergent ultrafilter α on E.*

Proof. If E is Tychonoff and $\beta(E)$ is homeomorphic to $R(E)$, then $R(E)$ is regular and $\alpha =$ cl (α) on nonconvergent ultrafilters by Theorem 1.

If $\alpha =$ cl (α) on nonconvergent ultrafilters, then $R(E)$ is regular, compact, principal and Hausdorff so, by Corollary 1, $R(E)$ is a topological, Hausdorff, Stone-Čech type compactification of E. Thus, $R(E)$ is homeomorphic to $\beta(E)$.

Remark. It is clear from results in [**4**] that "$\alpha =$ cl (α) on nonconvergent ultrafilters" agrees with compactness on minimal regular spaces. Any discrete space shows that this condition and compactness need not agree in general, even on completely regular topological spaces. Nondiscrete examples appear in section 3.

3. Consequences for topological E.

The first result below generates nondiscrete, noncompact E for which $R(E) = \beta(E)$. (For example, by starting with E discrete.)

THEOREM 2. *Suppose $R(E) = \beta(E)$. If $F = E \cup A$ is an extension of E in $\beta(E)$ with $A \subset \beta(E) - E$, A closed in $\beta(E)$, then $R(F) = \beta(F)$.*

Proof. Let $\mathbf{\Phi}$ be a nonconvergent ultrafilter on $E \cup A$ and let $B \in \mathbf{\Phi}$. Then $A \notin \mathbf{\Phi}$ so the restriction $\mathbf{\Phi}(E)$ of $\mathbf{\Phi}$ to E exists as a nonconvergent ultrafilter on E. By assumption

(1) cl $(W, E) \subset B - A$ for some $W \in \Phi(E)$.

Now $\mathbf{\Phi}(E)$ converges in $\beta(E)$ to some p, $p \notin A$. Hence, if $q \in A$ then q cannot be in the closure of every V, $V \subset W$, $V \in \mathbf{\Phi}(E)$. So, when $q \in A$, there exists $V(q) \in \mathbf{\Phi}(E)$ such that $q \notin$ cl $(V(q), R(E))$. By compactness of A in $\beta(E)$ there is a finite subset Q of A such that

$$\bigcap \text{ (cl V}((q), R(E) : q \in Q) \subset \beta(E) - A.$$

Then $U = \bigcap \, (V(q) : q \in Q) \in \mathbf{\Phi}(E)$ so

(2) There exists $U \in \mathbf{\Phi}(E)$ such that $U \subset W$ and cl $(U, E \cup A) \subset$ cl (U, E).

Finally, $U \cup A \in \mathbf{\Phi}$ and, using (1), (2),

$$\begin{aligned} \text{cl}\ (U \cup A, E \cup A) = \text{cl}\ (U, E \cup A) \cup A \subset \text{cl}\ (U, E) \cup A \\ \subset (B - A) \cup A = B. \end{aligned}$$

It has been shown that cl $(\mathbf{\Phi}, E \cup A) = \mathbf{\Phi}$ so $\beta(E \cup A) = R(E \cup A)$.

THEOREM 3. *If $R(E) = \beta(E)$ then E is normal and $\beta(E) = W(E)$, $W(E)$ the Wallman compactification of E.*

Proof. The result follows if $W(E)$ is Hausdorff. To show this, notice first that when $\lambda \in W(E)$, then $[\lambda] \in R(E)$, $[\lambda]$ the filter on E generated by the maximal closed filter λ. Next, the map $i : E \to R(E)$, $i(x) = \dot{x}$, extends to a continuous $f : W(E) \to R(E)$ by defining, for $\lambda \in W(E) - E$, $f(\lambda)$ to be the unique cluster point of the closed filter

$$\mathbf{\Psi}(\lambda) = (A : A \text{ closed in } R(E), i^{-1}(A) \in \lambda).$$

However $\mathbf{\Psi}(\lambda) \to [\lambda]$ in $R(E)$ because, if $F \in \lambda$,

$$\text{cl}\ (i(F), R(E)) \subset \text{cl}\ (\hat{F}, R(E)) \subset (\text{cl}\ [F, E))\hat{} = \hat{F}.$$

Hence $f(\lambda) = [\lambda]$, thus f is one-to-one, and since $R(E)$ is Hausdorff the same holds for $W(E)$.

THEOREM 4. *If $R(E) = \beta(E)$ then $\beta(E)$ is equivalent to $\alpha(E)$, $\alpha(E)$ the Fomin H-closed extension of E.*

Proof. According to the [**3**, Corollary, p. 245] we must show that bdry (G) is compact whenever G is open in E. Suppose this is not so for some open G. Then there exists a non-convergent ultrafilter α which contains bdry (G). By assumption $\alpha \in R(E) = \beta(E)$. Considering $\beta(E)$ as the space of maximal completely regular filters as in [**1**], α is generated by a maximal completely regular filter on E. So α has an open base. But this means bdry (G) contains a non void open set, a contradiction.

Concluding remarks. The class $\mathscr{C}$ of spaces E for which $R(E) = \beta(E)$ is not finitely productive. For, by Theorem 2, $N \cup \{p\} \in \mathscr{C}$, $p \in \beta(M) - N$, N the natural numbers. But the boundary of $N \times N$ in $N \times (N \cup \{p\})$ is not compact so $N \times (N \cup \{p\}) \notin \mathscr{C}$ by Theorem 4. Moreover, if E is any non-compact member of $\mathscr{C}$, then some power X of E is not normal (see [**5**]) so $X \notin \mathscr{C}$ by Theorem 3.

The class $\mathscr{C}$ is not hereditary: If N is the natural numbers and A is an infinite closed subset of $\beta(N)$, $A \subset \beta(N) - N$, then $N \cup A \in \mathscr{C}$. Define $F = N \cup A - \{p\}$, where p is a point of A which is the limit of a filter in $A - \{p\}$. As a subspace of $N \cup A$, $F \notin \mathscr{C}$ because the boundary of N in F is not compact. Closed subspaces of members of $\mathscr{C}$ are in $\mathscr{C}$, but a discrete space E is not closed in $E \cup \{p\}$ and yet $E \in \mathscr{C}$.

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