On Certain Formulae connected with the Function $(\cos n\alpha - \cos n\theta)/(\cos\alpha - \cos\theta)$.

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§ 1. This paper deals with certain formulae which, though probably not all new, have not appeared in the text-books. They were suggested to the writer while engaged in discussing the expression for the intensity of the transmitted beam in the *Lummer-Gehrcke* Interference Spectroscope, *viz.*

$$J = \frac{1 - 2p^n\cos n\theta + p^{2n}}{1 - 2p\cos\theta + p^2},$$

where p is the square of the reflexion-coefficient, which in practice falls not far short of unity, $2n$ is the number of internal reflexions, and θ is the phase-difference of successive elementary beams.* If we put $k = \log_e p$, we have

$$J = p^{n-1}\frac{chnk - \cos n\theta}{chk - \cos\theta},$$

which at once suggests the function in the title.

The function J has a very high maximum value for $\theta = 0, 2\pi, \ldots$, yielding sharp bright bands with feebly luminous interspaces; and the sharpness of the bands is measured by the value of $d^2J/d\theta^2$ for these values of θ. The most useful formula, obtained by direct differentiation, is

$$\left(\frac{d^2J}{d\theta^2}\right)_{\theta=0} = -2p\frac{(1-p^n)^2 - n^2p^{n-1}(1-p)^2}{(1-p)^4},$$

and the limit of this expression for $p \rightarrow 1$ is $-\frac{1}{6}n^2(n^2-1)$.

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* See Dr E. Gehrcke, *Die Anwendung der Interferenzen in der Spectroskopie und Metrologie*, (Braunschweig, 1906), p. 69. I have modified the notation. The subject was brought to my notice by Prof. A. L. Selby.

The following specimen values were obtained by this method : †

n	p	$(d^2J/d\theta^2)_{\theta=0}$
15	·8	535
15	·883	1649
15	·9	2097
15	1·0	8400
20	·883	3070
20	1·0	26600

But it is clear that J could be differentiated or integrated any number of times if it were expressed as a finite Fourier series, in cosines of multiples of θ.

§ 2. The expression

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} \qquad\qquad (1)$$

can be written

(i) as a product of $(n-1)$ factors each linear in $\cos\theta$, thus:

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} = 2^{n-1}\prod_{r=1}^{n-1}\left\{\cos\theta - \cos\left(\alpha + \frac{2r\pi}{n}\right)\right\},$$

(*cf.* Hobson, *Plane Trig.*, § 188);

(ii) as a finite series of powers of $\cos\theta$, up to the $(n-1)^{\text{th}}$; to be obtained by multiplying out the factors above, or from the relation (*ibid.* § 78)

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} \ldots ;$$

(iii) in the form we propose to consider, as a linear function of cosines of multiples of θ, up to the $(n-1)^{\text{th}}$.

† $n = 15$, $p = {\cdot}883$ are the actual values in a typical case.

§ 3. *To find the coefficients in the identity*

$$\frac{chnk - \cos n\theta}{chk - \cos\theta} = \tfrac{1}{2}a_0 + \sum_{r=1}^{n-1} a_r \cos r\theta.$$

On multiplying up we have

$$a_{r+1} - 2chk \,.\, a_r + a_{r-1} = 0, \;\; (r = 1,\ 2,\ \ldots (n-1)\,), \;\ldots\ldots\ldots\ldots (2)$$

$$a_0 chk - a_1 = 2chnk, \;\; a_{n-1} = 2\,; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

from (2), putting $p = e^k$,

$$a_r = \mathrm{A}p^r + \mathrm{B}p^{-r},$$

where A, B are independent of r; and from (3),

$$\mathrm{B} = \frac{2p^n}{p - p^{-1}}, \quad \mathrm{A} = \frac{-2p^{-n}}{p - p^{-1}},$$

whence

$$a_r = \frac{2(p^{n-r} - p^{-n+r})}{p - p^{-1}} = \frac{2sh(n-r)k}{shk},$$

and

$$\frac{chnk - \cos n\theta}{chk - \cos\theta} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1} sh(n-r)k\cos r\theta. \;\ldots\ldots\ldots\ldots (4)$$

We may note that k is positive or negative according as p is greater or less than unity.

§ 4. From (4) we at once deduce

$$\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} = \frac{\sin n\alpha}{\sin\alpha} + \frac{2}{\sin\alpha}\sum_{r=1}^{n-1} \sin(n-r)\alpha\cos r\theta,$$

$$= \frac{\sin n\theta}{\sin\theta} + \frac{2}{\sin\theta}\sum_{r=1}^{n-1} \sin(n-r)\theta\cos r\alpha\,; \;\ldots\ldots\ldots\ldots (5)$$

$$\frac{chnk - chnu}{chk - chu} = \frac{shnk}{shk} + \frac{2}{shk}\sum_{r=1}^{n-1} sh(n-r)kchru,$$

$$= \frac{shnu}{shu} + \frac{2}{shu}\sum_{r=1}^{n-1} sh(n-r)uchrk\,; \;\ldots\ldots\ldots\ldots\ldots (6)$$

$$\frac{1 - \cos n\theta}{1 - \cos\theta} = n + 2\sum_{r=1}^{n-1} (n-r)\cos r\theta,$$

$$= \frac{\sin n\theta}{\sin\theta} + \frac{2}{\sin\theta}\sum_{r=1}^{n-1} \sin(n-r)\theta\,; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\frac{1-chnu}{1-chu}= \quad n \quad +2\sum_{r=1}^{n-1}(n-r)chru,$$

$$=\frac{shnu}{shu}+\frac{2}{shu}\sum_{r=1}^{n-1}sh(n-r)u\,; \quad \ldots\ldots\ldots\ldots\ldots(8)$$

$$\frac{chnk-\cos n\theta}{chk-\cos\theta}=\frac{shnk}{shk}+\frac{2}{shk}\sum_{r=1}^{n-1}sh(n-r)k\cos r\theta,$$

$$=\frac{\sin n\theta}{\sin\theta}+\frac{2}{\sin\theta}\sum_{r=1}^{n-1}\sin(n-r)\theta chrk \ldots\ldots\ldots\ldots(9)$$

§5. Replacing θ by $\theta+\pi$, and u by $u+i\pi$, we have

$$\frac{\cos n\alpha-(-)^n\cos n\theta}{\cos\alpha+\cos\theta}=\frac{\sin n\alpha}{\sin\alpha}+\frac{2}{\sin\alpha}\sum_{r=1}^{n-1}(-)^r\sin(n-r)\alpha\cos r\theta,$$

$$=-\frac{2}{\sin\theta}\sum_{r=1}^{n-1}(-)^r\cos(n-r)\alpha\sin r\theta-\frac{(-)^n\sin n\theta}{\sin\theta}\,; \quad \ldots(5')$$

$$\frac{chnk-(-)^n chnu}{chk+chu}=\frac{shnk}{shk}+\frac{2}{shk}\sum_{r=1}^{n-1}(-)^r sh(n-r)kchru,$$

$$=-\frac{2}{shu}\sum_{r=1}^{n-1}(-)^r ch(n-r)kshru-\frac{(-)^n shnu}{shu}\,; \quad \ldots\ldots(6')$$

$$\frac{1-(-)^n\cos n\theta}{1+\cos\theta}=n+2\sum_{r=1}^{n-1}(-)^r(n-r)\cos r\theta,$$

$$=-\frac{2}{\sin\theta}\sum_{r=1}^{n-1}(-)^r\sin r\theta-\frac{(-)^n\sin n\theta}{\sin\theta}\,; \quad \ldots\ldots\ldots(7')$$

$$\frac{1-(-)^n chnu}{1+chu}=n+2\sum_{r=1}^{n-1}(-)^r(n-r)chru,$$

$$=-\frac{2}{shu}\sum_{r=1}^{n-1}(-)^r shru-\frac{(-)^n shnu}{shu}\,; \quad \ldots\ldots\ldots(8')$$

$$\frac{chnk-(-)^n\cos n\theta}{chk+\cos\theta}=\frac{shnk}{shk}+\frac{2}{shk}\sum_{r=1}^{n-1}(-)^r sh(n-r)k\cos r\theta,$$

$$=-\frac{2}{\sin\theta}\sum_{r=1}^{n-1}(-)^r ch(n-r)k\sin r\theta-\frac{(-)^n\sin n\theta}{\sin\theta}\ldots\ldots(9')$$

Similarly, by substituting $\frac{1}{2}\pi\pm\theta$, $\frac{1}{2}\pi-\alpha$, $\frac{1}{2}i\pi\pm u$, $\frac{1}{2}i\pi-k$ for θ, α, u, k respectively, a large variety of formulae can be deduced, proceeding on the right by alternate sines and cosines of multiples of θ (or alternate hyperbolic sines and cosines of multiples of u), and having for denominators on the left such quantities as

$$\cos\alpha\pm\sin\theta,\ \sin\alpha\pm\sin\theta,\ chk\pm shu,\ shk\pm shu,$$
$$1\pm\sin\theta,\ 1\pm shu,\ chk\pm\sin\theta,\ shk\pm\sin\theta,\ \sin\alpha\pm shu.$$

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§6. We have (*cf. Hobson*, §74, *ex.* 1).

$$\frac{\sin n\alpha}{\sin\alpha}=2\cos(n-1)\alpha+2\cos(n-3)\alpha+\ldots,\qquad\ldots\ldots(10)$$

where, however, if n is odd, the final term is *unity*. Also

$$\frac{\cos(2n-1)\alpha}{\cos\alpha}=2\cos(2n-2)\alpha-2\cos(2n-4)\alpha\ldots+(-1)^{n-1},$$

$$\frac{(-1)^{n-1}+\cos 2n\alpha}{\cos\alpha}=2\cos(2n-1)\alpha-2\cos(2n-3)\alpha\ldots+(-)^{n-1}2\cos\alpha,$$

$$\frac{\sin 2n\alpha}{\cos\alpha}=2\sin(2n-1)\alpha-2\sin(2n-3)\alpha\ldots,$$

$$\frac{1-\cos 2n\alpha}{\sin\alpha}=2\sin(2n-1)\alpha+2\sin(2n-3)\alpha\ldots,$$

with similar results in hyperbolics. Applying (10) to the coefficients in (5), we obtain an expansion of our function which is linear in the cosines of multiples of α, and also in those of θ; in fact, a *finite double Fourier series*, thus:

$$\frac{\cos n\alpha-\cos n\theta}{\cos\alpha-\cos}=4\sum_{r,s}\cos r\alpha\cos s\theta+2\sum_{t}(\cos t\alpha+\cos t\theta)+c,\qquad\ldots\ldots(11)$$

the summations extending to all positive integral non-zero values of r, s, t for which $n-r-s$, $n-t$ are odd positive integers, and c being zero or unity according as n is even or odd. Thus *e.g.*

$$(\cos 5\alpha-\cos 5\theta)/(\cos\alpha-\cos\theta)$$
$$=4(\cos 3\alpha\cos\theta+\cos 2\alpha\cos 2\theta+\cos\alpha\cos 3\theta+\cos\alpha\cos\theta)$$
$$+2(\cos 4\alpha+\cos 4\theta+\cos 2\alpha+\cos 2\theta)+1.$$

Similar results will follow for the other formulae of §§4, 5. And while the formulae of these articles are suitable for repeated differentiation and integration with regard to one or other of the symbols involved, those of the form (11) will admit of these processes being applied with respect to *both* variables.

§7. *Repeated Differentiations.—Expansions.*

By (10),

$$\left\{\frac{d^m}{d\alpha^m}\left(\frac{\sin n\alpha}{\sin\alpha}\right)\right\}_{\alpha=0}=2n_m\,.\cos\tfrac{1}{2}m\pi,$$

where $$n_m=(n-1)^m+(n-3)^m+\ldots,$$

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the last term being 1^m or 2^m according as n is even or odd; hence

$$\frac{\sin n\alpha}{\sin\alpha} = n + 2\sum_{m=1}^{\infty}(-)^m n_{2m}\frac{\alpha^{2m}}{(2m)!}.$$

Similarly let

$$n_m' = (n-1)^m - (n-3)^m + \dots;$$

then *when n is even*,

$$\frac{(-)^{\frac{1}{2}n-1} + \cos n\alpha}{\cos\alpha} = c' + 2\sum_{m=1}^{\infty}(-)^m n'_{2m}\frac{\alpha^{2m}}{(2m)!},$$

c' being zero or 2 according as $\frac{1}{2}n$ is even or odd; and *when n is odd*,

$$\frac{\cos n\alpha}{\cos\alpha} = 1 + 2\sum_{m=1}^{\infty}(-)^m n'_{2m}\frac{\alpha^{2m}}{(2m)!}.$$

By (7),

$$\left\{\frac{d^m}{d\theta^m}\left(\frac{1-\cos n\theta}{1-\cos\theta}\right)\right\}_{\theta=0} = 2a_{n,m}\cos\tfrac{1}{2}m\pi,$$

where $$a_{n,m} = \sum_{r=1}^{n-1}(n-r)r^m;$$

hence $$\frac{1-\cos n\theta}{1-\cos\theta} = n^2 + 2\sum_{m=1}^{\infty}(-)^m a_{n,2m}\frac{\theta^{2m}}{(2m)!}.$$

By (9),

$$\left\{\frac{d^m}{d\theta^m}\left(\frac{chnk - \cos n\theta}{chk - \cos\theta}\right)\right\}_{\theta=0} = \frac{2\cos\frac{1}{2}m\pi}{shk}\sum_{r=1}^{n-1} r^m sh(n-r)k;$$

let us put

$$k_{n,m} = \sum_{r=1}^{n-1}\frac{r^m sh(n-r)k}{shk} = \sum_{r=1}^{n-1} r^m\frac{p^{n-r} - p^{-n+r}}{p - p^{-1}}\,(m>0),$$

and

$$k_{n,0} = \frac{chnk - 1}{chk - 1} = \frac{1}{p^{n-1}}\left(\frac{p^n - 1}{p-1}\right)^2,\ p = e^k,$$

then

$$\frac{chnk - \cos n\theta}{chk - \cos\theta} = k_{n,0} + 2\sum_{m=1}^{\infty}(-)^m k_{n,2m}\frac{\theta^{2m}}{(2m)!}.$$

§8. *Integration.*

(i) From the formulae of §6 we have

$$\int_0 \frac{\sin n\theta d\theta}{\sin\theta} = \frac{2\sin(n-1)\theta}{n-1} + \frac{2\sin(n-3)\theta}{n-3} + \dots + c\theta,$$

c being zero or unity according as n is even or odd;

$$\int_0 \frac{\cos(2n-1)\theta d\theta}{\cos\theta} = \frac{2\sin(2n-2)\theta}{2n-2} - \frac{2\sin(2n-4)\theta}{2n-4} \dots + (-)^{n-1}\theta,$$

$$\int_0 \frac{(-1)^{n-1} + \cos 2n\theta}{\cos\theta} d\theta = \frac{2\sin(2n-1)\theta}{2n-1} - \frac{2\sin(2n-3)\theta}{2n-3} \dots + (-)^{n-1} 2\sin\theta$$

$$\int_{\frac{1}{2}\pi} \frac{\sin 2n\theta d\theta}{\cos\theta} = -\frac{2\cos(2n-1)\theta}{2n-1} + \frac{2\cos(2n-3)\theta}{2n-3} - \dots,$$

$$\int_{\frac{1}{2}\pi} \frac{1-\cos 2n\theta}{\sin\theta} d\theta = -\frac{2\cos(2n-1)\theta}{2n-1} - \frac{2\cos(2n-3)\theta}{2n-3} - \dots.$$

From these we obtain

$$\int_0^\pi \frac{\sin 2n\theta d\theta}{\sin\theta} = 0, \qquad \int_0^\pi \frac{\sin(2n-1)\theta d\theta}{\sin\theta} = \pi,$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta d\theta}{\sin\theta} = 2\left(1 - \tfrac{1}{3} + \tfrac{1}{5} \dots + \frac{(-)^{n-1}}{2n-1}\right), \quad \int_0^{\frac{1}{2}\pi} \frac{\sin(2n-1)\theta d\theta}{\sin\theta} = \tfrac{1}{2}\pi;$$

$$\int_0^\pi \frac{\cos(2n-1)\theta d\theta}{\cos\theta} = (-)^{n-1}\pi, \qquad \int_0^{\frac{1}{2}\pi} \frac{\cos(2n-1)\theta d\theta}{\cos\theta} = (-)^{n-1}\frac{\pi}{2},$$

$$\int_0^\pi \frac{(-)^{n-1} + \cos 2n\theta}{\cos\theta} d\theta = 0, \quad \int_0^{\frac{1}{2}\pi} \frac{(-)^{n-1} + \cos 2n\theta}{\cos\theta} d\theta$$

$$= (-)^{n-1} 2\left(1 + \tfrac{1}{3} + \dots + \frac{1}{2n-1}\right);$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta d\theta}{\cos\theta} = \int_{\frac{1}{2}\pi}^{\pi} \frac{\sin 2n\theta d\theta}{\cos\theta} = 2\left(\frac{1}{2n-1} - \frac{1}{2n-3} + \dots\right),$$

$$\int_0^{\frac{1}{2}\pi} \frac{1-\cos 2n\theta}{\sin\theta} d\theta = \int_{\frac{1}{2}\pi}^{\pi} \frac{1-\cos 2n\theta}{\sin\theta} d\theta = 2\left(1 + \tfrac{1}{3} + \dots + \frac{1}{2n-1}\right).$$

(ii) From (5), (7), (9) we have

$$\int_0 \frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} d\theta = \frac{\sin n\alpha}{\sin\alpha}\theta + \sum_{r=1}^{n-1} \frac{2\sin(n-r)\alpha \sin r\theta}{r\sin\alpha},$$

$$\int_0 \frac{1-\cos n\theta}{1-\cos\theta} d\theta = n\theta + 2\sum_{r=1}^{n-1} \frac{n-r}{r}\sin r\theta,$$

$$\int_0 \frac{chnk - \cos n\theta}{chk - \cos\theta} d\theta = \frac{shnk}{shk}\theta + \sum_{r=1}^{n-1} \frac{2sh(n-r)k \sin r\theta}{rshk};$$

hence

$$\int_0^{\pi} \frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta} d\theta = \frac{\pi \sin n\alpha}{\sin\alpha},$$

$$\int_0^{\pi} \frac{1 - \cos n\theta}{1 - \cos\theta} d\theta = n\pi,$$

$$\int_0^{\pi} \frac{ch nk - \cos n\theta}{chk - \cos\theta} d\theta = \frac{\pi sh nk}{shk}.$$

In the last write $p = e^k$, and obtain

$$\int_0^{\pi} \frac{1 - 2p^n \cos n\theta + p^{2n}}{1 - 2p\cos\theta + p^2} d\theta = \pi \frac{p^{2n} - 1}{p^2 - 1},$$

where p may have any real value.

(iii) Again from (5), (7), (9), on multiplying throughout by $\cos r\theta$ and integrating from 0 to π, as in finding the coefficients of a Fourier series, we have

$$\int_0^{\pi} \frac{(\cos n\alpha - \cos n\theta)\cos r\theta d\theta}{\cos\alpha - \cos\theta} = \frac{\pi \sin(n-r)\alpha}{\sin\alpha},$$

$$\int_0^{\pi} \frac{(1 - \cos n\theta)\cos r\theta d\theta}{1 - \cos\theta} = (n - r)\pi,$$

$$\int_0^{\pi} \frac{(ch nk - \cos n\theta)\cos r\theta d\theta}{chk - \cos\theta} = \frac{\pi sh(n-r)k}{shk};$$

and with p as before,

$$\int_0^{\pi} \frac{(1 - 2p^n \cos n\theta + p^{2n})\cos r\theta d\theta}{1 - 2p\cos\theta + p^2} = \frac{\pi}{p^r} \frac{p^{2n} - p^{2r}}{p^2 - 1} = \pi p^r . \frac{p^{2(n-r)} - 1}{p^2 - 1}.$$

(iv) The integral

$$\int \frac{\cos n\theta d\theta}{\cos\alpha - \cos\theta}$$

is improper if the range of integration includes α (or more generally $2r\pi \pm \alpha$); but we can write

$$\int_0^{\pi} \frac{\cos n\theta d\theta}{chk - \cos\theta} = ch nk \int_0^{\pi} \frac{d\theta}{chk - \cos\theta} - \int_0^{\pi} \frac{ch nk - \cos n\theta}{chk - \cos\theta} d\theta.$$

Now the first integral on the right is essentially positive, hence its value is $\pm \dfrac{\pi ch nk}{shk}$, according as k is positive or negative.

Let k be positive, then $p = e^k > 1$, and we have

$$\int_0^{\pi} \frac{\cos n\theta d\theta}{chk - \cos\theta} = \frac{\pi e^{-nk}}{shk}, \quad \text{...(12)}$$

whence

$$\int_0^\pi \frac{\cos n\theta d\theta}{1-2p\cos\theta+p^2}=\frac{\pi p^{-n}}{p^2-1}\cdot(p>1).$$

If we put $q=p^{-1}=e^{-k}<1$, we shall have

$$\int_0^\pi \frac{\cos n\theta d\theta}{1-2q\cos\theta+q^2}=\frac{\pi q^n}{1-q^2}\cdot(0<q<1).$$

(*cf.* Bromwich, *Infinite Series*, p. 167).

Now writing $chk=\sec\beta$, so that $shk=\tan\beta$, and

$$p=e^k=\sec\beta+\tan\beta=\frac{1+\tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\beta},\quad (0<\beta<\tfrac{1}{2}\pi),$$

we have from (12)

$$\int_0^\pi \frac{\cos n\theta d\theta}{1-\cos\beta\cos\theta}=\frac{\pi}{\sin\beta}\left(\frac{1-t}{1+t}\right)^n,\ t=\tan\tfrac{1}{2}\beta. \qquad\qquad(13)$$

The companion formula

$$\int_0^\pi \frac{\cos n\theta d\theta}{1+\cos\beta\cos\theta}=\frac{(-)^n\pi}{\sin\beta}\left(\frac{1-t}{1+t}\right)^n \qquad\qquad(14)$$

may be obtained from (13) either by changing θ into $\pi-\theta$, or β into $\pi-\beta$. The formula (13) is a good example of a discontinuous function. The integral becomes improper when β takes the value 0 or π, and therefore, as a function of β, is discontinuous at those values. Thus it is inadmissible to change β into $-\beta$ or into $\pi+\beta$, though it is allowable to change it into $\pi-\beta$.

Another form is obtained by putting $\cos\beta=e$,

$$\int_0^\pi \frac{\cos n\theta d\theta}{1\mp e\cos\theta}=\frac{\pi}{\surd(1-e^2)}\left\{\frac{\pm e}{1+\surd(1-e^2)}\right\}^n.$$

(v) Finally from (ii),

$$\int_0^\pi\int_0^\pi \frac{\cos n\alpha-\cos n\theta}{\cos\alpha-\cos\theta}d\alpha d\theta=0 \text{ or } \pi^2$$

according as n is even or odd, while

$$\int_0^\pi\int_0^\pi \frac{(\cos n\alpha-\cos n\theta)\cos r\alpha\cos s\theta d\alpha d\theta}{\cos\alpha-\cos\theta}=0 \text{ or } \pi^2$$

according as $n-r-s$ is even or odd.