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RESEARCH ARTICLE

Geometric structures for maximal representations and pencils

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Abstract
We study fibrations of the projective model for the symmetric space associated with $\mathrm{SL}(2n, \mathbb{R})$ by codimension 2 projective subspaces, or pencils of quadrics. In particular we show that if such a smooth fibration is equivariant with respect to a representation of a closed surface group, the representation is quasi-isometrically embedded, and even Anosov if the pencils in the image contain only nondegenerate quadrics. We use this to characterize maximal representations among representations of a closed surface group into $\mathrm{Sp}(2n, \mathbb{R})$ by the existence of an equivariant continuous fibration of the associated symmetric space, satisfying an additional technical property. These fibrations extend to fibrations of the projective structures associated to maximal representations by bases of pencils of quadrics.

Contents

1. Introduction

In this paper we study representations of the fundamental group Γ_g of a closed oriented surface of genus $g \geq 2$ into semi-simple Lie groups. Our goal is to characterize a special class of representations into $\mathrm{Sp}(2n, \mathbb{R})$, *maximal representations*, by the existence of an equivariant fibration of the projective model for the symmetric space associated to $\mathrm{SL}(2n, \mathbb{R})$, restricting to a fibration of the symmetric space associated to $\mathrm{Sp}(2n, \mathbb{R})$. To illustrate these ideas we begin by discussing quasi-Fuchsian representations in $\mathrm{SL}(2, \mathbb{C})$.

1.1. Fibrations of $\mathbb{H}^3$ by geodesics

Let $\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{C})$ be the composition of a Fuchsian, that is, discrete and faithful, representation and the inclusion $\mathrm{SL}(2, \mathbb{R}) \subset \mathrm{SL}(2, \mathbb{C})$. The locally symmetric space $\mathbb{H}^3/\rho(\Gamma_g)$ is a fiber bundle over S_g whose fibers are geodesics. One can construct such a fibration by taking the geodesics orthogonal to the totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^3$ preserved by the action of $\mathrm{SL}(2, \mathbb{R})$. This fibration extends to a fibration of an open domain in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.

Such a fibration is described by ρ-equivariant map $u : \widetilde{S_g} \to \mathcal{G}$ where $\mathcal{G}$ is the space of geodesics in $\mathbb{H}^3$. We say that an immersion $u : \widetilde{S_g} \to \mathcal{G}$ is *fitting* if the corresponding geodesics locally define a smooth fibration of $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.

Let $\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{C})$ be *nearly Fuchsian*, that is, suppose that it admits an equivariant immersion $h : \widetilde{S_g} \to \mathbb{H}^3$ with principal curvature in $(-1, 1)$. Epstein showed that the locally symmetric space $\mathbb{H}^3/\rho(\Gamma_g)$ admits a fibration described by the fitting immersion $\mathcal{G}h$ that associates to $x \in \widetilde{S_g}$ the geodesic orthogonal to $h(\widetilde{S_g})$ at $h(x)$ [Eps86]. The map $\mathcal{G}h$ is sometimes referred to as the *Gauss map* of h.

Nearly Fuchsian representations are quasi-Fuchsian, that is, are quasi-isometric embeddings [Eps86]. We generalize this fact to any representation that admits an equivariant fitting immersion.

Theorem 1.1. *Let $\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{C})$ be a representation that admits an equivariant fitting immersion $u : \widetilde{S_g} \to \mathcal{G}$. The representation ρ is quasi-Fuchsian.*

This theorem is a consequence of Theorem 5.4. There may a priori exist representations with equivariant fitting maps that are not nearly Fuchsian, see Remark A.4. However, Theorem 1.1 does not provide a characterization of quasi-Fuchsian representations in general because of the following result.

Theorem 1.2 (Theorem A.5). *For a genus g large enough, there exist quasi-Fuchsian representations $\rho : \Gamma_g \to S_g$ that admit no equivariant fitting immersions $u : \widetilde{S_g} \to \mathcal{G}$.*

As a corollary we prove that there exist quasi-Fuchsian representations that are not nearly Fuchsian.

1.2. Fibrations of a convex set

Let $V = \mathbb{R}^{2n}$. We denote by S^2V be the space of symmetric tensors in $V \otimes V$, or equivalently the space of symmetric bilinear forms on V^*. Let $S^2V^{>0}$ be the convex cone of positive tensors, that is, tensors that define positive symmetric bilinear forms on V^*.

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The projective convex domain $\mathbb{P}(S^2V^{>0})$ is the projective model for the symmetric space $\mathbb{X}_{\mathrm{SL}} = \mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2n,\mathbb{R})$. We study fibrations of this convex domain by projective codimension 2 subspaces.

A codimension 2 projective subspace of $\mathbb{P}(S^2V)$ corresponds to a dimension 2 subspace of the dual vector space S^2V^*, that can be interpreted as the space of *quadrics* on V. We will write $\mathcal{Q} = S^2V^*$. A plane in $\mathcal{Q}$ is called a *pencil of quadrics* and we will denote by $\mathrm{Gr}_2(\mathcal{Q})$ the space of such planes. Let $\mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ be the set of *mixed* pencils, that is, pencils P whose corresponding codimension 2 projective subspace intersects nontrivially the convex domain $\mathbb{P}(S^2V^{>0})$, or equivalently the pencils that do not contain any nonzero semi-positive quadric (in other words the signature of all quadrics of the pencil is mixed).

In this setting, in a way that is analog as for immersions in the space $\mathcal{G}$ of geodesics in $\mathbb{H}^3$, we say that an immersion $u : S \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ from a surface S is *fitting* is the corresponding codimension two subsets define locally a smooth fibration of the closure $\mathbb{P}(S^2V^{\geq 0})$ of the convex domain $\mathbb{P}(S^2V^{>0})$. As in $\mathbb{H}^3$ one can construct examples of such maps by taking the Gauss map of some totally geodesic surfaces in $\mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2n,\mathbb{R})$, see Proposition 7.1.

Let $\rho : \Gamma \to \mathrm{SL}(2n,\mathbb{R})$ be a representation and $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ a ρ-equivariant fitting immersion, then u is an embedding and defines a fibration of an open domain in $\mathbb{P}(S^2V^{\geq 0})$ that contains $\mathbb{P}(S^2V^{>0})$, see Proposition 5.5.

Moreover let $\mathrm{Gr}_2^{(n,n)}(\mathcal{Q}) \subset \mathrm{Gr}_2(\mathcal{Q})$ be the set of pencils of quadrics P such that every non zero $q \in P$ has signature (n,n), or equivalently the pencils containing no degenerate nonzero quadric.

Theorem 1.3. *Let $\rho : \Gamma_g \to \mathrm{SL}(2n,\mathbb{R})$ be a representation that admits an equivariant fitting immersion $u : \widetilde{N} \to Gr_2^{(n,n)}(\mathcal{Q})$. The representation ρ is $\{n\}$-Anosov.*

This theorem can be generalized by replacing S_g by a closed manifold of some dimension d and considering equivariant maps into $\mathrm{Gr}_d(\mathcal{Q})$, see Theorem 5.4.

In order to show this result we introduce the notion of a fitting flow. Let $\mathcal{E}$ be the tautological rank 2 vector bundle over $\mathrm{Gr}_2(\mathcal{Q})$, that is,

$$\mathcal{E} = \{(P,q) \mid P \in \mathrm{Gr}_2(\mathcal{Q}),\ q \in P\}.$$

Let $\pi : \mathcal{E} \to \mathcal{Q}$ be the tautological projection, that is, the projection onto the second factor. The pullback $u^*\mathbb{S}\mathcal{E}$ of the circle bundle $\mathbb{S}\mathcal{E}$ for a map $u : S \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ is the space of pairs (x,H) where $x \in S$ and H is a co-oriented projective hyperplane in $\mathbb{P}(S^2V)$ containing the codimension 2 projective subspace associated to $u(x)$. Note that a co-oriented projective hyperplane intersecting $\mathbb{P}(S^2V^{>0})$ defines a half-space in the convex set $\mathbb{P}(S^2V^{\geq 0})$

A *fitting flow* for the map $u : S \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ is a flow Φ on this circle bundle $u^*\mathbb{S}\mathcal{E}$ over S such that along flow lines the corresponding projective half-spaces in $\mathbb{P}(S^2V^{\geq 0})$ are nested, see Definition 4.1.

We show that a fitting immersion always admits a smooth fitting flow, see Proposition 4.8, that flow lines must exist between the fibers over any pair of points in S and must be quasi-geodesic, see Proposition 4.5. Finally we use the nestedness of the half-spaces when following a flow line to show uniform contraction occuring in projective space that is sufficient to imply that the representation is Anosov, using an argument similar to the criterion on nested multicones from Bochi-Potrie-Sambarino [BPS19].

1.3. Maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$

Let us now focus on representations into the group $\mathrm{Sp}(2n,\mathbb{R})$ of linear transformations of $\mathbb{R}^{2n}$ preserving a symplectic form.

Maximal representations of a closed surface group in $\mathrm{Sp}(2n,\mathbb{R})$ are representation whose Toledo invariant takes its maximal possible value [BIW11]. Equivalently maximal representations can be characterized as *positive representations* [BIW03], or again equivalently as representations that admit

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an equivariant continuous and transverse map $\xi_\rho^n : \partial\Gamma \to \mathcal{L}_n$ having a specific homotopy type, with $\mathcal{L}_n$ the space of Lagrangians in $(\mathbb{R}^{2n}, \omega)$, see Theorem 2.5.

Maximal representations form connected components of the space of representations, and such representations are all discrete and faithful [BIW03]. The space of maximal representations is therefore called a *higher rank Teichmüller space*. The classical Teichmüller space as well as some higher rank Teichmüller spaces, can be interpreted as spaces of geometric structures. For instance, Hitchin representations in $\mathrm{SL}(3, \mathbb{R})$ can be interpreted as spaces of convex projective structures on the considered surface. Maximal representations in $\mathrm{Sp}(4, \mathbb{R})$ or $\mathrm{SO}_o(2, n)$ can be interpreted as spaces of projective or photon structures on a bundle over the surface that admit a special fibration [CTT19]. In these two examples these structures modeled on flag manifolds can be interpreted as a part of a natural compactification of the locally symmetric space associated with the representation. In the present paper we focus on the locally symmetric structures and ask the following:

Question 1.4. Can maximal representations be characterized by the existence of some fibration of the associated locally symmetric space?

We provide some affirmative answer to this question. In order to study the symmetric space for $\mathrm{Sp}(2n, \mathbb{R})$ we embed it into the projective model for the symmetric space of $\mathrm{SL}(2n, \mathbb{R})$.

1.4. Characterization of maximal representations

The symmetric space $\mathbb{X}_{\mathrm{Sp}}$ associated to $\mathrm{Sp}(2n, \mathbb{R})$ can be identified with a totally geodesic submanifold of $\mathbb{X}_{\mathrm{SL}}$. Let $\mathrm{Gr}_2^{\omega}(\mathcal{Q}) \subset \mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$ be the set of pencils P such that every nonzero $q \in P$ is positive on some Lagrangian and negative on some other Lagrangian in $\mathbb{R}^{2n}$. We show that the projectivization of the codimension 2 projective subspace of $\mathbb{P}(S^2V)$ corresponding to any $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ intersects transversely the totally geodesic symmetric subspace $\mathbb{X}_{\mathrm{Sp}} \subset \mathbb{P}(S^2V^{>0})$, see Lemma 7.4.

The set $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ is open in $\mathrm{Gr}_2(\mathcal{Q})$, but we show that it is disconnected. We select a special union of connected components that we denote by $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ and we show the following:

Theorem 1.5 (Theorem 6.5). *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a representation. If it admits a ρ-equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ it is maximal for some orientation of S_g.*

If a representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ admits an equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$ the image of u lies necessarily in $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$. We then use that the homotopy type of the boundary map of ρ is determined by the connected component of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ in which the image of u lies to define $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ and to prove this theorem, see Section 6.1.

For $n = 2$ using results of [CTT19] we show a converse to this statement, which therefore provides a characterization of maximal representations, see Corollary B.5. We compare this characterization with the one from Collier-Tholozan-Toulisse in Remark B.6. For $n \geq 3$ we prove a weaker converse to this theorem. We construct equivariant maps $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ for maximal representations that locally define a fibration of $\mathbb{P}(S^2V^{\geq 0})$ but are only continuous and not smooth.

Theorem 1.6 (Theorem 6.5). *A representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ is maximal if and only if it admits a ρ-equivariant continuous map of pencils that admits an equivariant fitting flow:*

$$u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q}).$$

A continuous equivariant map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ for a representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ with a fitting flow defines a ρ-equivariant continuous fibration of $\mathbb{P}(S^2V^{\geq 0})$, but also of $\mathbb{X}_{\mathrm{Sp}} \subset \mathbb{P}(S^2V^{>0})$ as $\mathbb{X}_{\mathrm{Sp}}$ intersects the fibers transversely. Thus u defines a continuous fibration of the locally symmetric space $\mathbb{X}_{\mathrm{Sp}}/\rho(\Gamma_g)$.

The space of rank one elements of $\mathbb{P}(S^2V^{\geq 0})$ can naturally be identified with $\mathbb{P}(V)$. Such a map u defines a continuous ρ-equivariant fibration of a domain in projective space, that is equal to the domain of discontinuity in projective space constructed by Guichard-Wienhard [GW12]. The quotient of this domain inherits a $(\mathrm{Sp}(2n,\mathbb{R}),\mathbb{RP}^{2n-1})$-structure, a contact projective structure. Theorem 1.6 implies the following characterization of the contact projective structures corresponding to maximal representations.

Corollary 1.7. *A contact projective structure on a fiber bundle M with fiber F over S_g corresponds to a maximal representation by the construction of Guichard-Wienhard if and only if, up to homeomorphisms of M that stabilize $\pi_1(F)$ and act trivially on $\pi_1(M)/\pi_1(F)\simeq\Gamma_g$, the fibers are mapped via the developing map onto the bases of maximal pencils of quadrics parametrized by a continuous map that admits an equivariant fitting flow.*

In order to construct continuous maps that admits a fitting flow for maximal representations, we show how to associate to a pair of transverse Lagrangians a quadric on $\mathbb{R}^{2n}$.

1.5. Organization of the paper

This paper begins with a recall of some facts about maximal and Anosov representations in Section 2.

The main definitions are introduced in Section 3 where fibrations of a projective convex set by projective subspaces are discussed, more precisely fibrations of the projective model for the symmetric space of $\mathrm{SL}(2n,\mathbb{R})$.

In Section 4 we introduce the notion of fitting flows.

In Section 5 we discuss how the existence of an equivariant continuous map with a fitting flow implies the Anosov property, Theorem 5.4, and describes a fibration of a domain of discontinuity in projective space.

In Section 6 we focus on representations into $\mathrm{Sp}(2n,\mathbb{R})$ and prove Theorem 6.5, which is the characterization of maximal representations by the existence of a locally fitting map of maximal pencils of quadrics that admits a fitting flow.

In Section 7 we prove two propositions relative to the symmetric space of $\mathrm{SL}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R})$.

In Appendix A we discuss fibrations of the hyperbolic 3-space with geodesic fibers.

In Appendix B we show how spacelike surfaces in $\mathbb{H}^{2,2}$ with a bound on their principal curvatures define a fitting immersion of pencils.

2. Maximal and Anosov representations

2.1. Maximal representations

Let us fix a symplectic form ω on $\mathbb{R}^{2n}$, that is, a nondegenerated bilinear antisymmetric pairing. A *symplectic basis* of $\mathbb{R}^{2n}$ is a basis $(x_1,\cdots,x_n,y_1,\cdots,y_n)$ in which:

$$\omega=\sum_{i=1}^{n}x_i^*\wedge y_i^*.$$

We define $\mathrm{Gr}_n(\mathbb{R}^{2n})$ as the space of n-dimensional subspaces of $\mathbb{R}^{2n}$. A *Lagrangian* in $(\mathbb{R}^{2n},\omega)$ is an element $\ell\in\mathrm{Gr}_n(\mathbb{R}^{2n})$ such that ω restricted to ℓ is equal to zero. We denote by $\mathcal{L}_n$ the space of Lagrangians in $(\mathbb{R}^{2n},\omega)$. We say that two Lagrangians are transverse if their intersection is trivial.

Let $\mathrm{Sp}(2n,\mathbb{R})$ be the subgroup of elements in $\mathrm{SL}(2n,\mathbb{R})$ that preserves ω. This group acts transitively on $\mathcal{L}_n$, as well as on the space of pairs of transverse Lagrangians. Given a triple (ℓ_1,ℓ_2,ℓ_3) of transverse Lagrangians, one can find a symplectic basis such that for some $(\epsilon_i)\in\{1,-1\}$:

$$\ell_1=\langle x_1,x_2,\cdots,x_n\rangle,$$

$$\ell_2=\langle x_1+\epsilon_1y_1,x_2+\epsilon_2y_2,\cdots,x_n+\epsilon_ny_n\rangle,$$

$$\ell_3=\langle y_1,y_2,\cdots,y_n\rangle.$$

The sum of the (ϵ_i) is an invariant of the triple of flags that is called the *Maslov index* $M(\ell_1, \ell_2, \ell_3)$. These facts can be found in [LV13, Section 1.5.7]. The group $\mathrm{Sp}(2n, \mathbb{R})$ acts transitively on the space of triples of transverse Lagrangians with a given Maslov index. We say that (ℓ_1, ℓ_2, ℓ_3) is *maximal* if the Maslov index of the triple is equal to n.

The Lie group $\mathrm{Sp}(2n, \mathbb{R})$ is of Hermitian type and tube type. Hence it admits a special class in its continuous cohomology group $[\tau] \in H^2_c(\mathrm{Sp}(2n, \mathbb{R}), \mathbb{Z})$. Let S_g be a closed oriented surface of genus $g \geq 2$. The fundamental class of S_g defines a cohomology class $[S_g] \in H^2(\pi_1(S_g), \mathbb{Z}) \simeq \mathbb{Z}$. Given a representation $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n, \mathbb{R})$ one can consider the pullback of this class $\rho^*[\tau] = T(\rho)[S_g]$. The integer $T(\rho)$ is called the *Toledo number* of ρ.

The Toledo number can take only finitely many values as the space of representations can only have finitely many connected components. More precisely:

Lemma 2.1 [BIW11]. *Let* $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n, \mathbb{R})$*, the Toledo number satisfies:*

$$(-2g + 2)n \leq T(\rho) \leq (2g - 2)n.$$

Such a representation is called *maximal* if its Toledo number is equal to $(2g - 2)n$.

2.2. Anosov representations

Let Γ be a finitely generated group. Anosov representations are representations with some exponential gaps between singular values.

Fix a word metric $|\cdot|$ on Γ and a scalar product on $\mathbb{R}^{2n}$ allowing us to define the singular values $\sigma_1(g) \geq \sigma_2(g) \geq \cdots \geq \sigma_{2n}(g)$ of $g \in \mathrm{SL}(2n, \mathbb{R})$ as the eigenvalues of $\sqrt{g^t g}$. The following definition is independent of these choices.

Definition 2.2 [BPS19]. We say that a representation $\rho : \Gamma \to \mathrm{SL}(2n, \mathbb{R})$ is $\{n\}$-Anosov if there exist $A, B > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_n(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))} \geq e^{A|\gamma|+B}.$$

If a group admits an Anosov representation, it must be Gromov hyperbolic [BPS19]. We denote by $\partial\Gamma$ its Gromov boundary. Anosov representations come with boundary maps.

Theorem 2.3 [GGKW17, Theorem 1.1]. *Let* $\rho : \Gamma \to \mathrm{SL}(2n, \mathbb{R})$ *be* $\{n\}$*-Anosov. There exists a unique* ρ*-equivariant continuous map* $\xi^n_\rho : \partial\Gamma \to \mathrm{Gr}_n(\mathbb{R}^{2n})$ *such that:*

- *for all distinct* $x, y \in \partial\Gamma$, $\xi^n_\rho(x) \oplus \xi^n_\rho(y) = \mathbb{R}^{2n}$ *(transverse),*
- *for all* $\gamma \in \Gamma$ *that admit an attracting fixed point* $\gamma^+ \in \partial\Gamma$, $\xi^n_\rho(\gamma^+)$ *is the attracting fixed point of the action of* $\rho(\gamma)$ *on* $\mathrm{Gr}_n(\mathbb{R}^{2n})$ *(dynamic preserving),*

If moreover $\rho(\Gamma) \subset \mathrm{Sp}(2n, \mathbb{R})$*, then* $\xi^n_\rho(x)$ *is a Lagrangian for all* $x \in \partial\Gamma$*.*

The fact that $\xi^n_\rho(x)$ is a Lagrangian is a consequence of the fact that an attracting fixed n-dimensional subspace for an element $g \in \mathrm{Sp}(2n, \mathbb{R})$ is necessarily Lagrangian, and every $x \in \partial\Gamma$ is a limit of attracting fixed points γ^+ of elements $\gamma \in \Gamma$.

Maximal representations have been characterized in [BILW05], [BIW03]:

Theorem 2.4. *A representation* $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n, \mathbb{R})$ *is maximal if and only if it is* $\{n\}$*-Anosov and for one and hence any positively oriented triple* $(x, y, z) \in \partial\pi_1(S)$ *the triple* $(\xi^n_\rho(x), \xi^n_\rho(y), \xi^n_\rho(z))$ *is a maximal triple of Lagrangians.*

One can also characterize maximal representations among $\{n\}$-Anosov representations by looking at the homotopy type of their boundary map. The fundamental group of the space of Lagrangians $\mathcal{L}_n$ is isomorphic to $\mathbb{Z}$ [Wig98] where a generator is:

$$\tau : \theta \in \mathbb{S}^1 \mapsto \langle \cos\left(\frac{\theta}{2}\right)x_1 + \sin\left(\frac{\theta}{2}\right)y_1, x_2, \cdots, x_n\rangle \in \mathcal{L}_n.$$

Theorem 2.5. *A representation $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n, \mathbb{R})$ is maximal if and only if it is $\{n\}$-Anosov and the free homotopy type of the curve ξ^n_ρ is equal to $n[\tau]$.*

Remark 2.6. In Theorem 2.4 and hence also in Theorem 2.5, one can relax the assumption that ρ is $\{n\}$-Anosov. In [BIW03, Theorem 8] it is shown that it is sufficient to assume that ρ admits an equivariant, continuous and transverse map $\xi^n_\rho : \partial\Gamma \to \mathcal{L}_n$ satisfying the additional property of sending positive triples to maximal triples.

Proof. Let $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n, \mathbb{R})$ be $\{n\}$-Anosov. Let (x, y, z) be a positively oriented triple in $\partial\pi_1(S_g)$. Up to changing the symplectic basis, we can assume that for some $(\epsilon_i) \in \{-1, 1\}$:

$$\xi^n_\rho(x) = \langle x_1, x_2, \cdots, x_n\rangle,$$
$$\xi^n_\rho(y) = \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 y_2, \cdots, x_n + \epsilon_n y_n\rangle,$$
$$\xi^n_\rho(z) = \langle y_1, y_2, \cdots, y_n\rangle.$$

Here the Maslov index of the triple $(\xi^n_\rho(x), \xi^n_\rho(y), \xi^n_\rho(z))$ is equal to the sum of the (ϵ_i).
Consider the following curve:

$$\tau_0 : \theta \mapsto \langle \cos\left(\frac{\theta}{2}\right)x_1 + \epsilon_1 \sin\left(\frac{\theta}{2}\right)y_1, \cdots, \cos\left(\frac{\theta}{2}\right)x_n + \epsilon_n \sin\left(\frac{\theta}{2}\right)y_n\rangle.$$

This loop is homotopic to the concatenation of the loops τ_i for $1 \leq i \leq n$:

$$\tau_i : \theta \mapsto \langle x_1, x_2, \cdots, \cos\left(\frac{\theta}{2}\right)x_i + \epsilon_i \sin\left(\frac{\theta}{2}\right)y_i, \cdots, x_n\rangle.$$

These loops are homotopic to τ or its inverse depending on the sign of ϵ_i. The homotopy type of τ_0 is hence equal to $(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)[\tau]$. Moreover the set of Lagrangians transverse to a given Lagrangian is contractible, so one can homotope τ_0 on the intervals $[0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \pi]$ and $[\pi, 2\pi]$ to coincide with ξ^n_ρ. Hence the free homotopy class of ξ^n_ρ is equal to the one of τ_0, which is equal to $M(\xi^n_\rho(x), \xi^n_\rho(y), \xi^n_\rho(z))[\tau]$. We therefore deduce that ρ is maximal if and only if it is $\{n\}$-Anosov and $[\xi^n_\rho] = n[\tau]$. □

3. An invariant convex domain and its fibrations

In this section we define globally fitting maps and fitting immersions, which are maps that parametrize fibrations of the projective model for the symmetric space of $\mathrm{SL}(2n, \mathbb{R})$ by projective subspaces of codimension d.

3.1. Pencils of quadrics

Let V be a finite even-dimensional vector space with a fixed volume form. Let S^2V be the space of symmetric bilinear tensors on V, which we interpret as maps $V^* \to V$. The dual space $\mathcal{Q} = S^2V^*$ is the space of symmetric bilinear forms on V, or the space of *quadrics* on V, that we interpret as maps $V \to V^*$.

We denote by $S^2V^{\geq 0}$ and $S^2V^{>0}$ respectively the space of semi-positive and positive symmetric tensors, that is, elements $s \in S^2V$ that define respectively a semi-positive and positive symmetric bilinear form on V^*. The Lie group $\mathrm{SL}(V)$ acts on S^2V, and preserves the properly convex set $\mathbb{P}(S^2V^{\geq 0})$. The convex domain $\mathbb{P}(S^2V^{>0})$ is a projective model for the symmetric space associated to $\mathrm{SL}(V)$.

The Grassmannian of d-dimensional linear subspaces of $\mathcal{Q}$ will be denoted by $\mathrm{Gr}_d(\mathcal{Q})$. An element of $\mathrm{Gr}_2(\mathcal{Q})$ is usually called a *pencil of quadrics* on V. We will here also call elements of $\mathrm{Gr}_d(\mathcal{Q})$ pencils of quadrics.

To an element $P \in \mathrm{Gr}_d(\mathcal{Q})$ one can associate its annihilator, or dual, codimension d subspace $P^\circ \subset S^2V$. This dual space can be described as the space of symmetric tensors $s \in S^2V$ on which one has $q(s) = \mathrm{Tr}(q \circ s) = 0$ for all $q \in P$. Note that the projectivization $\mathbb{P}(P^\circ)$ also has codimension d in $\mathbb{P}(S^2V)$.

The subspace $\mathbb{P}(P^\circ)$ does not necessarily intersect the convex $\mathbb{P}(S^2V^{>0})$.

Definition 3.1. We say that a pencil $P \in \mathrm{Gr}_d(\mathcal{Q})$ is *mixed* if P° contains a positive element, that is, if $P^\circ \cap S^2V^{>0} \neq \{0\}$. We call the set of mixed pencils $\mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$.

Proposition 3.2. *A pencil $P \in \mathrm{Gr}_d(\mathcal{Q})$ is mixed if and only if there is no semi-positive quadric $0 \neq q \in P$.*

Morally this proposition is a consequence of the fact that the dual of the cone of positive elements $S^2V^{>0}$ is the cone of semi-positive bilinear forms in $\mathcal{Q}$, in the sense that $S^2V^{>0}$ is the set of tensors s such that $q(s) > 0$ for all semi-positive $q \in \mathcal{Q}$.

Proof. Fix $P \in \mathrm{Gr}_d(\mathcal{Q})$ and suppose that there is a positive element $s \in E^\circ$. We write s as a finite sum of positive rank one elements of the form $v_i \otimes v_i$ with $v_i \in V$ nonzero for $i \in I$ in a finite set. Indeed s defines a positive bilinear form on V^* and hence is a sum of rank one symmetric positive bilinear forms. Let $q \in P$ be a semi-positive element. One has $q(s) = 0$ since $s \in P^\circ$ but $q(v_i \otimes v_i) = q(v_i, v_i) \geq 0$ since q is positive and hence for all $i \in I$, $q(v_i, v_i) = 0$.

Note that for all $v \in V$ nonzero we can choose a decomposition of s such that v is one of the v_i, since $s - \epsilon v \otimes v$ is still positive for ϵ small enough.

We proved that $q(v, v) = 0$ for all $v \in V$ so q is necessarily equal to 0. Hence P does not contain any nonzero semi-positive element.

Now suppose that P° is disjoint from the cone $S^2V^{>0}$, so there must be a hyperplane H in S^2V containing P° and disjoint from $S^2V^{>0}$. This hyperplane corresponds to $\langle q\rangle^\circ$ for some nonzero $q \in P$. This element has the property that $q(s) \neq 0$ for all $s \in S^2V^{>0}$. Up to exchanging q by $-q$ one can assume that $q(s) > 0$ on $S^2V^{>0}$, so $q(s) \geq 0$ on $S^2V^{\geq 0}$. In particular for all $v \in V$, one has $q(v \otimes v) = q(v, v) \geq 0$, so q is semi-positive. ◻

3.2. Fitting pairs

To a pencil $P \in \mathrm{Gr}_d(\mathcal{Q})$ we associate the codimension d projective subspace $\mathbb{P}(P^\circ)$, and its intersection $\mathbb{P}(P^\circ \cap S^2V^{\geq 0})$ in the convex set $\mathbb{P}(S^2V^{\geq 0})$.

Definition 3.3. We say that two elements $P_1, P_2 \in \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ form a *fitting pair* if the associated subspaces $\mathbb{P}(P_1^\circ \cap S^2V^{\geq 0})$ and $\mathbb{P}(P_2^\circ \cap S^2V^{\geq 0})$ are disjoint.

The structure of the convex set $\mathbb{P}(S^2V^{\geq 0})$ is involved, but the set of its extreme points $S^2\mathbb{P}(V) \subset \mathbb{P}(S^2V^{\geq 0})$ is the projectivization of the set of rank one tensors, which is in one-to-one correspondence with $\mathbb{P}(V)$. Here the rank of a symmetric tensor is also the rank of the corresponding map $V^* \to V$. The *extreme points* of a closed properly convex subset C of projective space are the points $x \in C$ such that if $s \subset C$ is a projective segment containing it, x is an endpoint of s. We show that the condition of being a fitting pair can be checked by looking only at $\mathbb{P}(V)$.

Given a symmetric bilinear form $q \in \mathcal{Q}$, we will write respectively $\{q = 0\}$, $\{q > 0\}$ and $\{q \geq 0\} \subset \mathbb{P}(V)$ the set of lines that are respectively null, positive and non-negative for q.

Proposition 3.4. *Let $P_1, P_2 \in \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be two mixed pencils.*

(i) *(P_1, P_2) form a fitting pair,*
(ii) *there exist $q_1 \in P_1$ and $q_2 \in P_2$ such that $q_2 - q_1$ is positive,*
(iii) *there exist $q_1 \in P_1$ and $q_2 \in P_2$ such that $\{q_1 \geq 0\} \subset \{q_2 > 0\}$.*

In order to prove this statement, the key fact that we will prove and use is that the convex set $\mathbb{P}(S^2V^{\geq 0})$ does not have any segment as a *face*, that is, the intersection of this convex with any hyperplane intersecting trivially the interior $\mathbb{P}(S^2V^{>0})$ is never a nontrivial segment.

The set $\{q = 0\}$ is identified via the identification $\mathbb{P}(V) \simeq S^2\mathbb{P}(V) \subset \mathbb{P}(S^2V)$ with the intersection $\langle q\rangle^\circ \cap S^2\mathbb{P}(V)$.

Lemma 3.5. *For all linear hyperplane H in S^2V the extreme points of $\mathbb{P}(H \cap S^2V^{\geq 0})$ are also extreme points of $\mathbb{P}(S^2V^{\geq 0})$. In particular:*

$$\mathbb{P}(H \cap S^2V^{\geq 0}) = \mathrm{Hull}(H \cap S^2\mathbb{P}(V)).$$

Furthermore if $H = \langle q\rangle^\circ$ for $q \in \mathcal{Q}$, $H \cap S^2V = S^2\{q = 0\}$.

Recall that the Krein–Milman theorem states that the convex Hull of the extreme points of a closed convex set C in a finite dimensional vector space is equal to C.

Proof. Let f be a face of $\mathbb{P}(S^2V^{\geq 0})$, that is, the intersection of $\mathbb{P}(S^2V^{\geq 0})$ with a projective hyperplane corresponding to $[q] \in \mathbb{P}(S^2V^*) = \mathbb{P}(\mathcal{Q})$ not intersecting $\mathbb{P}(S^2V^{>0})$ trivially. It is the intersection of this convex with a projective hyperplane corresponding to $[q] \in \mathbb{P}(S^2V^*) = \mathbb{P}(\mathcal{Q})$. The fact that this hyperplane intersects $\mathbb{P}(S^2V^{>0})$ trivially implies that there is a lift $q \in \mathcal{Q}$ that is a semi-positive element, by Proposition 3.2. Let $W \subset V$ be the vector subspace of isotropic vectors for q. The corresponding face is equal to $\mathbb{P}(S^2W^{\geq 0})$.

Hence faces of $\mathbb{P}(S^2V^{\geq 0})$ are of the form $\mathbb{P}(S^2W^{\geq 0})$ for $W \subset V$ a linear subspace. This has dimension 0 or at least 2, and therefore no face is a segment.

Suppose that for some general projective hyperplane, some extreme point $[s]$ of $\mathbb{P}(H \cap S^2V^{\geq 0})$ is not an extreme point of $\mathbb{P}(S^2V^{\geq 0})$, then it belongs to the interior of a face of $\mathbb{P}(S^2V^{\geq 0})$, that has dimension at least 2 as stated previously. The intersection of this face with H contains therefore an open segment containing s, so s is not an extreme point of $H \cap \mathbb{P}(S^2V^{\geq 0})$. $\square$

Proof of Proposition 3.4. Let us prove that (i) implies (ii). The set $P_1^\circ \cap P_2^\circ$ is disjoint from $\mathbb{P}(S^2V^{\geq 0})$ if and only if there exists an element in $P_1 + P_2$ that belongs to the dual of $\mathbb{P}(S^2V^{\geq 0})$, that is, if there exists a positive bilinear form $q \in P_1 + P_2$. This form can be written as $q = q_1 - q_2$ with $q_1 \in P_1$ and $q_2 \in P_2$, therefore (i) implies (ii).

Moreover (ii) implies (iii). Indeed, if $q_2 - q_1$ is positive then $\{q_1 \geq 0\} \subset \{q_2 > 0\}$.

It only remains to show that (iii) implies (i). Lemma 3.5 implies that:

$$\mathbb{P}(\langle q_1\rangle^\circ \cap S^2V^{\geq 0}) = \mathrm{Hull}(S^2\{q_1 = 0\}).$$

Hence $q_2 \in P_2 \subset \mathcal{Q} = S^2V^*$ is positive on the cone $\langle q_1\rangle^\circ \cap S^2V^{\geq 0} \subset S^2V$ and therefore $\mathbb{P}(P_2^\circ)$ does not intersect $\mathbb{P}(\langle q_1\rangle^\circ \cap S^2V^{\geq 0})$. $\square$

3.3. *Fitting directions*

The space $\mathrm{Gr}_d(\mathcal{Q})$ inherits the structure of a smooth manifold. A chart around a point $P \in \mathrm{Gr}_d(\mathcal{Q})$ can be constructed given a subspace Q such that $P \oplus Q = \mathcal{Q}$. We denote by $U_Q \subset \mathrm{Gr}_d(\mathcal{Q})$ the open set of elements transverse to Q. Every element of U_Q can be written uniquely as the graph $\{x + u(x) | x \in P\}$ for some linear map $u : P \longrightarrow Q$. Hence U_Q can be identified with the vector space $\mathrm{Hom}(P, Q)$.

The tangent space $T_P\,\mathrm{Gr}_d(\mathcal{Q})$ can be naturally identified with $\mathrm{Hom}(P, \mathcal{Q}/P)$, so that for each chart U_Q containing P, the tangent space identifies with the tangent space in the chart via the identification $\mathrm{Hom}(P, Q) \simeq \mathrm{Hom}(P, \mathcal{Q}/P)$.

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Because of the identification $T_P \operatorname{Gr}_d(\mathcal{Q}) \simeq T_{P^\circ} \operatorname{Gr}_{N-d}(\mathcal{Q}^*)$ where $N = \dim(\mathcal{Q})$, to such an element v corresponds an element:

$$\mathrm{v}^\circ \in \operatorname{Hom}(P^\circ, S^2V/P^\circ).$$

Let $\mathrm{v}^\circ \in T_P \operatorname{Gr}_d(\mathcal{Q}^*)$, that we see as an element of $\operatorname{Hom}(P^\circ, \mathcal{Q}/P^\circ)$. Let $(P_t^\circ)_{t\geq 0}$ be a curve of elements of $\operatorname{Gr}_{N-d}(\mathcal{Q}^*)$ with $P_0^\circ = P^\circ$ and derivative v° at $t = 0$. One can interpret $\operatorname{Ker}(\mathrm{v}^\circ)$ as the set of vectors in P° which remain in P_t° at first order around $t = 0$. More precisely the lines $[s]$ in $\operatorname{Ker}(\mathrm{v}^\circ)$ are exactly the ones such that for any Riemannian metric on $\mathbb{P}(V)$ and any such curve $(P_t^\circ)_{t>0}$ one has $d(\mathbb{P}(P_t^\circ), [s]) = o(t)$ close to $t = 0$.

We will call *fitting directions* in the Grassmannian $\operatorname{Gr}_d(\mathcal{Q})$ the tangent directions such that if $(P_t) \in \operatorname{Gr}_d(\mathcal{Q})$ moves in this direction, the corresponding codimension d subspaces $\mathbb{P}\big(P_t^\circ \cap S^2V^{\geq 0}\big)$ are disjoint from $\mathbb{P}\big(P_0^\circ \cap S^2V^{\geq 0}\big)$, at order one, see Proposition 3.7.

Definition/Proposition 3.6. We say that a vector $\mathrm{v} \in T_P \operatorname{Gr}_d(\mathcal{Q})$ is *fitting* if one of the following equivalent statements holds:

- $\operatorname{Ker}(\mathrm{v}^\circ) \subset P^\circ$ intersects trivially $S^2V^{\geq 0}$,
- $\operatorname{Im}(\mathrm{v}) \subset \mathcal{Q}/P$ contains $[q]$ where q is a positive element.

We now check that these two statements in the definition are indeed equivalent.

Proof. We first show that $\operatorname{Ker}(\mathrm{v}^\circ) = \operatorname{Im}(\mathrm{v})^\circ$, where we use the natural identification $(\mathcal{Q}/P)^* \simeq P^\circ$.

We prove this by writing an equation that relates v and v°. By definition, $\operatorname{Tr}(qs) = 0$ for $q \in P$ and $s \in P^\circ$. Let us fix $q \in P$ and $s \in P^\circ$ and choose some representatives $\overline{\mathrm{v}(s)} \in S^2V$ and $\overline{\mathrm{v}^\circ(q)} \in \mathcal{Q}$ for $\mathrm{v}(s) \in S^2V/P^\circ$ and $\mathrm{v}^\circ(q) \in \mathcal{Q}/P$. If (P_t) is a smooth curve with $P_0 = P$ and with derivative v at $t = 0$, $s + t\overline{\mathrm{v}^\circ(p)} + o(t) \in P_t$ and $q + t\overline{\mathrm{v}(q)} + o(t) \in P_t^\circ$. Hence we get:

$$\operatorname{Tr}\Big(q\overline{\mathrm{v}^\circ(s)} + \overline{\mathrm{v}(q)}s\Big) = 0.$$

An element $s \in P^\circ$ satisfies $s \in \operatorname{Ker}(\mathrm{v}^\circ)$ if and only if $\operatorname{Tr}(\overline{\mathrm{v}(q)}s) = 0$ for all $q \in \mathcal{Q}$, hence if and only if the corresponding linear form on Q/P belongs to $\operatorname{Im}(\mathrm{v})^\circ$.

Now we prove the equivalence of the two definitions. If there exists $[q] \in \operatorname{Im}(\mathrm{v})$ with q positive, then for any $s \in \operatorname{Ker}(\mathrm{v}^\circ)$, $\operatorname{Tr}(q \circ s) = 0$. Hence s is not a positive tensor. Therefore $\operatorname{Ker}(\mathrm{v}^\circ) \subset P^\circ$ intersects trivially $S^2V^{\geq 0}$.

Conversely if $\operatorname{Ker}(\mathrm{v}^\circ)$ intersects trivially $S^2V^{\geq 0}$, there exists $q \in \mathcal{Q}$ that do not vanish on $S^2V^{\geq 0}$, that is, q is positive. The class $[q] \in \mathcal{Q}/P$ belongs to $\operatorname{Im}(\mathrm{v})$. $\square$

Fitting directions are related to fitting pairs. More precisely:

Proposition 3.7. *A vector* $\mathrm{v} \in T_P \operatorname{Gr}_d(\mathcal{Q})$ *is fitting if and only if, for every* $\mathcal{C}^1$ *curve* $\gamma : [0, 1] \to \operatorname{Gr}_d(\mathcal{Q})$ *with* $\gamma_0 = P$ *and* $\gamma_0' = \mathrm{v}$*, the pair* (γ_t, γ_0) *is fitting for all* $t > 0$ *small enough.*

Moreover in this case for any Riemannian metric on $\mathbb{P}(S^2V)$ *there exists an* $\epsilon > 0$ *such that for any* $t > 0$ *small enough the Riemannian distance between* $\mathbb{P}(\gamma_0^\circ \cap S^2V^{\geq 0})$ *and* $\mathbb{P}(\gamma_t^\circ \cap S^2V^{\geq 0})$ *is greater than* ϵt.

Proof. Let us fix a complement H of P° in S^2V. Let $N = \dim(\mathcal{Q}) = \frac{n(n+1)}{2}$. We identify a neighborhood of $P^\circ \subset \operatorname{Gr}_{N-d}(\mathcal{Q})$ with $\operatorname{Hom}(P^\circ, H)$. If there exists a nonzero element $s \in \operatorname{Ker}(\mathrm{v}^\circ) \cap S^2V^{\geq 0}$, one can consider the curve where γ_t corresponds to

$$q \in P^\circ \mapsto t\mathrm{v}(q).$$

The nonzero element $s \in S^2V^{\geq 0}$ belongs to $P^\circ = \gamma_0^\circ$ and γ_t° hence the pair (γ_0, γ_t) is not fitting for any $t > 0$.

In general the element corresponding to γ_t is equal for t close to 0 to the element of $\mathrm{Gr}_{N-d}(\mathcal{Q})$ which is the graph of the map $P^\circ \to H$:

$$q \mapsto t\mathrm{v}(q) + o(t).$$

The pair (γ_o, γ_t) is fitting if and only if $\mathrm{Ker}(\mathrm{v}) \cap S^2V^{\geq 0} = \{0\}$, hence if v is fitting.

In this case the distance between $\mathbb{P}(\gamma_0^\circ \cap S^2V^{\geq 0})$ and $\mathbb{P}(\gamma_t^\circ \cap S^2V^{\geq 0})$ grows at least linearly in t for t close to 0. □

Fitting vectors can be thought of analogs of spacelike vectors in a pseudo-Riemannian manifold of signature (d, N) for some $N > 0$. For instance, the set of spacelike vectors is the union of a family of cones parametrized by $\mathbb{S}^{d-1}$, and so is the set of fitting vectors, as shown in the following proposition. This analogy is also emphasized by Remark A.3 and Theorem B.1.

For a vector space W we write $\mathbb{S}W = (W \setminus \{0\})/\mathbb{R}_{>0}$

Proposition 3.8. *The set of fitting vectors in $T_P \mathrm{Gr}_d(\mathcal{Q})$ is equal to the union:*

$$\bigcup_{[q]\in\mathbb{S}\mathcal{E}_P} C_{[q]}.$$

Here $C_{[q]}$ is the convex open cone of elements $\mathrm{v} \in T_P \mathrm{Gr}_d(\mathcal{Q})$ such that there exists a positive element in the class $\mathrm{v}(q) \in \mathcal{Q}/P$.

Recall that we use the identification $\mathrm{v} \in T_P \mathrm{Gr}_d(\mathcal{Q}) \simeq \mathrm{Hom}(P, \mathcal{Q}/P)$.

Proof. Because of the second part of Definition 3.6 a vector is well-fitting if and only if it belongs to $C_{[q]}$ for some $[q] \in P$. We just check that the sets $C_{[q]}$ are indeed open convex cones.

Let us fix a complement H of P in $\mathcal{Q}$ to identify $T_P \mathrm{Gr}_d(\mathcal{Q})$ with $\mathrm{Hom}(P, H)$. If $\mathrm{v}_1, \mathrm{v}_2$ lie in $C_{[q]}$ and if $\lambda, \mu \in \mathbb{R}_{>0}$, then for some $q_1, q_2 \in P$ one has $\mathrm{v}_1(q) + q_1$ and $\mathrm{v}_2(q) + q_2$ positive. Therefore $(\lambda \mathrm{v}_1 + \mu \mathrm{v}_2)(q) + \lambda q_1 + \mu q_2$ is positive so $\lambda \mathrm{v}_1 + \mu \mathrm{v}_2$ belongs to $C_{[q]}$. □

3.4. Fibration of a convex set and globally fitting maps

We consider continuous and smooth fibrations of the $\mathrm{SL}(V)$-invariant convex set $\mathbb{P}(S^2V^{\geq 0})$ by projective codimension d subspaces. Let M be a connected manifold of dimension d. We are interested in continuous injective maps, or smooth immersions $u : M \to \mathrm{Gr}_d(\mathcal{Q})$.

We write $u^\circ(x) = (u(x))^\circ$. The map u determines a collection of projective subsets of codimension d:

$$\Big(\mathbb{P}\Big(u^\circ(x) \cap S^2V^{\geq 0}\Big)\Big)_{x\in M}. \tag{1}$$

Note here that for any linear subspace $A \subset V$ the projective codimension of $P(A)$ in $\mathbb{P}(V)$ is equal to the codimension of A in V.

If the image of u contains only mixed elements, then all the submanifolds in (1) are nonempty.

Definition 3.9. We call a continuous map $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ a *globally fitting map* if the subsets in the collection (1) are disjoint.

A continuous map $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ is a *locally fitting map* if for all $x \in M$ there is a neighborhood $U \subset M$ of x such that $u_{|U}$ is a globally fitting map.

Since $\dim(M) = d$, the invariance of domain implies that the sets (1) for a globally fitting map form a fibration for all $x \in M$ of a neighborhood in $\mathbb{P}(S^2V^{\geq 0})$ of $\mathbb{P}\big(u^\circ(x) \cap S^2V^{\geq 0}\big)$.

We now consider immersions from a manifold M of dimension d whose tangent directions are all fitting.

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Definition 3.10. A smooth immersion $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ is a *fitting immersion* if $\mathrm{d}u(v)$ is fitting for all $v \in TM$.

Remark 3.11. In Appendix A.1 we discuss the simpler case when the pencils are hermitian pecils of quadrics on $\mathbb{C}^2$. In this cases, fitting immersions are fibrations of $\mathbb{H}^3$ by geodesics.

Because of proposition 3.7, fitting immersions are locally fitting maps.

The following proposition is the infinitesimal equivalent of Proposition 3.4. We write the statement in a way to emphasize this analogy.

Consider the tautological rank d vector bundle $\mathrm{pr} : \mathcal{E} \to \mathrm{Gr}_d(\mathcal{Q})$. The fiber at $P \in \mathrm{Gr}_d(\mathcal{Q})$ of bundle $\mathcal{E}$ is identified with the vector subspace $P \subset \mathcal{Q}$. Since all the fibers are naturally identified with subsets of $\mathcal{Q}$, there is a tautological projection $\pi : \mathcal{E} \to \mathcal{Q}$ such that for (P, q) in the tautological bundle, that is, $P \in \mathrm{Gr}_d(\mathcal{Q})$ and $q \in P$ with $q \in P$, $\pi(P, q) = q$. We still denote by $\pi : u^*\mathcal{E} \to \mathcal{Q}$ the corresponding projection with a slight abuse of notations.

Note that that an element in $T\mathcal{Q}$ is a pair $(q, \dot{q}) \in T_q\mathcal{Q}$ and we say that it is positive if the symmetric bilinear form $\dot{q} \in \mathcal{Q}$ is positive.

Proposition 3.12. *Given an immersion $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$, let $x_0 \in M$. The following are equivalent:*

(i) *the manifolds $(\mathbb{P}(u^\circ(x)))_{x\in M}$ define locally a smooth fibration of an open neighborhood of $\mathbb{P}\big(u^\circ(x_0) \cap S^2V^{\geq 0}\big)$,*
(ii) *for all $v \in T_{x_0}M$, $\mathrm{d}u(v)$ is fitting,*
(iii) *for all $v \in T_{x_0}M$ there exists $\mathrm{w} \in T\mathcal{E}$ such that $\mathrm{d}\,\mathrm{pr}(\mathrm{w}) = \mathrm{d}u(v)$ and $\mathrm{d}\pi(\mathrm{w}) \in T\mathcal{Q}$ is positive.*

Proof. Note that since M has dimension d and $u^\circ(x_0)$ has codimension d, the statement (i) is equivalent to having for any Riemannian distance d_R on $\mathbb{P}(S^2V)$ and d_M on M, for some $\epsilon > 0$ when x is close to x_0:

$$d_R\Big(\mathbb{P}\Big(u^\circ(x) \cap S^2V^{\geq 0}\Big), \mathbb{P}\Big(u^\circ(x_0) \cap S^2V^{\geq 0}\Big)\Big) \geq \epsilon d_M(x, x_0).$$

Proposition 3.7 shows that the fitting condition is equivalent to having this distance growing linearly, hence the statements (i) and (ii) are equivalent.

A pair $(P, q_0) \in \mathrm{Gr}_d(\mathcal{Q}) \times \mathcal{Q}$ correspond to an element in $\mathcal{E}$ if and only if $q_0 \in P$. A pair $(\mathrm{v}, \dot{q}) \in T_P\,\mathrm{Gr}_d(\mathcal{Q}) \times T_q\mathcal{Q}$ can be written as $(\mathrm{d}\,\mathrm{pr}(\mathrm{w}), \mathrm{d}\pi(\mathrm{w}))$ for some $\mathrm{w} \in T\mathcal{E}$ if and only if $\dot{q}$ belongs to the class defined by $\mathrm{v}(q_0)$.

Let us show this last claim. Let H be a complement of P in $\mathcal{Q}$ let $u : [0, 1] \to \mathrm{Hom}(P, H)$ and $q : [0, 1] \to \mathcal{Q}$ be smooth curves with derivative v and $\dot{q}$ at $q = 0$ and such that $q(t)$ belongs to the graph of $u(t)$ for $t \in [0, 1]$. For some smooth curve $\tilde{q} : [0, 1] \to P$ with $\tilde{q}(0) = q_0$ and all $t \in [0, 1]$:

$$q(t) = \tilde{q}(t) + u(t)(\tilde{q}(t)).$$

Differentiating this at $t = 0$ and we get exactly $\mathrm{v}(q_0) + \tilde{q}'(0) = \dot{q}$, so $\dot{q}$ belongs to the class defined by $\mathrm{v}(q_0)$, since $\tilde{q}'(0) \in P$. Reciprocally if this holds, one can construct such a curve $\tilde{q}$, so the pair corresponds to an element of $T\mathcal{E}$.

We conclude that (iii) is equivalent to the second characterization of fitting vectors in Definition 3.6: one can find such a positive lift w if and only if one can find a class in $\mathrm{Im}(\mathrm{v})$ that contains a positive element. □

In Proposition 7.1 we show how to construct some examples of fitting maps from a totally geodesic immersion in the symmetric space.

Remark 3.13. The definition of a fitting immersion and the previous two propositions can be generalized to the more general setup when S^2V is replaced by a vector space W and $S^2V^{\geq 0}$ is replaced by a closed proper convex cone C in W. In this setup positive quadrics should be replaced by elements in the dual cone of C in W^*.

4. Fitting flows

In this section we define the notion of a fitting flow, and study the consequence of the existence of such a flow. We show next that such flows always exist for fitting immersions. In this section let us fix a map $u : M \to \mathrm{Gr}_d(\mathcal{Q})$.

4.1. Definition and application of fitting flows

The pullback $u^*\mathcal{E}$ of the tautological bundle $\mathrm{pr} : \mathcal{E} \to \mathrm{Gr}_d(\mathcal{Q})$ defines a rank d vector bundle over M. We define the sphere bundle $\mathbb{S}u^*\mathcal{E}$ as the quotient of the vector bundle $u^*\mathcal{E}$ by the action of positive scalars.

Recall that M has dimension d. We consider flows on $\mathbb{S}u^*\mathcal{E}$ so that some form of contraction occurs along the flow lines. We denote also by pr the bundle maps $u^*\mathcal{E} \to M$, $\mathbb{S}u^*\mathcal{E} \to M$, with a slight abuse of notations.

The fiber at $P \in \mathrm{Gr}_d(\mathcal{Q})$ of the bundle $\mathcal{E}$ is identified with the vector subspace $P \subset \mathcal{Q}$. Recall that there is a natural projection $\pi : \mathcal{E} \to \mathcal{Q}$ defined by $\pi(P, q) = q$. We still denote by $\pi : u^*\mathcal{E} \to \mathcal{Q}$ the corresponding projection with a slight abuse of notations.

Definition 4.1. A *fitting flow* for a continuous map $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ is a continuous flow $(\Phi_t)_{t\in\mathbb{R}}$ on $\mathbb{S}u^*\mathcal{E}$ such that

one can choose representatives q of $[q]$ and q' of $[q'] = \Phi_t([q])$ such that $\pi(q') - \pi(q) \in \mathcal{Q}$ is positive.

Note that the last condition is equivalent to asking that $\{\pi(q) \geq 0\} \subset \{\pi(q') > 0\}$ in $\mathbb{RP}^{2n-1}$. Along such flows, the associated quadric hypersurfaces are nested into one another. In particular if u admits a fitting flow it is locally a fitting map.

Lemma 4.2. *Let $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a continuous map that admits a fitting flow. The projection to $\mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ of the flow lines of the fitting flow are embedded.*

Proof. Assume by contradiction that for some $t_0 > 0$ and $q \in \mathbb{S}u^*\mathcal{E}$ one has $u(x) = \mathrm{pr}(q) = \mathrm{pr}(\Phi_{t_0}(q))$. The fact that the flow is fitting implies that for some $\lambda > 0$, $\lambda\pi(\Phi_{t_0}(q)) - \pi(q) \in \mathcal{Q}$ is positive. This positive quadric belongs to $u(x)$, contradicting the fact that $u(x) \in \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ by Proposition 3.2. □

Some fitting flows can be constructed by taking a geodesic flow on M for some Riemannian metric and identifying $u^*\mathcal{E}$ with the tangent bundle to M. In general the projections of the flow lines of a fitting flow satisfy the following topological property, which is clearly satisfied for geodesic flows.

Lemma 4.3. *Let $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a continuous map equipped with a fitting flow Φ in a neighborhood of $x \in M$. For $t > 0$ small enough the sphere $S_t : [q] \in \mathbb{S}u^*\mathcal{E}_{|x} \mapsto p \circ \Phi_t([q]) \in M$ is homotopic to a generator of the homology of $U \setminus \{x\}$ for any open neighborhood U of x in M that is diffeomorphic to $\mathbb{R}^d$.*

The proof relies on the fact the dimension of M is equal to d, and hence the manifolds $\mathbb{P}(u^\circ(x) \cap S^2V^{\geq 0})$ locally define a fibration of $\mathbb{P}(S^2V^{\geq 0})$.

Proof. Let $P = u(x)$. Let $s_0 \in P^\circ \subset S^2V$ be a positive tensor, which exists since the pencil P is assumed to be in $\mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$. Let $P' \in \mathrm{Gr}_d(S^2V)$ be a supplement of P°. Since u is continuous for all y close enough to x the vector subspace $u^\circ(y)$ is transverse to P', therefore there exists a unique vector $\phi(y) \in P'$ such that:

$$s_0 + \phi(y) \in (s_0 + P') \cap u^\circ(y) \subset S^2V^{>0}.$$

This defines a continuous map ϕ from a neighborhood U of $x \in M$ to P'.

Let $[q] \in \mathbb{S}u^*\mathcal{E}_x$. For all t such that $s \circ \Phi_t(q) \in U$, the linear form $\pi(\Phi_t(q)) \in \mathcal{Q} = S^2V^*$ vanishes on $\phi(s \circ \Phi_t(q)) \in S^2V$ since this point belongs to $u^\circ(\mathrm{pr} \circ \Phi_t(q))$. Moreover $\pi(\Phi_t(q)) - \pi(q)$ is a positive bilinear form since Φ is a fitting flow.

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In particular $\pi(q) \in \mathcal{Q} = S^2V^*$ is always negative on $\phi(\mathrm{pr} \circ \Phi_t(q)) \in S^2V$. Hence for t small enough $[\phi \circ S_t] : \mathbb{S}u^*\mathcal{E}_x \to \mathbb{S}P'$ has the same degree as $[\pi] : \mathbb{S}u^*\mathcal{E}_x \to \mathbb{S}P'^* \simeq \mathbb{S}P$ which associates to $[q] \in \mathbb{S}u^*\mathcal{E}_x$ the class $[\pi(q)] \in \mathbb{S}P$. The map $[\pi]$ is a diffeomorphism, so in particular $1 = |\deg([\pi])| = |\deg(\phi)\deg(S_t)|$. Hence S_t is a generator of the homotopy group of $U \setminus \{x\}$. □

Remark 4.4. In particular for an immersion $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ that admits a fitting flow, choosing continuously an orientation of the pencils $u(x)$ for $x \in M$ is equivalent to choosing an orientation of M.

Let $M = \widetilde{N}$ where N is compact and $\Gamma = \pi_1(N)$. Let $\rho : \Gamma \to \mathrm{SL}(V)$ be a representation. Recall that a (C, D)-quasi-geodesic in a metric space (X, d) is a curve $\gamma : \mathbb{R} \to X$ such that for all $t_1, t_2 \in \mathbb{R}$:

$$1/C|t_1 - t_2| - D \leq d(\gamma(t_1), \gamma(t_2)) \leq C|t_1 - t_2| + D.$$

Proposition 4.5. *Let $u : \widetilde{N} \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a ρ-equivariant continuous map that admits a ρ-equivariant fitting flow Φ. There exist $C, D > 0$ such that the projection to $\widetilde{N}$ of the flow lines of Φ are (C, D)-quasi-geodesics. Moreover for every $(x, y) \in \widetilde{N}^2$ there exists a flow line whose projection to $\widetilde{N}$ starts at x and ends at y.*

In the previous statement one can consider any fixed Riemannian metric on the compact space $\mathbb{S}u^*\mathcal{E}/\rho(\Gamma)$.

Remark 4.6. Note that since there exists a flow line between any pair of points in $\widetilde{N}$, the map u is necessarily a globally fitting map.

In this proof we will use the *Hilbert distance* d_H on the properly convex domain $\mathbb{P}(S^2V^{>0})$. It is defined which is defined using the cross ratio as $d_H([s_1], [s_2]) = \log(\mathrm{cr}([s^-], [s_1], [s_2], [s^+]))$ where $[s_1], [s_2] \in \mathbb{P}(S^2V^{>0})$ and $[s^-]$, $[s^+]$ are the intersection of the projective line through $[s_1]$ and $[s_2]$ with the boundary of the domain $\mathbb{P}(S^2V^{>0})$. If the closure in $\mathbb{P}(S^2V)$ of two sets $A, B \subset \mathbb{P}(S^2V^{>0})$ are disjoint, then the two sets are at positive distance for the Hilbert distance. Indeed for every Riemannian metric $d_{\mathbb{S}}$ on the compact manifold $\mathbb{P}(S^2V)$, the Hilbert distance between any two points in $\mathbb{P}(S^2V^{>0})$ is bounded from below by some uniform multiple of $d_{\mathbb{S}}$.

Proof. For all $[q] \in \mathbb{S}u^*\mathcal{E}$ we consider the convex $\mathbb{P}(\langle \pi(q) \rangle^\circ \cap S^2V^{>0})$.

We choose a continuous and ρ-equivariant map $\sigma : \mathbb{S}u^*\mathcal{E} \to \mathbb{P}(S^2V^{>0})$ with the property that $\sigma([q]) \in \mathbb{P}(\langle \pi(q) \rangle^\circ \cap S^2V^{>0})$ for all $[q] \in \mathbb{S}u^*\mathcal{E}$. One can always find such a choice, as it amounts to finding a section of an open ball bundle over a compact set. Indeed $\mathbb{P}(\langle \pi(q) \rangle^\circ \cap S^2V^{>0})$ is a nonempty convex set of codimension 1 that forms an open ball for all $[q] \in \mathbb{S}u^*\mathcal{E}$.

We fix a Γ-invariant Riemannian metric g on $\mathbb{S}u^*\mathcal{E}$ with associated distance d_g. We set C_1 to be the supremum of $d_H(\sigma([q_1]), \sigma([q_2]))$ for all $[q_1], [q_2] \in \mathbb{S}u^*\mathcal{E}$ such that $d_g([q_1], [q_2]) \leq 1$, which exists since Γ acts cocompactly on $\widetilde{N}$. The following inequality follows for all $[q_1], [q_2] \in \mathbb{S}u^*\mathcal{E}$ from the triangular inequality:

$$d_H(\sigma([q_1]), \sigma([q_2])) \leq C_1 d_g([q_1], [q_2]) + C_1.$$

Let n be the unique integer $d_g([q_1], [q_2]) \leq n < d_g([q_1], [q_2]) + 1$. One can find elements $x_0 = [q_1], x_1, \cdots, x_n = [q_2]$ in $\mathbb{S}u^*\mathcal{E}$ such that for all $1 \leq i \leq n$ one has $d_g(x_{i-1}, x_i) \leq 1$. The triangular inequality for the Hilbert distance implies that:

$$d_H(\sigma([q_1]), \sigma([q_2])) \leq nC_1.$$

We now set C_2 to be the supremum of $d_H(\sigma([q_1]), \sigma(\Phi_t([q])))$ for all $[q_1] \in \mathbb{S}u^*\mathcal{E}$ and $0 \leq t \leq 1$. Similarly we get the following inequality for all $[q] \in \mathbb{S}u^*\mathcal{E}$ and $t \geq 0$:

$$d_H(\sigma([q]), \sigma(\Phi_t([q]))) \leq C_2 t + C_2.$$

Let K be a compact fundamental domain for the action of Γ on $\mathbb{S}u^*\mathcal{E}$ and let ϵ be the infimum of the Hilbert distance for any $[q] \in K$ between $\mathbb{P}\big(\langle \pi(q)\rangle^\circ \cap S^2V^{>0}\big)$ and $\mathbb{P}\big(\langle \pi(\Phi_1(q))\rangle^\circ \cap S^2V^{>0}\big)$. Since the flow is fitting, the closures of these two sets in $\mathbb{P}(S^2V)$ are disjoint for any $[q]$, and hence their Hilbert distance is positive. Since K is compact, the infimum ϵ is also positive.

The Hilbert distance between $\sigma(q)$ and $\sigma(\Phi_t(q))$ for $t > 0$ and $q \in \mathbb{S}u^*\mathcal{E}$ is greater than $\epsilon(t-1)$. Indeed for all integer $0 \leq n \leq t$ the projective segment between $\sigma(q)$ and $\sigma(\Phi_t(q))$, which is a geodesic for the Hilbert distance, intersects $\mathbb{P}\big(\langle \pi(\Phi_n(q))\rangle^\circ \cap S^2V^{>0}\big)$ in exactly one point x_n. Hence the Hilbert distance between x_n and x_{n+1} for $0 \leq n \leq t-1$ is at least ϵ.

Putting all of these inequalities together we get that for all $t \geq 0$ and $[q] \in \mathbb{S}u^*\mathcal{E}$:

$$\frac{\epsilon}{C_1}(t-1) - 1 \leq d_g(\sigma([q]), \sigma(\Phi_t([q]))) \leq C_2(t+1)$$

Hence the flow lines are quasi-isometric embeddings.

We now check that flow lines exist between any pair of points. Let $x \in M$. Given $t \in \mathbb{R}$ we consider the d-sphere $S_t : q \in \mathbb{S}u^*\mathcal{E}_{|x} \mapsto \Phi_t(q) \in u^*\mathcal{E}$. Suppose that some $y \in M$, avoids the sphere S_t for all $t > 0$. Consider a curve η between x and y. The homological intersection between this segment and the spheres S_t in $M \setminus \{x, y\}$ is constant, and is equal to zero for t large enough since the spheres S_t are then uniformly far from x. However for t small enough, the homotopy class of S_t is the one of any small sphere encircling x by Lemma 4.3. This leads to a contradiction since such a sphere will have homological intersection equal to 1 or -1 with the curve η. Hence there exists a flow line joining any pair of points. □

As a consequence, we get the following.

Corollary 4.7. *Let $u : \widetilde{N} \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a ρ-equivariant continuous map that admits an equivariant fitting flow, then it is an embedding. In particular it is a globally fitting map. Moreover $\rho : \Gamma \to \mathrm{SL}(V)$ is a quasi-isometric embedding.*

Proof. Since there are flow lines between any pair of points in $\widetilde{N}$, for every $x \neq y \in M$ one can find $q \in u(x)$ and $q' \in u(y)$ such that $q - q'$ is positive. Since the pencils $u(x)$ and $u(y)$ are mixed, one cannot have $q - q' \in u(x) = u(y)$ so $u(x) \neq u(y)$. Furthermore the pair $(u(x), u(y))$ is a fitting pair for all $x \neq y \in M$, hence u is a globally fitting map by Proposition 3.4.

For any Γ-invariant Riemannian metric g on $\mathbb{S}\mathcal{E}$, any map σ as the proof of Proposition 4.5 is a quasi-isometric embedding. Indeed let δ be the maximum of $d_g(x, x')$ or $d_H(\sigma(x), \sigma(x'))$ for any x, x' in the same fiber of $\mathrm{pr} : \mathbb{S}\mathcal{E} \to \widetilde{N}$. For every $x, y \in \mathbb{S}\mathcal{E}$ one can find x', y' in the same fibers respectively as x, y and in the same flow line for Φ. Hence for the constants C, D from Proposition 4.5:

$$\frac{1}{C} d_g(x,y) - D - \left(\frac{2}{C} + 2\right)\delta \leq d_H(\sigma(x), \sigma(y)) \leq C d_g(x,y) + D + (2C+2)\delta.$$

Hence ρ is a quasi-isometric embedding. □

4.2. *Existence of fitting flows*

We now prove that the existence of a fitting flow is guaranteed on compacts for fitting immersions. It is not clear if it is the case in general for locally fitting maps.

Proposition 4.8. *Let M be a manifold of dimension d.*

(i) *A fitting immersion $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ admits a fitting flow.*
(ii) *An equivariant fitting immersion $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ for a representation $\rho : \Gamma \to \mathrm{SL}(V)$, and a proper action of Γ on M admits a ρ-equivariant fitting flow.*

Recall that $\mathrm{pr} : \mathcal{E} \to \mathrm{Gr}_d(\mathcal{Q})$ is the bundle map and $\pi : \mathcal{E} \to \mathcal{Q}$ is the tautological map.

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In order to construct the fitting flow we construct the vector field W on $\mathbb{S}u^*\mathcal{E}$ that generates the flow. This first step of the proof uses crucially the hypothesis that $\dim(M) = d$.

Lemma 4.9. *Let $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ be a fitting immersion with $\dim(M) = d$. For every $x \in M$ and any $q \in u(x)$, there exists a lift $\mathrm{w} \in T_{(u(x),q)}u^*\mathcal{E}$ such that $\mathrm{d}\pi(\mathrm{w}) \in T\mathcal{Q}$ is positive.*

We say that an element $(q, \dot{q}) \in T\mathcal{Q}$ is positive if $\dot{q} \in \mathcal{Q}$ is positive. Given any quadric in the pencil $u(x)$, we want to find an infinitesimal direction in which to move this quadric as well as the pencil containing it inside the image of u so that the derivative of the quadrics is positive.

This lemma is an inverse of point (iii) of Proposition 3.12: here we fix an element $q \in u(x)$ whereas before we were fixing a $v \in TM$. In order to construct this inverse we will use the fact that a continuous odd map between spheres of equal dimensions must have odd degree [Hat02, Proposition 2B.6] and therefore it must be surjective, as any nonsurjective map into a sphere is contractible.

Proof. Let us fix $x \in M$. We construct a continuous map:

$$\phi : T_xM \setminus \{0\} \to u^*\mathcal{E}_x \setminus \{0\}.$$

We require that this map satisfies $\lambda\phi(v) = \phi(\lambda v)$ for all $v \in T_xM \setminus \{0\}$ and $\lambda \in \mathbb{R}$. Note that in particular ϕ defines an odd map:

$$\overline{\phi} : \mathbb{S}T_xM \to \mathbb{S}u^*\mathcal{E}_x.$$

We furthermore construct a lift:

$$\psi : T_xM \setminus \{0\} \to Tu^*\mathcal{E}.$$

In other words we assume that $\psi(v) \in T_{\phi(v)}(u^*\mathcal{E})$ for $v \in T_xM \setminus \{0\}$. We require that $\psi(\lambda v) = \lambda(\mathrm{d}m_\lambda)(\psi(v))$ for all $v \in T_xM \setminus \{0\}$ and $\lambda \in \mathbb{R}$ where $m_\lambda : u^*\mathcal{E} \to u^*\mathcal{E}$ is the multiplication by λ. We require $\mathrm{d}\pi(\psi(v)) \in T\mathcal{Q}$ to be positive for all $v \in T_xM \setminus \{0\}$.

Finally we will make this construction so that in addition $\mathrm{d}\,\mathrm{pr}(\psi(v)) = v$ for $v \in T_xM \setminus \{0\}$, but this property will be only used during the construction.

The following diagram illustrates the situation.

$$\begin{tikzcd}
Tu^*\mathcal{E} \arrow[loop left, "\mathrm{d}m_\lambda"] \arrow[r, "\mathrm{d}\pi"] \arrow[d, shift right, "\mathrm{d}\,\mathrm{pr}"'] \arrow[dr] & T\mathcal{Q} \arrow[dr] & \\
TM \arrow[u, dashed, shift right, "\psi"'] \arrow[r, dashed, "\phi"] \arrow[dr] & u^*\mathcal{E} \arrow[loop right, "m_\lambda"] \arrow[r, "\pi"] \arrow[d, "\mathrm{pr}"] & \mathcal{Q} \\
& M &
\end{tikzcd}$$

If we can construct such continuous maps, the fact that $\overline{\phi}$ is an odd map between two spheres of the same dimension implies that it is homotopically nontrivial and therefore surjective. In particular for all $[q] \in \mathbb{S}\mathcal{E}_{u(x)} \simeq \mathbb{S}u(x)$ there exists a $v \in T_xM$ such that $\phi(v) = q$. The element $\mathrm{w} = \psi(v)$ then satisfies the required conditions, so this finishes the proof.

Now let us construct the maps ϕ and ψ. We first show that for all $v \in T_xM \setminus \{0\}$ we can define $\phi(v)$ and $\psi(v)$, and then we explain how to glue these maps together to get a continuous map.

Since u is a fitting immersion, the point (iii) of Proposition 3.12 implies that given $v_0 \in T_xM$ one can construct $\phi(v_0) \in u^*\mathcal{E}_x$ and $\psi(v_0) \in T_{\phi(v_0)}u^*\mathcal{E}$ such that $\mathrm{d}\pi(\psi(v_0))$ is positive and $\mathrm{d}\,\mathrm{pr}(\psi(v_0)) = v_0$.

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Note that the condition that $\mathrm{d}\pi(\psi(v))$ is positive is an open condition and the condition that $\mathrm{d}\,\mathrm{pr}(\psi(v)) = v$ requires that ψ is a section of an affine sub-bundle. Hence for every $v \in T_xM \setminus \{0\}$ we can find a small neighborhood $\mathcal{S}$ in a sphere in T_xM containing v_0 on which we can define ϕ and a lift ψ such that $\mathrm{d}\pi(\psi(v))$ is positive and $\mathrm{d}\,\mathrm{pr}(\psi(v)) = v$ for all $v \in \mathcal{S}$. We take $\mathcal{S}$ small enough so that it does not contain any antipodal pair of points.

We define U to be the set of nonzero elements λv for all $\lambda \in \mathbb{R}$ and $v \in \mathcal{S}$, and we extend ϕ and ψ to U in a homogeneous way. We define ϕ on U so that $\phi_i(\lambda v) = \lambda\phi_i(v)$ for all $\lambda \in \mathbb{R}$ nonzero and $v \in \mathcal{S}$. We set $\psi(\lambda v) = \lambda(\mathrm{d}m_\lambda)\psi_i(v)$ for $\lambda \in \mathbb{R}$, where m_λ is the multiplication by λ on $u^*\mathcal{E}_x$. Note that $\mathrm{d}\pi(\psi_i(\lambda v)) = \lambda^2\mathrm{d}\pi(\psi_i(v))$ is positive and $\mathrm{d}\,\mathrm{pr}(\psi_i(\lambda v)) = \lambda v$ for all $\lambda v \in U$. Indeed $\pi \circ m_\lambda = \lambda\pi$ so $\mathrm{d}\pi \circ \mathrm{d}m_\lambda = \lambda \mathrm{d}\pi$ and $p \circ m_\lambda = p$ so $\mathrm{d}\,\mathrm{pr} \circ \mathrm{d}m_\lambda = \mathrm{d}\,\mathrm{pr}$.

We therefore can construct an open cover $\{U_i\}_{i\in I}$ of T_xM and continuous maps $\phi_i : U_i \to \mathbb{S}u^*\mathcal{E} \subset u^*\mathcal{E}$, and lifts $\psi_i : U_i \to Tu^*\mathcal{E}$ such that $\mathrm{d}\pi(\psi_i(v))$ is positive and $\mathrm{d}\,\mathrm{pr}(\psi_i(v)) = v$ for any $i \in I$ and $v \in U_i$. The U_i can be assumed invariant by scalar multiplication, and ϕ and ψ satisfy the aforementioned homogeneity conditions.

We now glue these maps together. Let $\chi_i : U_i \to [0, 1]$ for $i \in I$ be a family of functions that forms a locally finite partition of the unit. We define:

$$\phi : v \in \mathbb{S}T_xM \mapsto \sum_{i\in I} \chi_i(v)\phi_i(v) \in (u^*\mathcal{E})_x.$$

Let us check that $\phi(v)$ is always nonzero. This is where we use that $\mathrm{d}\,\mathrm{pr}(\psi_i(v)) = v$ for $i \in I$, and we also use that the fitting immersion u is defining a smooth fibration of the cone $S^2V^{>0}$. Let $\gamma : \mathbb{R} \to M$ be a curve such that $\gamma(0) = x$ and $\gamma'(0) = v$. For all $t \in \mathbb{R}$ the intersection $u^\circ(\gamma(t)) \cap S^2V^{>0}$ is a nonempty convex set, so we can construct a section $s : t \in \mathbb{R} \mapsto S^2V^{>0}$ of the fibration, that is, such that for all $t \in \mathbb{R}$, $s(t) \in u^\circ(\gamma(t))$.

Let us fix $i \in I$. Let $q : \mathbb{R} \to \mathbb{S}u^*\mathcal{E}$ be such that $q'(0) = \psi(v)$, which implies that $p \circ q(t)$ is equal to $\gamma(t)$ at the first order around $t = 0$ since $\gamma'(0) = \mathrm{d}\,\mathrm{pr}(\psi(v)) = v$. Since $s(t) \in u^\circ(\gamma(t))$, one has $\pi(q(t))\cdot s(t) = 0$ at the first order around $t = 0$. Taking the first derivative at $t = 0$ of this equation we get:

$$\pi(\phi_i(v)) \cdot s'(0) + \mathrm{d}\pi(\psi_i(v)) \cdot s(0) = 0.$$

Since $\mathrm{d}\pi(\psi_i(v))$ is positive and $s(0)$ is a positive tensor:

$$\mathrm{d}\pi(\psi_i(v)) \cdot s(0) > 0.$$

Hence for all $i \in I$, $\pi(\phi_i(v)) \cdot s'(0) < 0$ and therefore $\pi(\phi(v)) \cdot s'(0) < 0$. In particular $\phi(v)$ does not vanish.

In order to glue the ψ_i we need to be careful since the vectors $\psi_i(v)$ do not belong to the same fiber of the tangent bundle $Tu^*\mathcal{E}$. Let Σ be the following map:

$$\Sigma : (q_i)_{i\in I} \in (u^*\mathcal{E})_x^I \mapsto \sum_{i\in I} q_i \in (u^*\mathcal{E})_x.$$

Given $v \in \mathbb{S}T_xM$ we set:

$$\psi(v) = \mathrm{d}\Sigma\big((\chi_i(v)\psi_i(v))_{i\in I}\big).$$

This combination still satisfies that $\mathrm{d}\pi(\psi)$ is positive, indeed:

$$\mathrm{d}\pi(\psi(v)) = \sum_{i\in I} \chi_i(v)\mathrm{d}\pi(\psi_i(v)).$$

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Note that we also get the following:

$$\mathrm{d}\,\mathrm{pr}(\psi(v)) = \sum_{i\in I} \chi_i(v)\mathrm{d}\,\mathrm{pr}(\psi_i(v)) = \left(\sum_{i\in I} \chi_i(v)\right) v = v.$$

This concludes the construction of ϕ and ψ, and hence this concludes the proof. □

To prove Proposition 4.8 we use the directions w from Lemma 4.9 and we glue these vectors into a vector field using once again a partition of the unit. We construct the vector field $W : \mathbb{S}u^*\mathcal{E} \longrightarrow T\mathbb{S}u^*\mathcal{E}$ in a similar manner as $\psi : \mathbb{S}TM \longrightarrow T\mathbb{S}u^*\mathcal{E}$. Morally "$W = \psi \circ \phi^{-1}$," but the map ϕ constructed previously is not a priori bijective.

Proof of Proposition 4.8. We construct a continuous vector field W over $u^*\mathcal{E}$, except the zero section that is homogeneous, that is, such that for all $x \in M$ and non zero $q \in u(x)$ one has $W_{\lambda q} = \mathrm{d}(m_\lambda)W_q$ for all $\lambda \in \mathbb{R}^{>0}$, where m_λ is the multiplication by λ on $\mathcal{E}$. Such a vector field defines a vector field $\overline{W}$ on $\mathbb{S}u^*\mathcal{E}$. We require moreover that $\mathrm{d}\pi(W)$ is always positive.

Given any nonzero $q_0 \in u^*\mathcal{E}$, Lemma 4.9 provides the existence of some $\mathrm{w}_0 \in T_{(u(x),q)}\mathcal{E}$ for all $(x, q) \in u^*\mathcal{E}$ such that:

(i) $\mathrm{d}\,\mathrm{pr}(\mathrm{w}_0) \in \mathrm{d}u(T_xM)$,
(ii) $\mathrm{d}\pi(\mathrm{w}_0)$ is positive.

Property (i) implies that w_0 defines a vector in $T_{x,q_0}u^*\mathcal{E}$. For each such q_0 one can find a neighborhood U of it in $u^*\mathcal{E}$ that is invariant by the $\mathbb{R}^{>0}$-action, and on which one can define a map W satisfying (i) as well as the homogeneity condition $W_{\lambda q} = \mathrm{d}(m_\lambda)W_q$. Condition (ii) being an open condition invariant by the action of m_λ it is satisfied automatically for U small enough.

Using a partition of the unit as in Lemma 4.9 we construct the desired vector field on $u^*\mathcal{E}$. Note that both property (i) and the homogeneity are preserved by linear combinations, and (ii) is preserved by positive combinations.

Finally note that the collection U_i as well as the partition of the unity can be chosen to be ρ-equivariant, so that the vector field W is also ρ-equivariant, and hence also the fitting flow Φ. □

5. The Anosov property and fibrations

In this section we show that the existence of an equivariant map of pencils that admits a fitting flow implies that the representation is Anosov. Moreover we describe the domain that is fibered in $\mathbb{RP}^{2n-1}$. Finally we apply this to show that some quasi-Fuchsian representations do not admit equivariant maps that admit a fitting flow.

5.1. The Anosov property

In order to show that a uniform contraction is taking place along the flow lines of the fitting flow, we define a way to measure the distance between two quadric hypersurfaces nested into one another. The characterization of Anosov representations that we use is similar to the characterization in terms of inclusion of multicones from [BPS19].

Let $\mathbb{S}\mathcal{Q}^{\mathrm{mix}}$ be the set of quadrics that are not semi-positive or semi-negative up to a positive scalar.

If we fix an affine chart of L so that $\ell_1, \ell_2, \ell_2', \ell_1'$ correspond to the real numbers $x_1' < x_2' \leq x_2 < x_1$, the cross ratio $[\ell_1, \ell_2, \ell_2', \ell_1']$ is defined as:

$$[\ell_1, \ell_2, \ell_2', \ell_1'] = \frac{x_2 - x_1'}{x_2 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'}. \tag{2}$$

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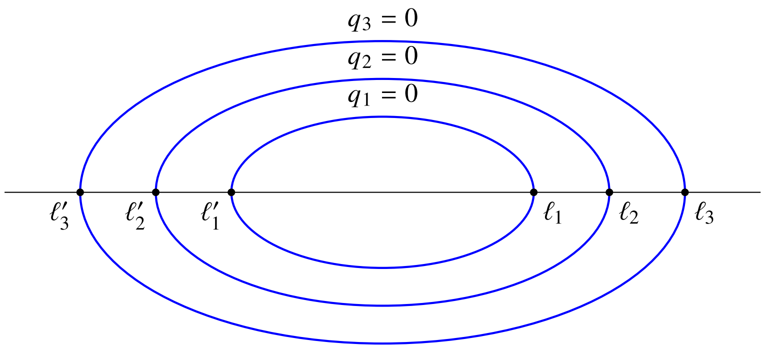

Figure 1. *Illustration of Proposition 5.2.*

Another equivalent definition, if we take nonzero representatives v_1, v_2, v_2', v_1' of $\ell_1, \ell_2, \ell_2', \ell_1'$:

$$[\ell_1, \ell_2, \ell_2', \ell_1'] = \frac{v_2 \wedge v_1'}{v_2 \wedge v_2'} \times \frac{v_1 \wedge v_2'}{v_1 \wedge v_1'}.$$

This expression makes sense as $\bigwedge^2 L \simeq \mathbb{R}$, and it is independent of the chosen identification. The equivalence betweens these formulations is due to the fact that if $v_1 = e_1 + x_1 e_2$ and $v_2 = e_1 + x_2 e_2$ then $v_2 \wedge v_1 = (x_2 - x_1) e_2 \wedge e_1$.

Definition/Proposition 5.1. Let $[q_1], [q_2] \in \mathbb{S}\mathcal{Q}^{\text{mix}}$ be such that for some choice of representatives, the difference $q_2 - q_1$ is positive. For every projective line L intersecting $\{q_1 > 0\}$ and $\{q_2 < 0\}$ we can define ℓ_1 and ℓ_1' are the zeros of q_1 on L and ℓ_1' and ℓ_2' and the zeros of q_2 on L so that the points $\ell_1, \ell_2, \ell_2', \ell_1' \in \mathbb{P}(V)$ are cyclically in this order on L. We define the *cross-ratio distance* $\mathrm{cr}([q_2], [q_1]) \in [1, \infty)$ between $[q_2]$ and $[q_1]$ as the minimum of the cross ratio $[\ell_1, \ell_2, \ell_2', \ell_1']$ for any such projective line L, which is well defined and in $(1, \infty)$.

We now verify that this minimum exists and is in $(1, \infty)$.

Proof. First note that the assumption $q_2 - q_1$ is positive implies that $\{q_1 \geq 0\} \subset \{q_2 > 0\}$ and $\{q_2 \leq 0\} \subset \{q_1 < 0\}$. If L intersects $\{q_1 > 0\}$ and $\{q_1 < 0\}$ then the quadratic forms q_1 and q_2 are both positive on some points and negative on some points of L, and hence they both admit exactly two zeroes on L. These zeroes, denoted respectively ℓ_1,ℓ_1' for q_1 and ℓ_2,ℓ_2' for q_2 are in the following cyclic order on L up to exchanging ℓ_1 and ℓ_1': $(\ell_1, \ell_2, \ell_2', \ell_1')$. One can refer to Figure 1 where $\{q_i > 0\}$ is the interior of the corresponding ellipse.

Consider the compact space X of projective lines L intersecting $\{q_1 \geq 0\}$ and $\{q_2 \leq 0\}$. For every such line we can define the lines ℓ_1, ℓ_1' and ℓ_2, ℓ_2' as in previously where $\ell_1, \ell_1' \neq \ell_2, \ell_2'$ but eventually $\ell_1 = \ell_1'$ or $\ell_2 = \ell_2'$ which occur respectively exactly if L intersects trivially $\{q_1 > 0\}$ or $\{q_2 > 0\}$. Indeed in these cases the quadratic forms q_i are semi-positive or semi negative on L and hence have a double zero. By looking at the formula (2) on an affine chart such that $x_1' < x_2' \leq x_2 < x_1$ we see the cross ratio $[\ell_1, \ell_2, \ell_2', \ell_1']$ is in $(1, \infty]$. This ratio is finite if and only if $\ell_1 \neq \ell_1'$ and $\ell_2 \neq \ell_2'$ which occurs exactly when L belongs to the interior of X, that is, when L intersects $\{q_1 > 0\}$ and $\{q_2 < 0\}$. Finally note that this ratio depends continuously on $L \in X$ and that the interior of X is nonempty since q_1 and q_2 are mixed. Therefore the minimum on X is well defined in $(1, \infty)$ and is reached on the interior of X. □

The logarithm of this quantity satisfies a reversed triangular inequality.

Proposition 5.2. *Let $[q_1], [q_2], [q_3] \in \mathbb{S}\mathcal{Q}^{\text{mix}}$ be such that $q_3 - q_2$ and $q_2 - q_1$ are positive. Then:*

$$cr([q_3], [q_1]) \geq cr([q_3], [q_2]) cr([q_2], [q_1]).$$

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We illustrate this proposition and its proof in Figure 1. This figure illustrates 3 quadrics of signature $(1,2)$ in $\mathbb{RP}^2$. In these pictures the quadrics are positive on the inside of the ellipse they define.

Proof. Pick any projective line that crosses $\{q_1 > 0\}$ and $\{q_3 < 0\}$. Let the intersections of L with the zeroes of q_1, q_2 and q_3 be respectively $(\ell_1, \ell_2, \ell_3, \ell_3', \ell_2', \ell_1')$, counted with multiplicity and cyclically ordered. We fix an affine chart for L such that this tuple corresponds to the tuple $x_3' < x_2' < x_1' \leq x_1 < x_2 < x_3$ of real numbers. This yields the following:

$$[\ell_1, \ell_3, \ell_3', \ell_1'] = \frac{x_3 - x_1'}{x_3 - x_3'} \times \frac{x_1 - x_3'}{x_1 - x_1'}.$$

$$[\ell_1, \ell_3, \ell_3', \ell_1'] = \left(\frac{x_3 - x_1'}{x_3 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'}\right) \times \left(\frac{x_3 - x_2'}{x_3 - x_3'} \times \frac{x_1 - x_3'}{x_1 - x_2'}\right).$$

Note that if $B > A > 0$, for all $c > 0$ one has $\frac{A}{B} \leq \frac{A+c}{B+c}$ and respectively $\frac{B+c}{A+c} \leq \frac{B}{A}$. Hence, due to the ordering of the tuple, we have the following inequalities by setting respectively $A = x_2 - x_1'$, $B = x_2 - x_2'$, $c = x_3 - x_2$ and $A = x_1 - x_2'$, $B = x_1 - x_3'$, $c = x_2 - x_1$:

$$\frac{x_2 - x_1'}{x_2 - x_2'} \leq \frac{x_3 - x_1'}{x_3 - x_2'}, \quad \frac{x_2 - x_3'}{x_2 - x_2'} \leq \frac{x_1 - x_3'}{x_1 - x_2'}.$$

Hence we obtain:

$$\mathrm{cr}([q_2],[q_1]) \leq [\ell_1, \ell_2, \ell_2', \ell_1'] = \frac{x_2 - x_1'}{x_2 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'} \leq \frac{x_3 - x_1'}{x_3 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'}.$$

$$\mathrm{cr}([q_3],[q_2]) \leq [\ell_2, \ell_3, \ell_3', \ell_2'] = \frac{x_3 - x_2'}{x_3 - x_3'} \times \frac{x_2 - x_3'}{x_2 - x_2'} \leq \frac{x_3 - x_2'}{x_3 - x_3'} \times \frac{x_1 - x_3'}{x_1 - x_2'}.$$

Hence one has $[\ell_1, \ell_3, \ell_3', \ell_1'] \geq \mathrm{cr}([q_2],[q_1])\mathrm{cr}([q_3],[q_2])$ for every such projective line L. Therefore $\mathrm{cr}([q_3],[q_1]) \geq \mathrm{cr}([q_3],[q_2])\mathrm{cr}([q_2],[q_1])$. □

A sequence of quadrics such that the cross ratio distance between the first and last quadric goes to $+\infty$ satisfies that the intersection of all half-spaces determined by the quadrics is a projective subspace.

Proposition 5.3. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of quadrics such that $q_{n+1} - q_n$ is positive for all $n \in \mathbb{N}$ and $cr([q_n],[q_0])$ goes to $+\infty$. Then $\bigcap_{n\in\mathbb{N}}\{q_n \leq 0\} \subset \mathbb{P}(V)$ is a projective subspace.*

This proposition is proven in [BG09]. We write a version of the argument here for the sake of completeness. We illustrate this proposition with quadrics of signature $(1,2)$ in $\mathbb{RP}^2$ in Figure 2.

Proof. The intersection $I = \bigcap_{n\in\mathbb{N}}\{q_n \leq 0\}$ is a compact nonempty subset. Let $x \neq y \in I$ and let L be the projective line from x to y. Suppose that there exists $z \in L$ such that $z \notin I$. Without any loss of generality one can assume that the open interval $S \subset L$ bounded by x, y and containing z does not intersect I. Indeed one can otherwise replace x, y by the points on $L \cap I$ closest to z on both sides.

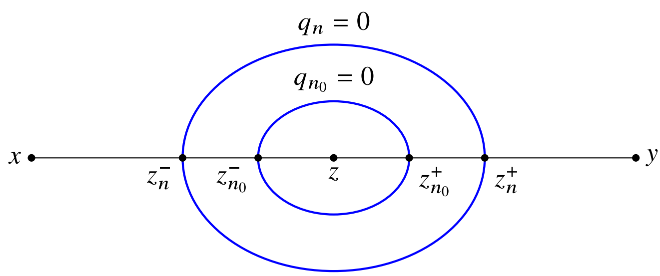

Figure 2. *Illustration of Proposition 5.3.*

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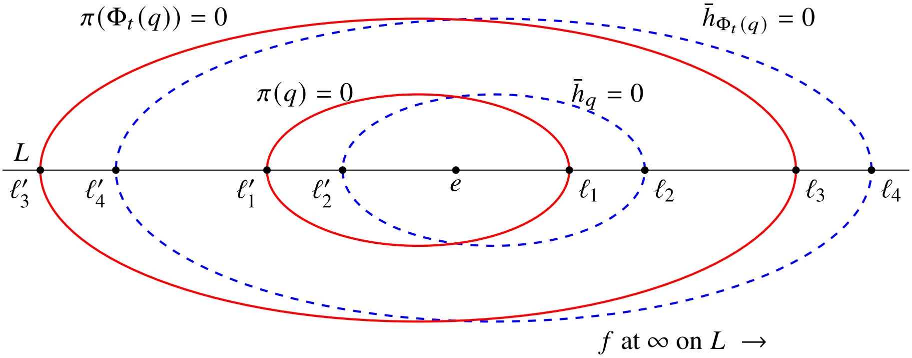

Figure 3. *Illustration of the proof of Theorem 5.4.*

Since $z \notin I$ there exist $n_0 \in \mathbb{N}$ such that q_{n_0} is positive on z. For $n \geq n_0$, let z_1, z_2 be the two intersections of L with the zeroes of q_n so that the points (x, z_n^-, z, z_n^+, y) are cyclically ordered. The sequences (z_n^-) and (z_n^+) are monotonic in S and must converge to x and y since $S \cap I = \emptyset$. The value of $\mathrm{cr}([q_{n_0}], [q_n])$ is bounded from above by the cross ratio $[z_{n_0}^+, z_n^+, z_n^-, z_{n_0}^-]$, which in turn converges to the cross ratio $[z_{n_0}^+, x, y, z_{n_0}^-] < \infty$ when n goes to $+\infty$. This contradicts the fact that $\mathrm{cr}([q_n], [q_0])$ goes to $+\infty$. Hence for every pair of points in I, the associated projective line is contained in I. In particular I is a projective subspace. □

Let $\mathrm{Gr}_d^{(n,n)}(\mathcal{Q}) \subset \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be the open subset of $\mathrm{Gr}_d(\mathcal{Q})$ of pencils P such that for all $q \in P$ nonzero, q has signature (n, n). We now apply the previous results to prove that representations that admit an equivariant fitting immersion in this open set, and more generally a fitting map with a fitting flow, are Anosov. For examples of such immersions, see Proposition 7.1. Here again $M = \widetilde{N}$ for a compact N, $\Gamma = \pi_1(N)$ and $\rho : \Gamma \to \mathrm{SL}(V)$ is a representation.

Theorem 5.4. *Let $u : \widetilde{N} \to \mathrm{Gr}_d^{(n,n)}(\mathcal{Q})$ be a continuous ρ-equivariant map that admits an equivariant fitting flow. The representation ρ is $\{n\}$-Anosov, and for any $q \in \mathbb{S}u^*\mathcal{E}$, the limit map $\xi_\rho^n(\zeta) \in \mathrm{Gr}_n(\mathbb{R}^{2n})$ at the limit $\zeta \in \partial\Gamma$ of the flow line $(\Phi_t(q))$ is the following projective subspace:*

$$\bigcap_{t \geq 0} \{\pi(\Phi_t(q)) \leq 0\}.$$

Note that the flow lines of an equivariant fitting flow are quasi-geodesics by Proposition 4.5, and $\mathbb{S}u^*\mathcal{E}$ is quasi-isometric to Γ, hence the limit point ζ is well-defined.

Proof. The flat V-bundle over $\mathbb{S}u^*\mathcal{E}$ associated with ρ admits a continuous splitting $E \oplus F$ where for $q \in \mathbb{S}u^*\mathcal{E}$:

$$\mathbb{P}(E_q) = \bigcap_{t \geq 0} \{\pi(\Phi_{-t}(q)) \geq 0\},$$
$$\mathbb{P}(F_q) = \bigcap_{t \geq 0} \{\pi(\Phi_t(q)) \leq 0\}.$$

This defines transverse vector subspaces by Proposition 5.3 since $\mathrm{cr}(\Phi_t(q), q)$ and $\mathrm{cr}(-\Phi_{-t}(q), -q)$ go to $+\infty$ when t goes to $+\infty$. Moreover the quadrics in the pencils in the image of u are of signature (n, n), one must have $\dim(E_q) = \dim(F_q) = n$ so this splitting is well-defined. This splitting is preserved by Φ.

We now construct a metric h on this flat V-bundle over $\mathbb{S}u^*\mathcal{E}$. Given $q \in \mathbb{S}u^*\mathcal{E}$ we define the symmetric bilinear form h_q on $V = E_q \oplus F_q$ so that this sum is orthogonal and h_q is equal to $\pi(q) \in \mathcal{Q}$ on E_q and $-\pi(q) \in \mathcal{Q}$ on F_q. Note that by definition of E_q and F_q, h_q is positive.

We also introduce an auxiliary symmetric bilinear form $\bar{h}$ of signature (n, n) on this flat V-bundle over $\mathbb{S}u^*\mathcal{E}$ so that the sum $E_q \oplus F_q$ is orthogonal and $\bar{h}_q$ is equal to $\pi(q) \in \mathcal{Q}$ on E_q and on F_q.

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Our first step is to compare the quadric $\bar{h}_q$ with $\pi(q)$. Let L be a projective line intersecting $\mathbb{P}(E_q)$ at some e and $\mathbb{P}(F_q)$ at some f. In Figure 3, we illustrate some $\mathbb{RP}^2 \subset \mathbb{RP}^{2n-1}$ containing the projective line L. Let ℓ_1, ℓ_1' be the zeroes of $\pi(q)$ on L and ℓ_2, ℓ_2' be the zeroes of $\bar{h}_q$ on L, so that ℓ_1, ℓ_2 lie in the same connected component of $L \setminus \{e, f\}$. Since N is compact, there exists a maximum $\delta < \infty$ for all $q \in \mathbb{S}u^*\mathcal{E}$ and all such projective line L of the quantity $|\log([\ell_1, \ell_2, f, e])|$.

Now we turn our attention to the contraction properties of Φ. Let $t > 0$ be a real number and let $q \in \mathbb{S}u^*\mathcal{E}$. Let $v \in E_q$ and $w \in F_q$. We are interested in the following ratio:

$$R = \frac{h_{\Phi_t(q)}(v) h_q(w)}{h_{\Phi_t(q)}(w) h_q(v)}.$$

Let e, f be the lines generated by v, w and L be the projective line joining them. Let ℓ_1, ℓ_1' be the zeroes of $\pi(q)$ on L, ℓ_2, ℓ_2' the zeroes of $\bar{h}_q$ on L, ℓ_3, ℓ_3' be the zeroes of $\pi(\Phi_t(q))$ on L and finally ℓ_4, ℓ_4' the zeroes of $h_{\Phi_t(q)}$ on L. We assume that $\ell_1, \ell_2, \ell_3, \ell_4$ all lie on the same component of $L \setminus \{e, f\}$.

The cross ratio $[\ell_2, \ell_4, f, e]$ is equal to $R^{\frac{1}{2}}$. Indeed ℓ_2 is generated by $h_q^{\frac{1}{2}}(w)v + h_q^{\frac{1}{2}}(v)w$ and ℓ_4 is generated by $h_{\Phi_t(q)}^{\frac{1}{2}}(w)v + h_{\Phi_t(q)}^{\frac{1}{2}}(v)w$, up to changing ℓ_i by ℓ_i' for $1 \le i \le 4$. Hence:

$$[\ell_2, \ell_4, f, e] = \frac{\left(h_q^{\frac{1}{2}}(w)v + h_q^{\frac{1}{2}}(v)w\right) \wedge w}{\left(h_q^{\frac{1}{2}}(w)v + h_q^{\frac{1}{2}}(v)w\right) \wedge v} \times \frac{\left(h_{\Phi_t(q)}^{\frac{1}{2}}(w)v + h_{\Phi_t(q)}^{\frac{1}{2}}(v)w\right) \wedge v}{\left(h_{\Phi_t(q)}^{\frac{1}{2}}(w)v + h_{\Phi_t(q)}^{\frac{1}{2}}(v)w\right) \wedge w} = R^{\frac{1}{2}}.$$

However due to our comparison of $\pi(q)$ and $\bar{h}_q$ one has:

$$e^{-\delta} \le [\ell_2, \ell_4, f, e], [\ell_1, \ell_3, f, e] \le e^{\delta}.$$

Therefore $[\ell_2, \ell_4, f, e]/[\ell_1, \ell_3, f, e] \ge e^{-2\delta}$. Hence $R^{\frac{1}{2}} > e^{-2\delta}[\ell_1, \ell_3, f, e]$. This last cross ratio is larger than:

$$[\ell_1, \ell_3, \ell_3', \ell_1'] \ge \operatorname{cr}(\pi(\Phi_t(q)), \pi(q),).$$

Since Φ is a fitting flow and since N is compact, there exist $\alpha > 0$ such that $\operatorname{cr}(\pi(\Phi_1(q)), \pi(q)) \ge e^{\alpha}$ for all $q \in \mathbb{S}u^*\mathcal{E}$. Hence by the triangular inequality from Proposition 5.2, for all $t > 0$:

$$\operatorname{cr}(\pi(\Phi_t(q)), \pi(q)) \ge e^{\alpha(t-1)}.$$

Hence we get the following estimate:

$$\frac{h_{\Phi_t(q)}(v) h_q(w)}{h_{\Phi_t(q)}(w) h_q(v)} \ge e^{2\alpha t - 4\delta - 2\alpha}.$$

This implies that the splitting $V = E_q \oplus F_q$ is $\{n\}$-contracting in the sense of [BPS19] with respect to the flow Φ for the metric h. Moreover Γ acts cocompactly on $\mathbb{S}u^*\mathcal{E}$ and every geodesic in Γ is at uniform distance from a flow line of Φ. The domination of this splitting implies an exponential gap for the singular values [BPS19, Theorem 2.2], which implies that ρ is $\{n\}$-Anosov. The vector subspace $\xi_\rho^n(\zeta)$ is the contracted subspace F_q. □

5.2. *Fibered domain of discontinuity*

Such an equivariant map into the space of pencils that admits a fitting flow induces a fibration of the Guichard-Wienhard domain of discontinuity.

Recall that $u^o(x) \subset S^2V$ is the annihilator subspace of $u(x) \subset \mathcal{Q} = \left(S^2V\right)^*$.

Proposition 5.5. *Let N be a compact manifold of dimension d with fundamental group Γ. Let $\rho : \Gamma \to \mathrm{SL}(V)$ and $u : \widetilde{N} \to \mathrm{Gr}_d^{(n,n)}(\mathcal{Q})$ be a ρ-equivariant continuous map that admits an equivariant fitting flow Φ on $\mathbb{S}u^*\mathcal{E}$. The union of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ for $x \in \widetilde{N}$ covers all of $\mathbb{P}(S^2V^{>0})$, and the closure of this union intersects the space of rank one points $\mathbb{P}(V) \simeq S^2\mathbb{P}(V) \subset \mathbb{P}(S^2V^{\geq 0})$ exactly at the domain of discontinuity for $\{n\}$-Anosov representations considered by Guichard-Wienhard [GW12]:*

$$\Omega = \mathbb{P}(V) \setminus \bigcup_{\zeta \in \partial\Gamma} \mathbb{P}(\xi_\rho^n(\zeta)). \tag{3}$$

The intersection of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ with the set of rank one points for $x \in \widetilde{N}$ defines a fibration over $\widetilde{N}$ of Ω.

In this argument we will use the Hilbert distance on $\mathbb{P}(S^2V^{>0})$ already introduced for the proof of Proposition 4.5. For a subset $A \subset \mathbb{P}(V)$ we write $S^2A \subset \mathbb{P}(S^2V)$ the corresponding set of rank one lines.

Proving first that $\mathbb{P}(S^2V^{>0})$ is fully covered helps us proving that the Guichard -Wienhard domain is also fully covered.

Proof. Let us first prove that all of $\mathbb{P}(S^2V^{>0})$ is covered by the union of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ for $x \in \widetilde{N}$. We fix a Riemannian metric on N that defines a Riemannian metric g on $\widetilde{N}$, with associated distance d_g. Since u is globally fitting, see Remark 4.6, for all $x \in \widetilde{N}$ there exists a neighborhood U of $\mathbb{P}(u^\circ(x) \cap S^2V^{\geq 0})$ in $\mathbb{P}(S^2V^{\geq 0})$ that is covered by the manifolds $\mathbb{P}(u^\circ(y) \cap S^2V^{\geq 0})$ for y in the ball for d_g of radius 1 centered at 0, by Proposition 3.12, part (i).

This neighborhood contains an ϵ-neighborhood of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ for the Hilbert metric on $\mathbb{P}(S^2V^{>0})$ for some $\epsilon > 0$. Since N is compact, this $\epsilon > 0$ can be chosen independently of x.

Now let us fix some $x_0 \in \widetilde{N}$ and some $[s_0] \in \mathbb{P}(u^\circ(x_0) \cap S^2V^{>0})$. Given any $[s'] \in \mathbb{P}(S^2V^{>0})$ one can find a finite sequence $s_0, s_1, \cdots, s_k = s'$ in $S^2V^{>0}$ so that the Hilbert distance between $[s_i]$ and $[s_{i+1}]$ is less than ϵ for all $0 \leq i < k$. By induction, and since the Riemannian metric d_g is complete, one can construct for all $1 \leq i \leq k$ a point $x_i \in \widetilde{N}$ such that $[s_i] \in \mathbb{P}(u^\circ(x_i) \cap S^2V^{>0})$ and $d_H(x_{i-1}, x_i) \leq 1$. Therefore the manifolds $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ cover all of $\mathbb{P}(S^2V^{>0})$.

Now let us consider the fibered domain in projective space. We first prove that the fibers are contained in the domain $S^2\Omega$. Consider a rank one line $[s] \in S^2\mathbb{P}(\xi_\rho^n(x))$ for some $\zeta \in \partial\Gamma$. Suppose that p belongs to $u^\circ(x)$ for some $x \in \widetilde{N}$. There exists a flow line $(\Phi_t([q]))_{t\geq 0}$ starting at $\mathrm{pr}([q]) = x$ and converging to $\zeta \in \partial\Gamma \simeq \partial\widetilde{N}$ by Proposition 4.5. Theorem 5.4 implies that $\pi(q) \in \mathcal{Q}$ must be negative on $\xi_\rho^2(\zeta)$ and hence p cannot belong to $u^\circ(x)$.

Conversely let us prove that every point in $S^2\Omega$ is covered by some fiber. Fix a point $x \in \widetilde{N}$ and take any rank one point $[s] \in S^2\Omega$ in the Guichard-Wienhard domain of discontinuity. There exists a sequence (x_n) such that $[s]$ belongs to the limit of $\mathbb{P}(u^\circ(x_n) \cap S^2V^{\geq 0})$, since these manifolds cover $\mathbb{P}(S^2V^{>0})$. We consider some $[q_n] \in \mathbb{S}\mathcal{E}_x$ such that $\Phi_t([q_n]) \in \mathbb{S}\mathcal{E}_{x_n}$ for some $t_n > 0$, which exist by Proposition 4.5. If t_n diverges when n varies, then the set $\mathbb{P}(u^\circ(x_n) \cap S^2\mathbb{P}(V))$ becomes arbitrarily close to $S^2\mathbb{P}(\xi_\rho^n(\zeta_n))$ where ζ_n is the limit when t goes to $+\infty$ of $\Phi_t([q_n])$. This would contradict the fact that $[s] \in S^2\Omega$, as in this case $[s] \in \mathbb{P}(\xi_\rho^n(\zeta))$ where ζ is a limit point of (ζ_n). Hence the sequence (x_n) is bounded and therefore converges up to subsequence to some $x_\infty \in \widetilde{N}$, and $[s] \in \mathbb{P}(u^\circ(x_\infty))$. □

6. Fitting maps and maximal representations

Let us consider representations $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$. We prove our main result, which is the characterization of maximal representations in terms of maps of pencils. The first part introduces ω-regular pencils, as well as a connected component of the space of ω-regular pencils. We then state the characterization, and then present the construction of a map of pencils with a fitting flow for any maximal representation. This construction relies on a map from the space of pairs of Lagrangians to the space of quadrics.

Throughout this section we set $d = 2$, and consider the case when $N = S_g$ is a surface.

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6.1. Definition of maximal pencils

We say that a quadric q in $\left(\mathbb{R}^{2n}, \omega\right)$ is *ω-regular* if it is positive on some Lagrangian ℓ_1 and negative on some Lagrangian ℓ_2. Note that if ℓ_1, ℓ_2 are only assumed to be n dimensional subspaces, this condition is exactly the condition of having signature (n, n). We call $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ the space of pencils hose nonzero elements are all ω-regular, which is an open subset of $\mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$. These pencils have in particular the property that the corresponding subsets of $\mathbb{P}(S^2V)$ intersect transversely the symmetric space of $\mathrm{Sp}(2n, \mathbb{R})$, see Lemma 7.4.

Remark 6.1. If a locally fitting map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$ admits a fitting flow which is equivariant with respect to a representation $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$, the image of u must lie in $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ as ρ is $\{n\}$-Anosov by Theorem 5.4, and the limit map of $\{n\}$-Anosov representations in $\mathrm{Sp}(2n, \mathbb{R})$ takes values in the space of Lagrangians. This last fact is a consequence of the fact that an attractive fixed n-dimensional subspace of an element in $\mathrm{Sp}(2n, \mathbb{R})$ is necessarily Lagrangian, see, for instance, [GW12].

There are nonmaximal representations admitting equivariant fitting immersions, for instance almost-Fuchsian representations in $\mathrm{SL}(2, \mathbb{C}) \subset \mathrm{Sp}(4, \mathbb{R})$, see Appendix A. In order to obtain the maximality property, we need to restrict ourselves to the correct union of connected component of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$.

Let $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ be a pencil, and fix an orientation for P. Recall that $\mathcal{L}_n$ in the space of Lagrangians in $\mathbb{R}^{2n}$. We construct a "boundary map" for an ω-regular pencil of quadrics, defined up to homotopy. Before defining this map, note the following:

Lemma 6.2. *Let $q \in \mathcal{Q}$ be an ω-regular element. The set of Lagrangians ℓ such that q is positive on ℓ is homeomorphic to an open ball.*

Proof. There exist some $\ell_+ \in \mathcal{L}_n$ on which q is positive. Moreover there exist some $\ell_- \in \mathcal{L}_n$ on which q is negative.

Every Lagrangian ℓ on which q is positive must be transverse to ℓ_-, hence it can be written as the graph $\{x + u(x) \mid x \in \ell_+\}$ of some linear map $u : \ell_+ \to \ell_-$, and one has for all $v \in \ell^+$:

$$q(v, v) + q(u(v), u(v)) + 2q(v, u(v)) = q(v + u(v), v + u(v)) > 0.$$

Since $q(v, v) > 0$ and $q(u(v), u(v)) < 0$, for all $0 < \lambda < 1$:

$$q(v + \lambda u(v), v + \lambda u(v)) = q(v, v) + \lambda^2 q(u(v), u(v)) + 2\lambda q(v, u(v)) > 0.$$

We can identify the elements of $\mathcal{L}_n$ transverse to ℓ_- as the vector subspace of the space of maps $u : \ell_+ \to \ell_-$. We just proved that in this chart the set of elements on which q is positive is open and star-shaped, hence it is an open ball. □

We now define the "boundary map" of the pencil.

Proposition 6.3. *Let $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$. There exists a continuous map $\xi_P : \mathbb{S}P \to \mathcal{L}_n$ such that for all $[q] \in \mathbb{S}P$, $q > 0$ on $\xi_P([q])$. Moreover any two such maps are homotopic, so the free homotopy type $[\xi_P]$ is well defined.*

Proof. A map ξ_P is exactly a section of the bundle $\{([q], \ell) \mid q_{|\ell} > 0\} \to \mathbb{S}P$, which is a fiber bundle whose fibers are open balls. Such sections always exist and are unique up to homotopy. □

We say that a pencil is *maximal* for some orientation if $[\xi_P] = n[\tau]$, where $[\tau]$ is the generator of $\pi_1(\mathcal{L}_n)$ introduced in Section 2.2. We denote by $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ the space of pencils that are maximal for some orientation. This is a union of connected components of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ as the homotopy type $[\xi_P]$ is locally invariant for $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$.

Remark 6.4. The previous discussion allows us to distinguish several connected components of the open subspace $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ by looking at the homotopy type of the boundary map ξ_P.

As a recall we have the following inclusions:

$$\mathrm{Gr}_2^{\max}(\mathcal{Q}) \subset \mathrm{Gr}_2^{\omega}(\mathcal{Q}) \subset \mathrm{Gr}_2^{(n,n)}(\mathcal{Q}) \subset \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q}) \subset \mathrm{Gr}_2(\mathcal{Q}).$$

All these inclusions are open, and the inclusion $Gr_2^{\max}(\mathcal{Q}) \subset \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ is a union of connected components.

6.2. *Statement of the characterization*

We obtain the following characterization of maximal representations in terms of the existence of locally fitting maps that admit a fitting flow.

Theorem 6.5. *A representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ admits a ρ-equivariant locally fitting map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that admits a ρ-invariant fitting flow if and only if it is maximal for some orientation of S_g.*

In this case the orientation of S_g for which ρ is maximal is induced by the orientation of the maximal pencils $u(x)$ for $x \in \widetilde{S_g}$ and Lemma 4.3.

In particular if a representation ρ admits an equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ then it is maximal because of Proposition 4.8. This theorem leaves the following question open:

Question 6.6. Given a maximal representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$, is there always an equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$?

We show in Section B that this is true for $\mathrm{Sp}(4, \mathbb{R})$, and in this case there exists a fitting immersion whose image lies in a single special $\mathrm{Sp}(4, \mathbb{R})$-orbit of $\mathrm{Gr}_2^{\max}(\mathcal{Q})$.

The following Lemma shows one direction of Theorem 6.5, the other direction is proven by Lemma 6.13.

Lemma 6.7. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a representation that admits an equivariant continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that admits an equivariant fitting flow. Then ρ is maximal for some orientation of S_g.*

Recall that the limit map of $\{n\}$-Anosov representations in $\mathrm{Sp}(2n, \mathbb{R})$ takes values in the space of Lagrangians, see Remark 6.1.

Proof. We first apply Theorem 5.4 which shows that ρ is $\{n\}$-Anosov. Let $x \in \widetilde{S_g}$, we prove that the homotopy type of the boundary curve ξ_ρ^n is the same as the homotopy type $[\xi_{u(x)}]$ from Proposition 6.3. We consider the map ζ_∞ that associates to $[q] \in \mathbb{S}u^*\mathcal{E}_x$ the limit of $\Phi_t([q])$ in $\partial\Gamma$. This map is homotopic in $\widetilde{S_g} \cup \partial\Gamma \setminus \{x\}$ to the map ζ_1 that associates the projection $p(\Phi_1)$. Hence by Lemma 4.3, ζ_1 defines a diffeomorphism of degree 1 between the circle $\mathbb{S}u(x)$ with its maximal orientation and the boundary $\partial\Gamma$ for the induced orientation.

The map $\xi_\rho^n \circ \zeta_\infty$ has the homotopy type $[\xi_{u(x)}]$ associated to $u(x)$: it defines a boundary map as in Proposition 6.3. The fact that ρ is maximal is then a consequence of the characterization of maximal representations from Theorem 2.5. Indeed $u(x) \in \mathrm{Gr}_2^{\max}(\mathcal{Q})$ implies that $[\xi_{u(x)}] = n[\tau]$, and we already know that the degree of ζ_∞ is equal to the degree of ζ_1 which is equal to 1. □

6.3. *Construction of a fitting flow*

In this section we study special quadrics in $\mathbb{R}^{2n}$ associated to pairs of transverse Lagrangians. These objects will allow us to construct fitting continuous embeddings of pencils.

Definition 6.8. Let ℓ_1, ℓ_2 be two transverse Lagrangians in $\mathbb{R}^{2n}$. We define q_{ℓ_1,ℓ_2} to be the symmetric bilinear form on $\mathbb{R}^{2n}$ such that if π_1, π_2 are the projections on ℓ_1, ℓ_2 associated to the direct sum $\ell_1 \oplus \ell_2 = V$:

$$q_{\ell_1,\ell_2}(v, v) = \omega(\pi_1(v), \pi_2(v)).$$

Note that $q_{\ell_2,\ell_1} = -q_{\ell_1,\ell_2}$.

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Remark 6.9. In particular q_{ℓ_1,ℓ_2} is characterized by the fact that ℓ_1 and ℓ_2 are isotropic and for all $v \in \ell_1, w \in \ell_2$:

$$q_{\ell_1,\ell_2}(v,w) = \omega(v,w).$$

Maximal triples of Lagrangians can be characterized as follows:

Lemma 6.10. *A triple of Lagrangians (ℓ_1, ℓ_2, ℓ_3) is maximal if and only if q_{ℓ_1,ℓ_3} is positive on ℓ_2.*

Proof. Let us write ℓ_1, ℓ_2, ℓ_3 as in Section 2.1 for some symplectic basis:

$$\ell_1 = \langle x_1, x_2, \cdots x_n \rangle,$$
$$\ell_2 = \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 y_2, \cdots x_n + \epsilon_n y_n \rangle,$$
$$\ell_3 = \langle y_1, y_2, \cdots y_n \rangle.$$

The form q_{ℓ_1,ℓ_3} can be written in this basis as:

$$q_{\ell_1,\ell_3} = \frac{1}{2} \sum_{i=1}^{n} x_i^* \otimes y_i^* + y_i^* \otimes x_i^*.$$

This form is positive on ℓ_2 if and only if all of the ϵ_i are positive, and hence if the triple is maximal. □

These quadrics have also the following remarkable properties for maximal quadruples of Lagrangians. A quadruple of Lagrangians $(\ell_1, \ell_2, \ell_3, \ell_4)$ is called *maximal* if each cyclic oriented subtriple is maximal.

Lemma 6.11. *Let $(\ell_1, \ell_2, \ell_3, \ell_4)$ be a maximal quadruple of Lagrangians. The bilinear form $q_{\ell_4,\ell_3} - q_{\ell_1,\ell_2}$ is positive. In particular the zero set of these quadrics define two disjoint quadric hypersurfaces in $\mathbb{P}(\mathbb{R}^{2n})$.*

Note that a triple of Lagrangians (ℓ_1, ℓ_2, ℓ_3) is maximal if and only if corresponding linear map $u \in \mathrm{Hom}(\ell_1, \ell_3)$ whose graph is equal to ℓ_2 is such that $\omega(\cdot, u(\cdot))$ is positive on ℓ_1.

Proof. Let us prove the first part of the statement. Since (ℓ_4, ℓ_1, ℓ_3) is a maximal triple of Lagrangians, ℓ_1 can be written as the graph of some linear map $u_1 : \ell_4 \to \ell_3$ such that $\omega(\cdot, u_1(\cdot))$ is a positive bilinear form on ℓ_4. Similarly since (ℓ_4, ℓ_2, ℓ_3) is a maximal triple of Lagrangians, ℓ_2 can be written as the graph of some linear map $u_2 : \ell_4 \to \ell_3$ such that $\omega(\cdot, u_2(\cdot))$ is a positive bilinear form on ℓ_4.

Let $v \in \mathbb{R}^{2n}$, it can be decomposed uniquely as $v = v_1 + v_2$ with $v_1 \in \ell_1$ and $v_2 \in \ell_2$. Moreover there exist some unique $x, y \in \ell_4$ such that $v_1 = x + u_1(x)$ and $v_2 = y + u_2(y)$. The vector v decomposes therefore as $v = x + y + u_1(x) + u_2(y)$. One computes that:

$$\begin{aligned} q_{\ell_4,\ell_3}(v,v) - q_{\ell_1,\ell_2}(v,v) &= \omega(x+y, u_1(x) + u_2(y)) - \omega(x + u_1(x), y + u_2(y)), \\ &= \omega(x, u_1(x)) + \omega(y, u_2(y)) + 2\omega(y, u_1(x)). \end{aligned}$$

Finally the fact that (ℓ_1, ℓ_2, ℓ_3) forms a maximal triple implies that the bilinear form $\omega(\cdot, \widetilde{u_2}(\cdot))$ is positive on ℓ_1, where $\widetilde{u_2} \in \mathrm{Hom}(\ell_1, \ell_3)$ corresponding to u_2. In particular for all nonzero $x \in \ell_4$:

$$\omega(x + u_1(x), \widetilde{u_2}(x + u_1(x)) = \omega(x, \widetilde{u_2}(x + u_1(x)) > 0.$$

However note that by definition $x + u_1(x) + \widetilde{u_2}(x + u_1(x)) = x + u_2(x)$ and in particular $\widetilde{u_2}(x + u_1(x)) = (u_2 - u_1)(x)$. Hence the symmetric bilinear form $\omega(\cdot, (u_2 - u_1)(\cdot))$ is positive on ℓ_4.

In particular for $y \neq 0$ the previous expression is strictly greater than the following $\omega(x, u_1(x)) + \omega(y, u_1(y)) + 2\omega(y, u_1(x))$, which is non-negative since $\omega(\cdot, u_1(\cdot))$ is positive. In the case when $y = 0$, this last inequality is strict for $x \neq 0$. Otherwise the previous inequality $\omega(y, (u_2 - u_1)(y)) > 0$ is strict. Therefore for $v \neq 0$, $q_{\ell_4,\ell_3}(v,v) - q_{\ell_1,\ell_2}(v,v) > 0$. □

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We now state an infinitesimal version of Lemma 6.11 that we will use in Section B.

Lemma 6.12. *Let $\ell^+, \ell^- : [0,1] \to \mathcal{L}_n$ be smooth and such that $\ell_0^+ = \ell^+(0)$ and $\ell_0^- = \ell^-(0)$ are transverse and the linear maps $\dot{u}^+ \in \mathrm{Hom}(\ell_0^+, \ell_0^-)$ and $\dot{u}^- \in \mathrm{Hom}(\ell_0^+, \ell_0^+)$ corresponding to $(\ell^+)'(0)$ and $(\ell^-)'(0)$ are such that $\omega(\cdot, \dot{u}^+(\cdot))$ and $\omega(\dot{u}^-(\cdot), \cdot)$ are positive respectively on ℓ_0^+ and ℓ_0^-. The derivative at $t = 0$ of $q_{\ell^+(t),\ell^-(t)}$ is positive.*

Proof. The proof is similar to the previous one. Let $u^+(t) \in \mathrm{Hom}(\ell_0^+, \ell_0^-)$ and $u^-(t) \in \mathrm{Hom}(\ell_0^-, \ell_0^+)$ be the linear maps whose graph is equal to $\ell^+(t)$ and $\ell^-(t)$ respectively. Let $v \in \mathbb{R}^{2n}$. The derivative of the evaluation of $q_t = q_{\ell^+(t),\ell^-(t)}$ to v can be written in term of the derivative $\dot{v}_t^+$, $\dot{v}_t^-$ of the vectors $v_t^+ \in \ell_+(t)$, $v_t^- \in \ell_-(t)$ such that $v = v_t^+ + v_t^-$:

$$\dot{q}_0(v, v) = \omega(v_0^+, \dot{v}_0^-) + \omega(\dot{v}_0^+, v_0^-).$$

One has $\dot{v}_0^+ = -\dot{v}_0^-$ since $v = v_t^+ + v_t^-$ does not vary with t. The fact that $v_t^+ \in \ell_+(t)$ for all t implies that $\dot{v}_0^+$ can be written $w^+ + \dot{u}^+(v_0^+)$ with $w^+ \in \ell_0^+$. Similarly $\dot{v}_0^-$ can be written $w^- + \dot{u}^-(v_0^+)$ with $w^- \in \ell_0^-$. Note that $\omega(v_0^+, w^+) = 0$ as they belong to the same Lagrangian ℓ_0^+. Similarly $\omega(v_0^-, w^-) = 0$ and hence:

$$\dot{q}_0 = 2\omega(v_0^+, \dot{u}^+(v_0^+)) + 2\omega(\dot{u}^-(v_0^-), v_0^-).$$

This is positive by our assumption. □

We now construct an equivariant continuous map that admits a well-fitting flow.

Proposition 6.13. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. There exists a continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that admits a fitting flow.*

We prove this lemma using some averaging argument, where the basic building blocks are the quadrics associated to pairs of Lagrangians in the limit curve. This construction is not unique, as we start by fixing a hyperbolic metric.

Proof. Fix a hyperbolic metric on S_g. Since S_g is oriented it admits an associated complex structure J. For $v \in T_x^1\widetilde{S_g}$ write $\ell_v = \xi_\rho^n(\zeta_v)$ where $\zeta_v \in \partial\Gamma$ is the limit point of the geodesic with initial derivative v. We furthermore define:

$$q_v^\circ = q_{\ell_{Jv}, \ell_{-Jv}}.$$

These quadrics for a fixed $x \in \widetilde{S_g}$ do not in general define a pencil of quadrics. We therefore define the following quadric associated to $v \in T_x\widetilde{S_g}$:

$$q_v = \int_{w \in T_x^1\widetilde{S}} \langle v, w\rangle q_w^\circ \mathrm{d}\lambda.$$

Here we take the integral for the measure λ on $T_x^1\widetilde{S}$ induced by the hyperbolic metric. For each $x \in \widetilde{S}$, we consider the pencil $u(x) = \{q_v | v \in T\widetilde{S_g}\} \in \mathrm{Gr}_2(\mathcal{Q})$ which is well defined since q_v depends linearly on $v \in T_xS$.

First we check that these pencils are in $\mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$, by proving that they are actually ω-regular.

Let $v \in T_x^1\widetilde{S_g}$ be nonzero for some $x \in \widetilde{S_g}$. For all $w \in T_x\widetilde{S_g}$ if $\langle w, v\rangle > 0$, the triple $(\zeta_{-Jw}, \zeta_v, \zeta_{Jw})$ is positively oriented and hence $(\ell_{-Jw}, \ell_v, \ell_{Jw})$ is maximal and hence q_w° is negative on ℓ_v. If $\langle w, v\rangle < 0$, the triple $(\ell_{Jw}, \ell_v, \ell_{-Jw})$ is maximal and hence q_w° is positive on ℓ_v. Hence q_v is negative on ℓ_v, and by a similar argument q_v is positive on ℓ_{-v}, which are Lagrangians. In particular q_v is ω-regular for all $v \in T_x^1\widetilde{S_g}$, and so $u(x)$ is ω-regular.

We consider the geodesic flow on $u^*\mathbb{S}\mathcal{E} \simeq T^1\widetilde{S_g}$, and we prove that this flow is fitting. Let $t > 0$, $x \in \widetilde{S_g}$ and $v \in T_x^1\widetilde{S_g}$. Let (y, v') be the image of (x, v) by the geodesic flow at time t, and let $\phi : T_x^1\widetilde{S_g} \to T_y^1\widetilde{S_g}$ be the identification given by the parallel transport along the geodesic between x and y.

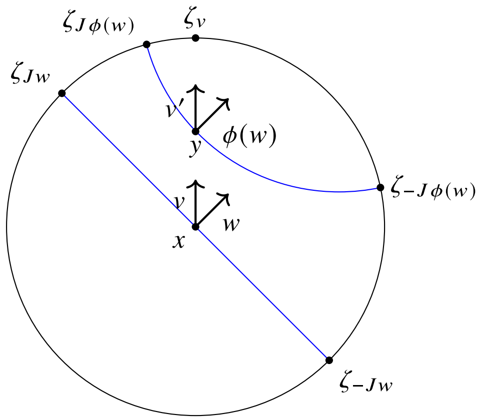

Figure 4. *Proof of Lemma 6.13.*

Let $w \in T^1_x\widetilde{S}$ be such that $\langle w, v\rangle > 0$. The following quadruple is positive $(\zeta_{Jw}, \zeta_{-Jw}, \zeta_{-J\phi(w)}, \zeta_{J\phi(w)})$ due to the negative curvature of the metric we put on $\widetilde{S_g}$, see Figure 4. Hence the corresponding quadruple of Lagrangians is maximal. Therefore $q^\circ_{\phi(w)} - q^\circ_w$ is positive by Lemma 6.11. When $\langle w, v\rangle < 0$, the following quadruple is positive $(\zeta_{-Jw}, \zeta_{Jw}, \zeta_{J\phi(w)}, \zeta_{-J\phi(w)})$ and therefore $q^\circ_{\phi(w)} - q^\circ_w$ is negative. Hence:

$$q_{v'} - q_v = \int_{w \in T^1_x\widetilde{S}} \langle v, w\rangle \left(q^\circ_{\phi(w)} - q^\circ_w\right) \mathrm{d}\lambda > 0.$$

We therefore have proven that the geodesic flow for the fixed hyperbolic metric is a fitting flow on $T^1\widetilde{S_g} \simeq \mathbb{S}u^{\mathcal{E}}$.

Finally as in the proof of Lemma 6.7 the homotopy type of ξ^n_ρ is equal to the homotopy type of $\xi_{u(x)}$ for all $x \in \widetilde{S_g}$. Hence these pencils are in $\mathrm{Gr}^{\max}_2(\mathcal{Q})$. □

Remark 6.14. These pencils always lie in the same connected component of $\mathrm{Gr}^\omega_2(\mathcal{Q})$. Indeed one can construct such a pencil $u(x) \in \mathrm{Gr}^\omega_2(\mathcal{Q})$ given any maximal continuous map ξ from $\partial\widetilde{S_g}$ into the space of Lagrangians. The space of such maximal continuous maps being path connected, any two such pencils can be joined by a path in $\mathrm{Gr}^\omega_2(\mathcal{Q})$. It is not clear if $\mathrm{Gr}^{\max}_2(\mathcal{Q})$ only contains this connected component.

7. Geometry of the symmetric space

In this section we prove Proposition 7.1 and Lemma 7.4 which are two facts independent from the main results of the paper. We show how to construct fitting immersions of pencils using totally geodesic surfaces in the symmetric space $\mathbb{P}(S^2V^{>0})$ as in [Dav23]. We then prove that the codimension d submanifolds corresponding to pencils in $\mathrm{Gr}^\omega_d(\mathcal{Q})$ intersect transversely the symmetric space of $\mathrm{Sp}(2n, \mathbb{R})$ embedded in $\mathbb{P}(S^2V^{>0})$.

Given an immersion $h : M \to \mathbb{P}(S^2V^{>0})$ from a manifold M of dimension d we define its *Gauss map* $\mathcal{G}h : M \to \mathrm{Gr}_d(\mathcal{Q})$, that associates to $x \in M$ the pencil P associated with the codimension d projective subspace of $\mathrm{Gr}_d(\mathcal{Q})$ orthogonal to $\mathrm{d}h(TM)$ at $h(x)$, for the $\mathrm{SL}(V)$-invariant Riemannian metric on $\mathbb{P}(S^2V)$.

The invariant Riemannian metric of the symmetric space associated to $\mathrm{SL}(V)$ can be described by a natural identification between $\mathbb{P}(S^2V^{>0})$ and its dual cone $\mathbb{P}(S^2(V^*)^{>0})$. We therefore reformulate the definition of $\mathcal{G}h$ as follows.

We first identify $\mathbb{P}(S^2V^{>0})$ with $\mathbb{P}(S^2(V^*)^{>0})$ via the map $[X] \mapsto [X^{-1}]$. Note as once again we view elements of S^2V and $S^2(V^*) = \mathcal{Q}$ respectively as maps $V^* \to V$ and $V \to V^*$. An immersion $h : M \to \mathbb{P}(S^2V^{>0})$ hence defines a dual immersion $h^* : M \to \mathbb{P}(S^2(V^*)^{>0})$. Fixing a volume form on V and V^* allows us to lift this map to a map $\overline{h}^*$ into the space of elements in $S^2(V^*)^{>0}$ whose corresponding map $V \to V^*$ has determinant 1. We define $\mathcal{G}h(x) = \mathrm{d}\overline{h}^*(T_xM) \subset S^2V^* = \mathcal{Q}$.

Proposition 7.1. *Let $h : M \to \mathbb{P}(S^2V^{>0})$ be a totally geodesic immersion. Suppose that the image of the Gauss map $\mathcal{G}h : M \to \mathrm{Gr}_d(\mathcal{Q})$ contains only regular pencils, that is, pencils containing only nondegenerate quadrics. The immersion $\mathcal{G}h$ is then a fitting immersion. If h is complete it is a globally fitting map.*

This proposition can be applied for instance to the totally geodesic surface associated to any representation of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{Sp}(2n, \mathbb{R})$. This can also be applied to the totally geodesic immersion of $\mathbb{H}^3$ induced by the inclusion $\mathrm{SL}(2, \mathbb{C}) \subset \mathrm{Sp}(4, \mathbb{R})$.

Remark 7.2. The fibration of a domain of $\mathbb{P}(V)$ induced by the fitting map in this proposition is a particular case of Theorem 6.3 from [Dav23].

Note that if $d \geq 2$, the signature (a, b) of the quadrics of a regular pencil must satisfy $a = b$ since $\mathbb{S}^{d-1}$ is connected. Hence the condition that the pencils are regular can be replaced in this case by the condition that the pencils belong to $\mathrm{Gr}_d^{(n,n)}(\mathcal{Q})$ where $\dim(V) = 2n$.

As a corollary one can construct fitting immersions for some representations that factor through $\mathrm{SL}(2, \mathbb{R})$. Indeed if one has a representation $\iota : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(V)$ there exists a ι-equivariant totally geodesic map $h : \mathbb{H}^2 \to \mathrm{SL}(V)$, see Proposition 7.1.

Proof. Let $\gamma : \mathbb{R} \to M$ be a geodesic for the metric induced by h. Recall that $\overline{h}(\gamma(t))$ for $t \in \mathbb{R}$ is an element of S^2V, and hence it defines a positive symmetric endomorphism of V^* and as such it is diagonalizable in an orthonormal basis with positive coefficients. We write the representative of $h(\gamma(o))$ with determinant 1 in a basis $(e_i)_{i \in I}$ such that for some $\lambda^i \in \mathbb{R}$, for all $t \in \mathbb{R}$:

$$\overline{h}(\gamma(t)) = \sum_{i \in I} e^{t\lambda_i} e_i \otimes e_i.$$

The dual immersion can be written as:

$$\overline{h}^*(\gamma(t)) = \sum_{i \in I} e^{-t\lambda_i} e_i^* \otimes e_i^*.$$

The element $q_t \in \mathcal{Q}$ corresponding to $(\overline{h}^* \circ \gamma)'(t)$ is the symmetric bilinear form:

$$q_t = \sum_{i \in I} -\lambda_i e^{-t\lambda_i} e_i^* \otimes e_i^*.$$

The derivative of (q_t) at $t = 0$ equals:

$$\sum_{i \in I} \lambda_i^2 e^{-t\lambda_i} e_i^* \otimes e_i^*.$$

This is a positive bilinear form if and only if all the λ_i are nonzero, which is the case if and only if the bilinear forms q_t are nondegenerate, that is, if the image of $\mathcal{G}h$ contains only regular pencils. In this case, the positivity of the derivative of (q_t) implies that the geodesic flow on $\mathbb{S}TM$ induces a fitting flow on $\mathcal{G}h^*\mathbb{S}\mathcal{E}$, so $\mathcal{G}h$ is a fitting immersion.

If moreover h is complete, it is ρ-equivariant for the discrete and faithful action of some closed surface group Γ. Hence Corollary 4.7 implies that h is a globally fitting map. □

Let us fix a symplectic form ω on $V = \mathbb{R}^{2n}$. Let $\mathbb{X}_{\mathrm{Sp}}$ be the subset of $\mathbb{P}(S^2V^{>0})$ consisting of tensors $[q^{-1}]$ that are compatible with ω, that is, such that for some complex structure J on $\mathbb{R}^{2n}$, the bilinear form $q + i\omega$ is a J-hermitian metric on V. Recall that $q : V \to V^*$ is a bilinear form, and $q^{-1} : V^* \to V$ is a tensor.

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The space $\mathbb{X}_{\mathrm{Sp}}$ is a copy of the symmetric space associated to $\mathrm{Sp}(2n,\mathbb{R})$, which is a totally geodesic subspace of the symmetric space associated to the Lie group $\mathrm{SL}(2n,\mathbb{R})$ whose model is $\mathbb{P}(S^2V^{>0})$. However it is not a projective subset: the closure of the projective convex hull of $\mathbb{X}_{\mathrm{Sp}}$ in $\mathbb{P}(S^2V^{>0})$ is equal to $\mathbb{P}(S^2V^{\geq 0})$ since it contains all the extreme points of $\mathbb{P}(S^2V^{\geq 0})$, that is, the rank one elements $S^2\mathbb{P}(\mathbb{R}^{2n})$.

Indeed, we can write matrices of the form $\mathrm{Diag}(\lambda, 1, \cdots, 1, \lambda^{-1}) \in \mathrm{Sp}(2n,\mathbb{R})$ for any $\lambda \in \mathbb{R}_{>0}$ and such that the first basis spans any fixed line $\ell \in \mathbb{RP}^{2n-1}$, one can observe that the closure of the orbit of any point of $\mathbb{X}_{\mathrm{Sp}}$ by these elements contains the rank one point corresponding to ℓ. Hence the closure of $\mathbb{X}_{\mathrm{Sp}}$ contains all rank one points.

The intersection of $\mathbb{X}_{\mathrm{Sp}}$ with a general linear subspace is not necessarily transverse. However it is the case for some special subspaces.

Definition 7.3. We say that an element $q \in \mathcal{Q}$ is *ω-regular* if for some Lagrangians ℓ^+, ℓ^- the bilinear form q is positive on ℓ^+ and negative on ℓ^-.

We denote by $\mathrm{Gr}_d^{\omega}(\mathcal{Q})$ the set of pencils whose nonzero elements are ω-regular.

In particular an ω-regular pencil q has signature (n, n).

Lemma 7.4. *Let $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ be an ω-regular pencil, that is, such that all its nonzero elements are ω-regular. The space $\mathbb{P}(P^\circ)$ intersects transversely the manifold $\mathbb{X}_{\mathrm{Sp}}$ in a codimension 2 submanifold.*

Proof. Let $q \in \mathcal{Q}$ be ω-regular, and let $x \in \mathbb{X}_{\mathrm{Sp}} \cap \mathbb{P}(\langle q\rangle^\circ)$ be an intersection point. Up to acting by $\mathrm{Sp}(2n,\mathbb{R})$, one can assume that $x = [X^{-1}] \in \mathbb{P}(S^2V)$ where X is the bilinear form whose associated matrix in some symplectic basis is:

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

The annihilator $(T\mathbb{X}_{\mathrm{Sp}})^\circ \subset S^2V$ of the tangent space to $\mathbb{X}_{\mathrm{Sp}}$ at this point, viewed as a linear subspace of $\mathcal{Q}$ can be identified with the space of following symmetric matrices where A is symmetric and B is antisymmetric:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Suppose that the intersection is not transverse, that is, that q can be written in this form. Let ℓ^+ and ℓ^- be Lagrangians on which q is respectively positive and negative. Since the maximal compact $U(n)$ acts transitively on the space of Lagrangians one can assume that $\ell^+ = \langle x_1, \cdots, x_n\rangle$. Let $\ell_0^- = \langle y_1, \cdots, y_n\rangle$ be its orthogonal for the fixed euclidean metric. The Lagrangian ℓ^- is transverse to ℓ^+ so for some symmetric matrix U, one can view ℓ^- as the image of:

$$\begin{pmatrix} U \\ I_n \end{pmatrix}.$$

The fact that q is positive on ℓ^+ implies that A is positive. The fact that q is negative on ℓ^- implies that the following is negative:

$$UAU + A - BU + UB.$$

But A and UAU are both positive and the bracket $[B, U]$ has trace zero so it cannot be negative. Hence the intersection must be transverse. $\square$

Proposition 7.5. *A tangent vector in $T\mathbb{X}_{\mathrm{Sp}} \subset T\mathbb{P}(S^2(V^*)^{>0})$ is ω-regular if and only if the corresponding element of $\mathcal{Q}$ is nondegenerate.*

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Proof. Given v $\in T\mathbb{X}_{\mathrm{Sp}}$ one can write the corresponding element $q \in \mathcal{Q}$ for some symplectic basis $(x_1, \cdots x_n, y_1, \cdots y_n)$ as:

$$\sum_{i=1}^{n} \lambda_i (x_i^* \otimes x_i^* - y_i^* \otimes y_i^*).$$

This bilinear form is regular if and only if all the λ_i are nonzero. Else we can assume that $\lambda_i > 0$ up to exchanging x_i by y_i and y_i by $-x_i$ in the symplectic basis. Then the bilinear form is positive on the Lagrangian $\langle x_1, \cdots, x_n\rangle$ and negative on the Lagrangian $\langle y_1, \cdots, y_n\rangle$. □

A. Immersions in the space of geodesics in $\mathbb{H}^3$

A.1. The space of geodesics

In this section we set $V = \mathbb{C}^2$, and restrict ourselves to pencils of Hermitian quadrics. Note that we can see this way $\mathrm{SL}(2,\mathbb{C})$ as a subgroup of $\mathrm{Sp}(4,\mathbb{R}) \simeq \mathrm{Sp}(V,\omega)$ for ω the real part of a volume form on $\mathbb{C}^2$. The results of this Section serve as an illustration of the previously introduced notions of fitting pairs and fitting directions.

Let $\mathcal{H} \subset \mathcal{Q}$ be the subset of Hermitian bilinear forms on $V = \mathbb{C}^2$. A Hermitian form $q \in \mathcal{H}$ that is not semi-positive or semi-negative is of Hermitian signature $(1, 1)$, and hence its zero set in $\mathbb{CP}^1$ is a circle.

Let $S^2V = S^2_hV \oplus S^2_aV$ be the eigenspace decomposition for the operator $J \otimes J$ where J is the complex conjugation. Here S^2_hV is the 4-dimensional eigenspace associated to 1 and $S^2_aV = \mathcal{H}^\circ$, the 6-dimensional space associated to -1. The intersection of $\mathbb{P}(S^2_hV)$ with the space of positive tensors is the projective Klein model for $\mathbb{H}^3$.

The annihilator of a pencil $P \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$, that is, a plane $P \subset \mathcal{H}$ such that no nonzero vectors are semi-positive, is equal to $P^\circ = \mathcal{H}^\circ \oplus H$ where H is a plane in S^2_hV, which in turns corresponds to a geodesic in $\mathbb{H}^3$ in the projective Klein model.

A pencil of quadrics $P \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ vanishes completely on two points in $\mathbb{CP}^1$, as in Figure 5 where the zero set of three elements of the pencil are depicted. The space of mixed pencils in $\mathcal{H}$ can be identified with the space $\mathcal{G}$ of unoriented geodesics in $\mathbb{H}^3$.

Remark A.1. The space $\mathcal{G}$ is a $\mathrm{SL}(2,\mathbb{C})$-homogeneous space. Note that however the larger spaces $P \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ or $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ from the rest of the paper are not $\mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{SL}(2n,\mathbb{R})$ homogeneous spaces.

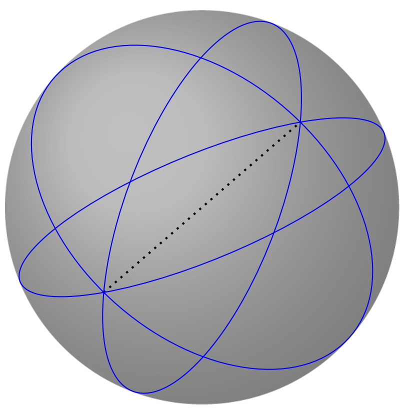

Figure 5. *Three Hermitian quadrics in a pencil and the corresponding geodesic in* $\mathbb{H}^3$.

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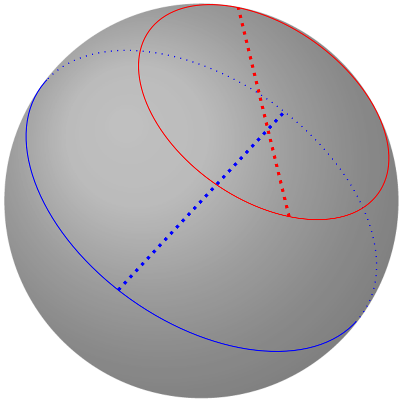

Figure 6. *Two disjoint geodesics in $\mathbb{H}^3$ and disjoint circles in $\mathbb{CP}^1$ between their endpoints.*

Two pencils in $\mathcal{H}$ form a fitting pair if and only if we can find a circle in each of the pencils that are disjoint. This is possible if and only if the two corresponding geodesics in $\mathbb{H}^3$ are disjoint, with disjoint endpoints. A fitting pair of pencils is illustrated in Figure 6.

The space $\mathcal{G}$ of unoriented geodesics in $\mathbb{H}^3$ admits a pseudo-Riemannian metric of signature $(2, 2)$: we now compare the notion of fitting vectors with the notion of spacelike vectors. The tangent space at a geodesic with endpoints $(x, y) \in \mathbb{CP}^1$ can be identified with $T_x\mathbb{CP}^1 \times T_y\mathbb{CP}^1$. The choice of a point in the geodesic provides an identification $\phi : T_x\mathbb{CP}^1 \to T_y\mathbb{CP}^1$ and a metric q_0 on $T_y\mathbb{CP}^1$. Taking a different point in the geodesic means replacing ϕ by $\lambda\phi$ and q_0 by $\lambda^{-1}q_0$ for some $\lambda \in R_{>0}$.

Hence we can consider the pseudo Riemannian metric that is invariant by the action of the isometry group of $\mathbb{H}^3$:

$$q : (T_x\mathbb{CP}^1 \times T_y\mathbb{CP}^1)^2 \to \mathbb{R}$$

$$(v_1, w_1), (v_2, w_2) \mapsto q_0(\phi(v_1), w_2) + q_0(\phi(v_2), w_1).$$

For this metric a vector $(v, w) \in T_x\mathbb{CP}^1 \times T_y\mathbb{CP}^1$ is spacelike if and only if $q_0(\phi(v), w) > 0$.

Proposition A.2. *A pair of geodesics is a fitting pair if and only if the corresponding geodesics are disjoint.*

A tangent vector (v, w) to $\mathcal{G} \simeq \mathrm{Gr}_2(\mathcal{H})$ is fitting if and only if $\phi(v)$ and w are not positively anti-colinear, that is, there are no $\lambda, \mu \in \mathbb{R}^{\geq 0}$ such that $\lambda\phi(v) = -\mu w$.

Remark A.3. In particular spacelike vectors are fitting, but not all fitting vectors are spacelike.

Proof. Let $P_1, P_2 \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$. One has $\mathbb{P}(S_a^2 V) \cap \mathbb{P}(S^2 V^{\geq 0}) = \emptyset$. Hence the subsets $\mathbb{P}(P_1^\circ \cap S^2 V^{\geq 0})$ and $\mathbb{P}(P_2^\circ \cap S^2 V^{\geq 0})$ for $P_1, P_2 \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ are disjoint if and only if the corresponding geodesics are disjoint.

Let (v, w) be a tangent vector to $\gamma \in \mathcal{G}$ and let γ_t be a curve in $\mathcal{G}$ with this derivative at $t = 0$. If $\phi(v)$ and w are not nonpositively colinear, the distance between $\gamma = \gamma_0$ and γ_t is greater than ϵt for some $\epsilon > 0$ and t small enough. Indeed there exist $z \in T_x\mathbb{CP}^1$ such that $q_0(\phi(z), \phi(x)), q_0(\phi(z), w) \geq 0$, and the totally geodesic disk in $\mathbb{H}^3$ through x, y normal to z at x contains γ_0 while being at distance ϵt to γ_t.

Hence the distance between the subsets $\mathbb{P}(P_0^\circ \cap S^2 V^{\geq 0})$ and $\mathbb{P}(P_t^\circ \cap S^2 V^{\geq 0})$ is also greater than $\epsilon' t$ for some $\epsilon' > 0$ and all t small enough. Therefore by Proposition 3.7 this direction is fitting.

Conversely if If $\phi(v)$ and w are nonpositively colinear, then there is such a curve γ_t such that the corresponding geodesics all have a common point. Therefore by Proposition 3.7 this direction is not fitting. □

Remark A.4. If S is a surface in $\mathbb{H}^3$ with principal curvature in $(-1, 1)$, then the set of normal geodesics forms spacelike surface, for the pseudo-Riemannian structure on the space of geodesics described

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in Section A.1. The corresponding map $\mathcal{G}u : S \to \mathcal{G}$ is called the *Gauss map*. Nearly Fuchsian representations are representations of a closed surface group Γ_g admitting an equivariant surface with principal curvature in $(-1,1)$. They are a priori a larger class of representations than almost Fuchsian representations, for which the equivariant surface with principal curvature in $(-1,1)$ is required to be minimal.

The space $\mathcal{G}$ also admits a special $\mathrm{SL}_2(\mathbb{C})$-invariant symplectic structure. An immersion in $\mathcal{G}$ is locally the Gauss map of an immersion with principal curvature in $(-1,1)$ if and only if it is spacelike and Lagrangian for this symplectic structure [ES22]. Therefore if the fitting immersion is not Lagrangian, it does not come as the Gauss map of a surface in $\mathbb{H}^3$. Hence there could be representations admitting fitting immersions that are not nearly Fuchsian.

A.2. *A quasi-Fuchsian representation with no fitting immersions*

Having a representation that is Anosov is not sufficient to ensure that there exist an equivariant fitting immersion. We show that there are quasi-Fuchsian representations that admit no such immersions of Hermitian pencils of quadrics. We use here the notations from Section A.1.

Theorem A.5. *There exists a quasi-Fuchsian representation $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C})$ for some genus g large enough that admits no ρ-equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H}) = \mathcal{G}$.*

Moreover it also admits no continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H}) = \mathcal{G}$ that admits a ρ-equivariant fitting flow.

A particular corollary is the following result on Nearly Fuchsian representations, introduced in Remark A.4.

Corollary A.6. *There exist quasi-Fuchsian representations that are not nearly Fuchsian.*

Almost Fuchsian representations were already known to not contain all quasi-Fuchsian representations see for instance [HW15], but a recent result of Nguyen, Schlenker and Seppi [NSS25] proves that nearly Fuchsian representations are not all almost Fuchsian, making this corollary new.

Here $V = \mathbb{C}^2$ and if $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C}) \subset \mathrm{Sp}(4,\mathbb{R})$, the Guichard-Wienhard domain of discontinuity Ω_ρ, mentioned in Proposition 5.5 corresponds to the pullback in $\mathbb{RP}^3 = \mathbb{P}_{\mathbb{R}}(\mathbb{C}^2)$, of a domain of discontinuity $\Omega_\rho^0 \subset \mathbb{CP}^1 = \mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$. This domain is the complement in $\mathbb{CP}^1$ of the limit set of ρ.

Since Γ_g is a surface group, this domain in $\mathbb{CP}^1$ is the union of two topological disks, and hence for each $x \in \widetilde{S_g}$ the geodesic corresponding to $u(x)$ has one endpoint in each of these discs. This fact will be used in the proof of Theorem A.5.

Another ingredient of the proof of Theorem A.5 is the following.

Proposition A.7 [HW15, Corollary 3.5]. *Given any $\mathcal{C}^1$ embedded circle γ in $\mathbb{CP}^1$, and any $\epsilon > 0$, there exists a quasi-Fuchsian representation $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C})$ for some genus g large enough whose limit set has Hausdorff distance at most ϵ with γ.*

Proof of Theorem A.5. We consider the Jordan curve γ from Figure 7. Let x, y, z be as in the figure. We consider a quasi-Fuchsian representation of Γ_g for a genus large enough such that its limit set Λ contains x, z and is close enough to γ using Proposition A.7. More precisely let s_x and s_z be the open arcs of the circle of $\mathbb{CP}^1$ passing through x, y, z respectively between x, y and y, z and let I be the interior of the Jordan curve Λ. We require that the union U_x of all the connected component of $I \setminus s_z$ whose closure contain x is disjoint from the union U_z of all the connected component of $I \setminus s_x$ whose closure contain z. These two disjoint sets are illustrated for the curve γ as the two gray regions.

Let $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C})$ then be such a quasi-Fuchsian representation. Suppose that it admits an equivariant continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ with an equivariant fitting flow Φ on $\mathbb{S}u^*\mathcal{E}$. By Proposition 4.5 there exists a flow line $(\Phi_t(q))_{t\in\mathbb{R}}$ such that its projection $\gamma : \mathbb{R} \to \widetilde{S_g}$ is a quasi-geodesic between the points of $\partial\Gamma$ corresponding to x and z in the limit set.

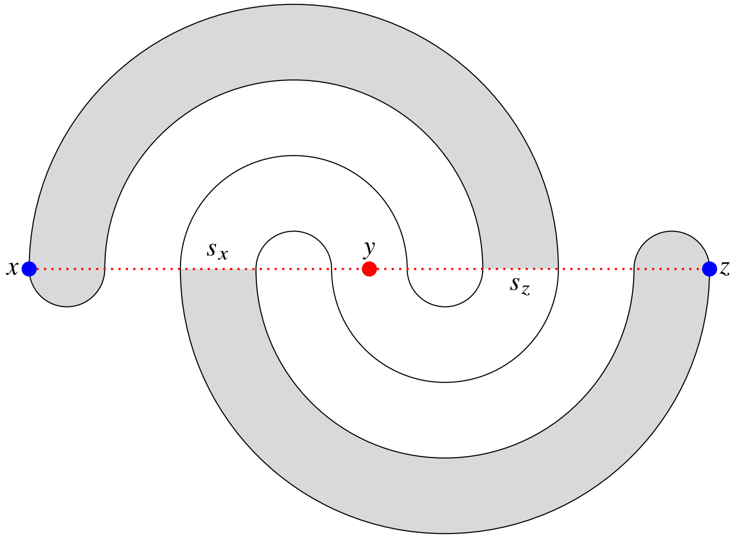

Figure 7. *A Jordan curve in* $\mathbb{CP}^1$.

For all $t \in \mathbb{R}$, let $x_t \in \mathbb{CP}^1$ be the endpoint of the geodesic corresponding to $u(\gamma(t))$ that belongs to the interior of the Jordan curve Λ. When t goes to $+\infty$, x_t converges to z and it converges to x for t going to $-\infty$. Moreover x_t always belong to the circle determined by $\Phi_t(q)$.

There exists a $t_0 \in \mathbb{R}$ such that y belongs to the circle $\Phi_{t_0}(q)$. Note that this great circle splits $\mathbb{CP}^1$ in two parts, one containing x and s_x and one containing z and s_z. In particular $(x_t)_{t \geq t_0}$ must lie in U_z and $(x_t)_{t \leq t_0}$ must lie in U_x, leading to a contradiction. Hence no such map u can exist. □

B. Fitting immersions and spacelike immersions for $\mathrm{Sp}(4,\mathbb{R})$

In this section we explain how the data of a maximal immersion into the Pseudo-Riemannian space $\mathbb{H}^{2,2}$ with principal curvature in $(-1,1)$ induces a fitting immersion. Combining this with a result from Collier-Tholozan-Toulisse we show that our characterization of maximal representations can be improved in $\mathrm{Sp}(4,\mathbb{R})$.

Note first that one has the following exceptional isomorphism:

$$\mathrm{PSp}(4,\mathbb{R}) \simeq \mathrm{SO}_o(2,3).$$

This isomorphism comes from the fact that $\mathrm{PSp}(4,\mathbb{R})$ preserves a subspace of dimension 5 of $\Lambda^2\mathbb{R}^4$, as well as a symmetric bilinear form of signature $(2,3)$ on this subspace. Hence $\mathrm{PSp}(4,\mathbb{R})$ acts naturally on the pseudo-Riemannian symmetric space with constant negative sectional curvature $\mathbb{H}^{2,2}$, which consists of vectors of norm -1 in $\mathbb{R}^{2,3}$. The space of Lagrangians $\mathcal{L}$ in $\mathbb{R}^4$ is naturally identified with the space of isotropic lines $\mathrm{Ein}^{1,2}$ in $\mathbb{R}^{2,3}$.

To a pointed totally geodesic spacelike plane (p,P) in $\mathbb{H}^{2,2}$ one can associate an element in a special G-orbit of $\mathrm{Gr}_2^{\max}(\mathcal{Q})$. For every geodesic in this plane passing through the base point, we consider the endpoints $\ell_1,\ell_2 \in \mathcal{L} \simeq \mathrm{Ein}^{1,2}$, and the space generated by all such quadrics q_{ℓ_1,ℓ_2} forms a plane which is an element of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$. Indeed for some symplectic basis (x_1,x_2,y_1,y_2) the Lagrangians corresponding to the boundary of the spacelike plane P are for $\theta \in [0,2\pi]$:

$$\ell(\theta) = \langle \cos\left(\frac{\theta}{2}\right)x_1 + \sin\left(\frac{\theta}{2}\right)y_1, \cos\left(\frac{\theta}{2}\right)x_2 + \sin\left(\frac{\theta}{2}\right)y_2\rangle.$$

The corresponding quadric $q_{\ell(0),\ell(\pi)}$ for $\theta = 0$ in the basis (x_1, y_1, x_2, y_2) is equal to:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let R_θ be the following rotation matrix:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The corresponding quadric $q_{\ell(\theta),\ell(\theta+\pi)}$ in the basis (x_1, y_1, x_2, y_2) is equal to:

$$\begin{pmatrix} R_{\frac{\theta}{2}} & 0 \\ 0 & R_{\frac{\theta}{2}} \end{pmatrix} q_{\ell(0),\ell(\pi)} \begin{pmatrix} R_{-\frac{\theta}{2}} & 0 \\ 0 & R_{-\frac{\theta}{2}} \end{pmatrix}.$$

Hence $q_{\ell(\theta),\ell(\theta+\pi)}$ is equal to:

$$\begin{pmatrix} \sin(\theta) & \cos(\theta) & 0 & 0 \\ \cos(\theta) & -\sin(\theta) & 0 & 0 \\ 0 & 0 & \sin(\theta) & \cos(\theta) \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \end{pmatrix}.$$

These quadrics span a plane in $\mathcal{Q}$ when θ varies, and these quadrics are ω-regular so this plane is in $\mathrm{Gr}_2^\omega(\mathcal{Q})$.

We define the Gauss map $\mathcal{G}u : S \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ of a spacelike immersion $u : S \to \mathbb{H}^{2,2}$ as the map that associates to $x \in S$ the pencils corresponding to the pointed totally geodesic spacelike plane $(u(x), P)$ where $T_{u(x)}P = \mathrm{d}u(T_xS)$. Let II_u be the second fundamental form of the immersion u.

Theorem B.1. *Let $u : S \to \mathbb{H}^{2,2}$ be a spacelike immersion such that for all $v \in TS$, $\|\mathrm{II}_u(v,v)\| < \|v\|^2$. The Gauss map $\mathcal{G}u : S \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ is a fitting immersion.*

Let S be a spacelike surface in $\mathbb{H}^{2,n}$. Let $\gamma : [0,1] \to S$ be a geodesic for the induced metric on S parametrized with unit speed and let $V : [0,1] \to TS$ be the unit orthogonal vector field to γ' in S along γ. We denote by $V^+, V^- : [0,1] \to \mathrm{Ein}^{1,2}$ the endpoints of the geodesic rays starting respectively at V and $-V$. Up to changing the sign of V one can assume that (V^+, γ^+, V^-) is a maximal triple where γ^+ is the endpoint of the geodesic ray starting at γ'.

We say that an immersed curve $c : \mathbb{R} \to \mathrm{Ein}^{1,2}$ is *spacelike* if the tangent vectors to one and hence any lift of c to $\mathbb{R}^{2,n+1}$ is spacelike. At every point $p \in \mathbb{H}^{2,2}$, the set of tangent vectors of space type decomposes into two cones, a positive and a negative cone. One can make such a global continuous choice of positive cones on all of $\mathrm{Ein}^{1,2}$, and this choice is preserved by the action of $\mathrm{SO}_o(2, n+1)$, but not $\mathrm{SO}(2, n+1)$. We say that a spacelike curve is *positive* if its derivative lies in the positive cones, and *negative* if it lies in the other cones.

The identification of $\mathcal{L}$ and $\mathrm{Ein}^{1,2}$ via the isomorphism $\mathrm{PSp}(4,\mathbb{R}) \simeq \mathrm{SO}_o(2,3)$ identifies a point $p \in \mathrm{Ein}^{1,2}$ with $\ell \in \mathcal{L}$ and identifies the positive cone of vectors of spacelike type at $p \in \mathrm{Ein}^{1,2}$ with the cone of $T_\ell\mathcal{L} \simeq \mathrm{Hom}(\ell, \mathbb{R}^4/\ell)$ of elements $u \in \mathrm{Hom}(\ell, \mathbb{R}^4/\ell)$ such that $\omega(\cdot, u(\cdot))$ is positive definite on ℓ.

Remark B.2. This last identification is a consequence of the fact that the boundary of these cones are exactly the directions $v \in T_\ell \mathrm{Ein}^{1,2} \simeq T_\ell\mathcal{L}$ tangent to the locus of flags that are not transverse, that is, in generic position, with ℓ. Note that changing the choice of positive cones in $T\,\mathrm{Ein}^{1,2}$ amounts to changing the symplectic form ω to $-\omega$.

Lemma B.3. *Suppose that $\|\mathrm{II}_u(V, \gamma')\| \leq 1$, then the curves $V^+, V^- : [0,1] \to \mathrm{Ein}^{1,2}$ are respectively spacelike positive and spacelike negative, up to reversing the choice of positive cones.*

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Note that in this lemma $\mathrm{II}_u(V,\gamma')$ is timelike, so its norm is the timelike norm that we see as a positive number.

Proof. We fix a orthogonal basis (e_i) of $\mathbb{R}^{2,n+1}$ such that e_1 and e_2 have norm 1 and e_i for $i \geq 3$ have norm -1. Without any loss of generality we suppose that $\gamma(0) = e_3$, $\gamma'(0) = e_1$ and $V(0) = e_2$. Let d be the flat connection on $\mathbb{R}^{2,n}$ and let ∇ be the Levi-Civita connection on $u(S)$ for the induced spacelike metric. We view V as a vector field along the curve γ. For t close to 0 we have $V(t) = V(0) + tdV(0)$ and along the curve γ we have $dV = \nabla V + \mathrm{II}_{\mathbb{H}^{2,n}}(V,\gamma') + \mathrm{II}_u(V,\gamma')$. Here the second fundamental form of $\mathbb{H}^{2,n}$ inside $\mathbb{R}^{2,n+1}$ is equal to $\mathrm{II}_{\mathbb{H}^{2,n}}(v_1,v_2) = \langle v_1, v_2\rangle v_0$ for $v_1, v_2 \in T_{v_0}\mathbb{H}^{2,n}$. Note also that since γ is a geodesic and V an orthogonal unit vector field along γ, $\nabla V = 0$.

Hence $V(t) = e_2 + t\mathrm{II}_u(V,\gamma') + o(t)$. Since V has norm 1, we can write a representative of the isotropic line $V^+(t)$ as $v^+(t) = V(t) + \gamma(t)$. Therefore:

$$v^+(t) = e_2 + t(\mathrm{II}_u(V,\gamma') + e_1) + o(t).$$

This curve is spacelike since e_2 is spacelike of norm 1 and the (timelike) norm of $\mathrm{II}_u(V,\gamma')$ is strictly less than 1. The same holds for V^-, $v^-(t) = -V(t) + \gamma(t)$, and we get:

$$v^-(t) = -e_2 + t(-\mathrm{II}_u(V,\gamma') + e_1) + o(t).$$

Now we check that these two tangent directions lie in different families of cones,:

$$\begin{aligned}(v^+)'(0) &= e_1 + \mathrm{II}_u(V,\gamma') \in (e_2+e_3)^\perp \simeq T_{[e_2+e_3]}\operatorname{Ein}^{1,2},\\ (v^-)'(0) &= e_1 - \mathrm{II}_u(V,\gamma') \in (-e_2+e_3)^\perp \simeq T_{[-e_2+e_3]}\operatorname{Ein}^{1,2}.\end{aligned}$$

For that we move continuously the second vector by the rotation of angle θ in the plane $\langle e_1, e_2\rangle$. When $\theta = \pi$, the vector $(v^-)'(0)$ is sent to:

$$-e_1 - \mathrm{II}_u(V,\gamma') \in (e_2+e_3)^\perp \simeq T_{[e_2+e_3]}\operatorname{Ein}^{1,2}.$$

This vector is in the opposite cone of spacelike vectors to $(v^+)'(0)$, hence $(v^-)'(0)$ and $(v^+)'(0)$ belong to opposite families of cones of spacelike vectors. The curves V^+, V^- are hence spacelike with one being positive and one negative. □

Proof of Theorem B.1. The Gauss map $\mathcal{G}u : S \longrightarrow \mathrm{Gr}_2^{\max}(\mathcal{Q})$ comes with an identification between $u^*\mathcal{E}$ and TS, $\mathcal{E}$ being the tautological bundle over $\mathrm{Gr}_2^{\max}(\mathcal{Q})$. We consider the geodesic flow Φ on $\mathbb{S}TS$ for the metric induced by u.

Lemma B.3 implies that the curves V^+ and V^- are spacelike. Up to exchanging the positive cones and negative cones, we can assume that for all starting point $\gamma(0)$ and tangent vector $\gamma'(0)$, the associated curve V^+ is positive and V^- is negative. Hence the corresponding curves of Lagrangians ℓ^+, ℓ^- in $\mathbb{R}^4$ satisfy the hypothesis of Lemma 6.12.

In particular the derivative of the associated quadrics along this flow is positive so Φ is a fitting flow. Since u admits a fitting flow it is in particular a well-fitting immersion. □

The existence of a maximal spacelike immersion was proven by Collier-Tholozan-Toulisse and a bound of its second fundamental form is a consequence of a result from Cheng.

Theorem B.4 [CTT19],[Che93]. *Every maximal representation $\rho : \Gamma_g \longrightarrow \mathrm{SO}_o(2,3)$ admits a unique ρ-equivariant maximal spacelike immersion $u : \widetilde{S_g} \longrightarrow \mathbb{H}^{2,2}$. Moreover it is an embedding and for all $v \in TS$, $\|\mathrm{II}_u(v,v)\| < \|v\|^2$.*

The bound on the second fundamental form is a consequence of a maximal principle, see [LT22, Corollary 5.2]. Note that in this reference it is written that the square norm of II_u is at most 2, but since u is maximal it implies that the principal values are at most equal to 1.

Putting together Theorem B.1, Theorem B.4 and Theorem 6.5 we obtain:

Corollary B.5. *Every maximal representation $\rho : \Gamma_g \to \mathrm{Sp}(4, \mathbb{R})$ admits a fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that is ρ-equivariant. This characterizes representations which are maximal for some orientation of S_g.*

Remark B.6. Note that a maximal spacelike surface in $\mathbb{H}^{2,2}$ always defines a fibration of the space of photons and of the symmetric space of $\mathrm{Sp}(4, \mathbb{R})$, without assuming any bound on the second fundamental form. In this case however the bound is useful to deduce that the fibration extends to a fibration of the symmetric space of $\mathrm{SL}(4, \mathbb{R})$.

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