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BOOLEAN ALGEBRAS OF PROJECTIONS IN (DF)- AND (LF)-SPACES

J. Bonet and W.J. Ricker

Conditions are presented which ensure that an abstractly σ-complete Boolean algebra of projections on a (DF)-space or on an (LF)-space is necessarily equicontinuous and/or the range of a spectral measure. This is an extension, to a large and important class of locally convex spaces, of similar and well known results due to W. Bade (respectively, B. Walsh) in the setting of normed (respectively metrisable) spaces.

1. Introduction and main results

The theory of Boolean algebras of projections in Banach (and more general) spaces is a natural extension of the notion of "resolution of the identity" for normal operators in Hilbert space. An underlying principle is to realise the Boolean algebra (whenever possible) as the range of some spectral measure defined on a σ-algebra of sets. The well developed theory of vector and projection-valued measures and integration with respect to them can then be invoked; see [**3, 4, 5, 6, 10, 15, 17**] and the references therein, for example. To make this realisation possible, there are two minimal but essential properties required of the Boolean algebra; it should be at least σ-complete as an abstract Boolean algebra and it should be uniformly bounded (that is, equicontinuous) as a family of continuous linear operators. In the setting of Banach (respectively Fréchet) spaces these two properties are intimately connected since abstract σ-completeness ensures *automatically* the uniform boundedness, [**1**], (respectively, equicontinuity, [**19**]) of the Boolean algebra of projections. For locally convex Hausdorff spaces which are non-metrisable, this is no longer the case in general, [**11**]. Accordingly, in this setting it becomes useful to identify properties of the underlying locally convex Hausdorff space or classes of such spaces, which lead to the conclusion that (certain) Boolean algebra's of projections on these spaces are automatically either equicontinuous and/or the range of a spectral measure;

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see [**16**, Section 3] for such a result. The aim of this note is to present some further results of this kind within the classes of (DF)-spaces and (LF)-spaces. It is time to be more precise.

Let X be a locally convex Hausdorff space and $L(X)$ be the space of all continuous linear operators of X into itself. To stress when $L(X)$ is equipped with the topology of uniform convergence on the finite (respectively, bounded) subsets of X we write $L_s(X)$ (respectively, $L_b(X)$). The zero and identity operators on X are denoted by 0 and I, respectively. The continuous dual space of X is written as X'. A family $\mathscr{M} \subseteq L(X)$ of commuting projections which contains 0 and I is a Boolean algebra if it contains $I - Q_1$ and $Q_1 \wedge Q_2 := Q_1Q_2$ and $Q_1 \vee Q_2 = Q_1 + Q_2 - Q_1Q_2$ whenever $Q_1, Q_2 \in \mathscr{M}$. The partial order $\leqslant$ in $\mathscr{M}$ is given by $Q_1 \leqslant Q_2$ (that is, $Q_1Q_2 = Q_1$) if and only if $Q_1X \subseteq Q_2X$.

Theorem 1.1. *Let X be a quasibarrelled (DF)-space. Then every Boolean algebra of projections on X which is σ-complete as an abstract Boolean algebra is necessarily equicontinuous.*

For certain classes of locally convex Hausdorff spaces X, particular types of Boolean algebra's of projections on X can be identified as equicontinuous. For instance, B. Walsh proved that if X is a strict inductive limit of Fréchet spaces (briefly, s(LF)-space) and $\mathscr{M} \subseteq L(X)$ is an abstractly σ-complete Boolean algebra of projections having the property that each set $\{Qx : Q \in \mathscr{M}\}$, for $x \in X$, is contained in one of the limitands in the inductive limit making up the space X, then $\mathscr{M}$ is necessarily equicontinuous, [**19**, pp. 298-299].

Theorem 1.2. *Let $X = \operatorname{ind}_n X_n$ be an (LF)-space and $\mathscr{M} \subseteq L(X)$ be an abstractly σ-complete Boolean algebra of projections.*

(i) *If, for every $x \in X$, there exists $n(x) \in \mathbb{N}$ such that*

$$\mathscr{M}(x) := \{Qx : Q \in \mathscr{M}\} \subseteq X_{n(x)}, \tag{1}$$

then $\mathscr{M}$ is necessarily equicontinuous.

(ii) *Suppose that X is regular. If $\mathscr{M}$ is equicontinuous then, for every $x \in X$, there exists $n(x) \in \mathbb{N}$ such that (1) holds.*

Since not every (LF)-space is an s(LF)-space (see Section 3) this is a genuine strengthening of Walsh's result.

A Boolean algebra of projections $\mathscr{M} \subseteq L(X)$ is called *Bade σ-complete* if it is σ-complete as an abstract Boolean algebra and if, for every sequence $\{Q_n\}_{n=1}^{\infty} \subseteq \mathscr{M}$, we have $(\wedge_n Q_n)X = \bigcap_{n=1}^{\infty} Q_nX$ and $(\vee_n Q_n)X = \overline{\operatorname{span}\left(\bigcup_{n=1}^{\infty} Q_nX\right)}$, where the bar denotes

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closure in X. This notion was introduced by W. Bade (without the term "Bade") for Banach spaces [**1**, **6**]. By a *spectral measure* in X is meant a σ-additive map $P : \Sigma \to L_s(X)$, defined on a σ-algebra of subsets Σ of some non-empty set Ω, satisfying $P(\Omega) = I$ and $P(E \cap F) = P(E)P(F)$, for every $E, F \in \Sigma$. The range $P(\Sigma)$, of P, is always a Bade σ-complete Boolean algebra of projections on X. In metrisable spaces X, the converse is also known to be true; see [**6**, p. 2204] for Banach spaces and [**19**] for the non-normable case. For non-metrisable locally convex Hausdorff spaces X the situation is far more subtle; see [**12**], for instance, and the references therein. The following result makes a further contribution in this direction.

Theorem 1.3. *Let X be a (DF)-space. Then every Bade σ-complete Boolean algebra of projections on X is the range of some spectral measure.*

Since X is not required to be quasibarrelled, the equicontinuity of Boolean algebra's of the kind in Theorem 1.3 is *not* guaranteed by Theorem 1.1.

2. Proof of Theorems

Recall that a locally convex Hausdorff space X possesses a *fundamental sequence of bounded sets*, if there exist bounded sets $B_1 \subset B_2 \subset \ldots$ in X such that every bounded subset of X is contained in some B_n. This is equivalent to the strong dual X'_β being metrisable; [**13**, p. 250] or [**7**, p. 257].

Lemma 2.1. *Let X be a locally convex Hausdorff space such that X'_β is metrisable. Then every abstractly σ-complete Boolean algebra of projections on X is a bounded subset of $L_b(X)$.*

Proof: Let $\mathcal{M} \subseteq L(X)$ be an abstractly σ-complete Boolean algebra of projections. Then the collection of adjoint operators $\mathcal{M}' := \{Q' : Q \in \mathcal{M}\}$ is a subset of $L(X'_\beta)$, [**18**, p. 130], and again forms a Boolean algebra of projections. The abstract σ-completeness of $\mathcal{M}'$ follows from that of $\mathcal{M}$; see [**14**, p. 290]. Moreover, the metrisability of X'_β then ensures that $\mathcal{M}'$ is equicontinuous in $L(X'_\beta)$, [**19**, Proposition 1.2].

A neighbourhood basis of 0 in X'_β is given by the polars

$$A^\circ := \Big\{x' \in X' : \big|\langle x, x'\rangle\big| \leqslant 1, \quad x \in A\Big\},$$

as A varies through the closed, absolutely convex, bounded subsets of X. Fix such a set A. By equicontinuity of $\mathcal{M}'$ there is another closed, bounded, absolutely convex subset $B \subseteq X$ such that $Q'(B^\circ) \subseteq A^\circ$ for all $Q \in \mathcal{M}$. Since $B = B^{\circ\circ}$, we have

$$\big|\langle Qx, x'\rangle\big| = \big|\langle x, Q'x'\rangle\big| \leqslant 1, \qquad x \in A, \quad x' \in B^\circ,$$

for every $Q \in \mathscr{M}$, that is, $Q(A) \subseteq B$ for every $Q \in \mathscr{M}$. This is precisely the statement that $\mathscr{M}$ is bounded in $L_b(X)$. ∎

Recall that a locally convex Hausdorff space X is a (DF)-*space* if it possesses a fundamental sequence of bounded sets and has the property that every bounded subset of X'_β which is the union of countably many equicontinuous subsets is itself equicontinuous; see [**7**, p. 253] or [**18**, p. 154], for example. This latter property is also called $\aleph_0$-*barrelledness*, [**7**, pp. 251-252]. In particular, every quasibarrelled locally convex Hausdorff space X such that X'_β is metrisable is a (DF)-space.

Proof of Theorem 1.1: Let $\mathscr{M} \subseteq L(X)$ be an abstractly σ-complete Boolean algebra of projections. By Lemma 2.1, $\mathscr{M}$ is bounded in $L_b(X)$. The quasibarrelledness of X then ensures that $\mathscr{M}$ is equicontinuous, [**9**, p. 137]. ∎

Let $\mathscr{M} \subseteq L(X)$ be a Boolean algebra of projections. A monotonic sequence $\{Q_n\}_{n=1}^\infty \subseteq \mathscr{M}$ is called σ-*small*, [**12**], if given any neighbourhood U of 0 in X there exists another neighbourhood V of 0 in X such that, for each $x \in V$, there exists $n(x) \in \mathbb{N}$ with the property that $Q_n x \in U$ for all $n \geqslant n(x)$. In particular, if $\{Q_n\}_{n=1}^\infty$ is either a convergent sequence in $L_s(X)$ or an equicontinuous subset of $L(X)$, then it is necessarily σ-small. We say that $\mathscr{M}$ itself is σ-small if every monotonic sequence in $\mathscr{M}$ is σ-small.

Lemma 2.2. *Let X be a (DF)-space. Then every abstractly σ-complete Boolean algebra of projections on X is necessarily σ-small.*

Proof: Let $\mathscr{M} \subseteq L(X)$ be an abstractly σ-complete Boolean algebra of projections. By Lemma 2.1, $\mathscr{M}$ is a bounded subset of $L_b(X)$. In particular, if $\{Q_n\}_{n=1}^\infty \subseteq \mathscr{M}$ is any monotonic sequence, then it is a bounded subset of $L_b(X)$. By [**7**, Theorem 12.2.1], $\{Q_n\}_{n=1}^\infty$ is an equicontinuous part of $L(X)$ and hence, as noted above, it is then σ-small. ∎

Proof of Theorem 1.3: Since every Bade σ-complete Boolean algebra of projections is also abstractly σ-complete, the conclusion follows immediately from Lemma 2.2 and [**12**, Theorem 2]. ∎

So, it remains to establish Theorem 1.2.

An inductive limit $X = \operatorname{ind}_n X_n$, of increasing subspaces $\{X_n\}_{n=1}^\infty$ of X is an (LF)-*space* if each X_n is a Fréchet space (for a locally convex Hausdorff topology τ_n), if the inclusion $\rho_n : (X_n, \tau_n) \hookrightarrow (X_{n+1}, \tau_{n+1})$ is continuous for each $n \in \mathbb{N}$, and where the lc-topology τ on X is the finest making the inclusions $(X_n, \tau_n) \hookrightarrow (X, \tau)$, for each $n \in \mathbb{N}$, continuous. It is *assumed* that (X, τ) is Hausdorff; this is not always so, [**13**, Observation 8.1.2(b)]. The (LF)-space $X = \operatorname{ind}_n X_n$ is called *strict* (that is, s(LF)) if ρ_n is an isomorphism of X_n into X_{n+1}, for each $n \in \mathbb{N}$. An (LF)-space $X = \operatorname{ind}_n X_n$ is called *regular*, [**13**, p. 285], if every bounded subset of (X, τ) is contained and bounded in some

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(X_n, τ_n). For an (LF)-space $X = \operatorname{ind}_n X_n$, if X_n is a Banach space, for each $n \in \mathbb{N}$, then X is called an (LB)-*space.*

Proof of Theorem 1.2: To establish part (ii), fix $x \in X$. The equicontinuity of $\mathscr{M}$ ensures that $\mathscr{M}(x)$ is a bounded subset of X and then regularity of X guarantees the existence of $n(x) \in \mathbb{N}$ such that (1) holds.

To prove (i), notice that X is barrelled, [**8**, p. 368] and so it suffices to show that $\mathscr{M}$ is bounded in $L_s(X)$. Suppose not. Then there exists $x \in X$ and a continuous seminorm p in X such that $\sup\{p(Qx) : Q \in \mathscr{M}\} = \infty$. Arguing as in the proof of Proposition 1.2. in [**19**], with $B := \{\lambda x : |\lambda| \leqslant 1\}$, there exist mutually disjoint projections $\{H_{j,k}\}_{k=1}^{\infty} \subseteq \mathscr{M}$, for each $j \in \mathbb{N}$, and projections $F_j := \vee_{k=1}^{\infty} H_{j,k}$ in $\mathscr{M}$ (because of abstract σ-completeness) such that

$$\lim_{k\to\infty} p\left(H_{j,k}F_j x\right) = \infty, \quad j \in \mathbb{N}.$$

For more details of such a construction see [**15**, pp. 43-45], where "property (α)" as defined there, for any $E \in \mathscr{M}$, now becomes $\alpha(E) := \sup\{p(Qx) : Q \in \mathscr{M}, Q \leqslant E\} = \infty$.

Let $n(x)$ satisfy (1) and let $\{U_j\}_{j=1}^{\infty}$ be a basis of neighbourhoods of 0 in the Fréchet space $X_{n(x)}$. For each $j \in \mathbb{N}$, let $a_j \in \left(0, (1/j)\right)$ satisfy $a_j F_j x \in U_j$ and then select $k(j)$ such that $p\left(H_{j,k(j)}F_j x\right) > (j/a_j)$. For each $j \in \mathbb{N}$ define $x_j := a_j F_j x$, in which case $x_j \in U_j$. Then $\{x_j\}_{j=1}^{\infty}$ is a null sequence in $X_{n(x)}$ and hence, also in X. On the other hand, the element $F := \vee_{j=1}^{\infty} H_{j,k(j)}$ of $\mathscr{M}$ satisfies

$$p\left(Fx_j\right) = p\left(a_j F_j F x\right) = a_j p\left(H_{j,k(j)}F_j x\right) > j, \qquad j \in \mathbb{N}.$$

Since $F \in L(X)$ this is impossible and the proof is complete. ∎

3. Concluding remarks

The class of locally convex Hausdorff spaces X such that X'_β is metrisable, which is relevant for Lemma 2.1, is quite extensive. It includes all (DF), (gDF) and df-spaces. Indeed, the df-spaces X are precisely those for which X'_β is a Fréchet space, [**7**, 12.4 Theorem 1]. The inclusion for the classes of spaces (DF) $\subsetneq$ (gDF) is proper, [**13**, p. 251], as in the inclusion (gDF) $\subsetneq$ *df*, [**7**, p. 258]. Spaces X also exist for which X'_β is metrisable but not complete, [**13**, Example 8.6.12].

In Lemma 2.2 and Theorem 1.3 the (DF)-space X is not required to be quasibarrelled (as *is* the case for Theorem 1.1). Many such (DF)-spaces exist; see [**13**, Observation 8.3.6], for example. Of course, if the (DF)-space is separable or has a fundamental sequence of metrisable bounded sets, then it is necessarily quasibarrelled, [**13**, Proposition 8.3.13].

Finally some comments in relation to Theorem 1.2 are in order. Every s(LF)-space is a regular (LF)-space, [**8**, Section 19.5], but not conversely; see the discussion prior

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to Proposition 8.5.18 of [**13**]. Since every s(LF)-space is complete, [**8**, p.255], there is also an abundance of (LF)-spaces which are not strict, [**13**, Corollary 8.7.10]. There also exist complete (LF)-spaces which are not s(LF), [**13**, Example 8.8.6]. The space $\mathscr{D}$ of test functions for distributions and the strong dual of the space of real analytic functions are classical examples of regular (LF)-spaces. For further (natural) examples and details about inductive limits we refer to [**2**].

Since every (LB)-space is both a (DF)-space, [**13**, Observation 8.3.6(a) and Proposition 8.3.16], and barrelled, [**13**, Proposition 4.2.6], it follows from Theorem 1.1 that every abstractly σ-complete Boolean algebra of projections on an (LB)-space is necessarily equicontinuous. If, in addition, the (LB)-space is regular, then (1) holds for every $x \in X$. To summarise, we have

PROPOSITION 3.1. *Let* $X = \mathrm{ind}_n X_n$ *be a regular (LB)- space. Then every abstractly* σ*-complete Boolean algebra of projections* $\mathscr{M} \subseteq L(X)$ *necessarily has property* (1), *for every* $x \in X$.

As noted above (even for s(LF)-spaces), every s(LB)-space is regular. The converse is not always valid, [**13**, Example 8.5.23(a)]. There also exist (LB)-spaces which are not regular; see [**13**, Example 7.3.6 and Observation 8.5.14(d)]. The strong duals of distinguished Fréchet spaces exhibit a large class of regular (LB)-spaces, [**13**, Observation 8.5.14(e)]. Classical examples of regular (LB)-spaces are the space of distributions of compact support and the Schwartz space of tempered distributions; see also [**2**] for further examples.

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Dpto. Matematica Aplicada
Universidad Politecnica de Valencia
E-46071 Valencia
Spain

Math.-Geogr. Fakultät
Katholische Universität Eichstätt-Ingolstadt
D-85072 Eichstätt
Germany