PERMUTATIONAL PRODUCTS OF LATTICE ORDERED GROUPS [1]

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Let H be a group, let $\{G_i : i \in I\}$ be a set of groups and, for each i, let θ_i be a a monomorphism: $H \to G_i$, with $H\theta_i = H_i$. We call such a system of groups and monomorphisms an *amalgam* and denote it by $[G_i; H; \theta_i; H_i]$. By an *embedding* of the amalgam into a group G is meant a set of monomorphisms $\varphi_i : G_i \to G$ such that $\theta_i\varphi_i = \theta_j\varphi_j$, for all i, j and $G_i\varphi_i \cap G_j\varphi_j = H\theta_k\varphi_k$, for all i, j, k.

It is known (B. H. Neumann [9]) that the amalgam $[G_i; H; \theta_i; H_i]$ can be embedded in a group. It is also known (J. M. Howie [7]) that if the G_i are just semigroups and the H_i are almost unitary subsemigroups then the amalgam may be embedded in a semigroup.

If G is an 1-group (lattice ordered group), a subgroup which is also a sublattice is called an 1-subgroup. If H is an 1-subgroup of G and if the mapping $C \to C \cap H$ is a one-to-one correspondence between the lattice of 1-subgroups of G and those of H then Conrad [2] called G an a-extension of H. In his discussion of a-extensions Conrad has shown that amalgams of the form $[G_i; H; \theta_i; H_i]$ are embeddable where the G_i are *a*-extensions of the H_i and the H_i all belong to one of a number of classes of 1-groups. In a sense, the second section of this note considers the other extreme to Conrad's results. In section two, we consider the problem of embedding the amalgam $[G_i; H; \theta_i; H_i]$, where H and G_i are 1-groups, the H_i are convex 1-subgroups and the θ_i are 1-monomorphisms (lattice and group monomorphisms), into an 1-group G. It is shown that if, for each i, H_i is in the centre of G_i then the amalgam is embeddable and if each G_i is abelian then the amalgam may be embedded in an abelian 1-group.

The approach combines the permutational product of groups with the representation of an 1-group as an 1-subgroup of the group of automorphisms of an ordered set due to Holland [6]. If the G_i are 0-groups (totally ordered groups) and the H_i are normal then the standard permutational product may be used. In general this does not yield an 0-group although it will if the G_i are abelian.

In more general situations (e.g. [10]), consideration has been given to a weaker amalgamation embedding. Let us say that an amalgam $[G_i; H; \theta_i; H_i]$, where the

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G_i are 1-groups and the H_i are convex 1-subgroups, is *weakly embeddable* if there exists an 1-group G and 1-monomorphisms $\theta_i : G \to G_i$ such that $\theta_i\varphi_i = \theta_j\varphi_j$, for all i,j: that is, we drop the intersection preserving requirements.

K. Pierce (University of Wisconsin) has provided an example (unpublished) to show that the amalgam $[G_i; H; \theta_i; H_i]$, where the H_i are abelian 1-subgroups of the 1-groups G_i, cannot, in general, be even weakly embedded in an 1-group.

Unfortunately, on account of the restrictions placed on the subgroups H_i to be amalgamated, these results do not permit the kind of interesting applications treated by B. H. Neumann in (9) and by Higman, Neumann and Neumann in [5].

Finally, I would like to thank R. Gregorac for his many comments and criticisms, especially for pointing out an over ambitious claim in Theorem 4.1 as originally stated.

1. Preliminary results

We refer the reader to [1] and [4] for basic results on 1-groups.

For any totally ordered set X, we denote by $P(X)$ the lattice ordered group of automorphisms (that is, order preserving permutations) of X.

A convex 1-subgroup A of an 1-group G is called *prime* if the set $R(A)$ of right cosets of A can be totally ordered by defining $A+x \leqq A+y$ if $a+x \leqq y$ for some $a \in A$. For further discussion of prime subgroups see [3] and [6]. Whenever A is a prime subgroup of an 1-group G, G/A will denote the set of right cosets of A ordered in this way. (Of course, in general a prime subgroup is not normal.) The mapping $\rho : g \to \rho_g$ of G into $P(G/A)$ defined by

$$(A+x)\rho_p = A+x+g$$

is an 1-homomorphism. We call ρ the *Holland representation* of G on G/A.

Prime subgroups are plentiful. If $g \in G$ and G_γ is a maximal convex 1-subgroup with respect to not containing g then G_γ is a prime subgroup. Denote by $\sum_{i\in I} \boxplus A_i$ the unrestricted cardinal sum of 1-groups A_i. If $\{G_\gamma : \gamma \in \Gamma(G)\}$ is the set of prime subgroups of G and $\theta : G \to \sum_{\gamma\in\Gamma(G)} \boxplus P(G/G_\gamma)$ is the product of the Holland representations on the G/G_γ then θ is an 1-isomorphism. (For details of this representation see Holland [6]).

If $\{G_\gamma : \gamma \in \Gamma(G)\}$ is the set of all prime subgroups of G, if we totally order $\Gamma(G)$ and then order the set of all left cosets $C(G)$ of all prime subgroups of G by $G_\gamma+a > G'_\gamma+a'$ if $\gamma > \gamma'$ or $\gamma = \gamma'$ and $G_\gamma+a > G_\gamma+a'$ (in the order defined in the previous paragraph) then we can obtain a faithful representation of G as an 1-subgroup of $P(C(G))$, (Holland [6]).

LEMMA 1.1 (Lloyd [8]). *Let $A \subseteq B$ be prime subgroups of a 1-group G. Then there exists an order isomorphism η' of G/A onto $B/A \times G/B$, where $B/A \times G/B$ is ordered lexicographically from the right. This induces an 1-isomorphism η of $P(G/A)$ onto $P(B/A \times G/B)$.*

The isomorphism η' in Lemma 1.1 is defined as follows. Let $\{b_i : i \in I\}$ be a set of representatives of the right cosets of B in G. Let $A+x \in G/A$ and $B+x = B+b_i$. Then $(A+x)\eta' = (A+x-b_i, B+b_i)$. Of course, if $A = B$ then $B/A \times G/B$ is trivially isomorphic to $G/B = G/A$.

An 1-*ideal* of an 1-group is a normal convex 1-subgroup. The following lemma is routine.

LEMMA 1.2. *Let G be an 1-group, A an 1-ideal and B a prime subgroup of G. Then there exists an order isomorphism ζ' of $(B+A)/B$ onto $A/A \cap B$.*

PROOF. We take ζ' to be the obvious isomorphism defined by $(B+x')\zeta' = (A \cap B)+x$, where $x \in A$ and $B+x' = B+x$.

Let G be an 1-group, H an 1-ideal and G_γ a prime subgroup of G. Let ξ_γ be the Holland representation of G on G/G_γ. Let η_γ be the 1-isomorphism of $P(G/G_\gamma)$ onto $P((G_\gamma+H)/G_\gamma \times G/(G_\gamma+H))$ defined as in Lemma 1.1. Let ζ_γ be the 1-isomorphism of $P((G_\gamma+H)/G_\gamma \times G/(G_\gamma+H))$ onto $P(H/H \cap G_\gamma \times G/(G_\gamma+H))$ induced by the order isomorphism of $(G_\gamma+H)/G_\gamma$ onto $H/H \cap G_\gamma$ defined as in Lemma 1.2. Thus $\xi_\gamma\eta_\gamma\zeta_\gamma$ is an 1-homomorphism of G into $P(H/H \cap G_\gamma \times G/(G_\gamma+H))$. We write $\Psi_\gamma = \xi_\gamma\eta_\gamma\zeta_\gamma$.

2. Permutational products

Let $\{G_i \quad i \in I\}$ be a set of distinct 1-groups and, for each i, let H_i be a convex 1-subgroup of G_i contained in the centre of G_i. Let H be an 1-group and, for each $i, \theta_i : H \to H_i$ be an 1-isomorphism. We wish to embed the amalgam $[G_i; H; \theta_i; H_i]$ in an 1-group. For any prime subgroup H_γ of H, $H_\gamma\theta_i$ is a prime subgroup of H_i and θ_i induces an order isomorphism of H/H_γ onto $H_i/H_\gamma\theta_i$, which we also denote by θ_i. We introduce a dummy symbol 1.

We denote by $\{H_\gamma : \gamma \in \Gamma(H)\}$ $(\{H_{i,\gamma_i} : \gamma_i \in \Gamma(H_i)\})$ the set of prime subgroups of $H(H_i)$. Let $\Gamma(H)$ be endowed with some total order, then we can order the set of right cosets $C(H)$ of the prime subgroups of H by defining $H_\gamma+x > H'_\gamma+x'$ if either $\gamma > \gamma'$ or $\gamma = \gamma'$ and $H_\gamma+x > H_\gamma+x'$. We extend the order on $C(H)$ to $C(H)^1 = C(H) \cup \{1\}$ by defining 1 to be the least element.

Similarly, let $C(H_i)$ denote the set of right cosets of the prime subgroups of H_i. Then θ_i determines a bijection of $C(H)$ onto $C(H_i)$ which is an order isomorphism when restricted to the cosets of an individual prime subgroup of H. Thus θ_i induces an ordering of $C(H_i)$ extending that on the cosets of the individual prime subgroups.

For each $i \in I$, let $\{G_{i,\gamma} : \gamma \in \Gamma(G_i)\}$ denote the set of prime subgroups of G_i and let $\{G_{i,\gamma} : \gamma \in \Gamma_1(G_i)\}$ denote the set of prime subgroups of G_i containing H_i. Order $\Gamma_1(G_i)$ and then the set of cosets $C(G_i)$ of the prime subgroups $G_{i,\gamma}$ with $\gamma \in \Gamma_1(G_i)$ just as $C(H)$ was ordered.

It has been shown by Conrad [3] that, for any convex 1-subgroup K of an l-group L, the mapping $\sigma : A \to A \cap K$ is a bijection from the set of prime subgroups of L that do not contain K to those of K. So, for each i, we denote by σ_i the correspondence between the prime subgroups of G_i not containing H_i and the prime subgroups of H_i.

We define a subset T of the cartesian product of $C(H)^1$ and the sets $C(G_i)$, considered as the set of (choice) functions p from $J = I \cup \{\alpha\}$ such that $p(\alpha) \in C(H)^1$ and $p(i) \in C(G_i)$, $i \in I$. A function p belongs to T if

either $\quad p(\alpha) = 1$

or $\quad p(\alpha) = H_\gamma + h, \; p(i) = G_{i,\gamma} + g_i$

and $\quad H_\gamma \theta_i \sigma_i^{-1} + H_i = G_{i,\gamma}$

Note: By the definition of $C(G_i)$, the only prime subgroups of G_i whose cosets appear as $p(i)$, for some p, are the prime subgroups of G_i containing H_i.

We choose a fixed set of representatives $g_{i\gamma}, g'_{i\gamma}, \cdots$ of the cosets in G_i of the prime subgroups $G_{i,\gamma}$ of G_i, as required for the application of Lemma 1.1.

Totally order T as follows: first well order I: then define $p > q$ if and only if either $p(i) > q(i)$ for the first element $i \in I$ for which $p(i) \neq q(i)$ or $p(i) = q(i)$, for all $i \in I$, and $p(\alpha) > q(\alpha)$.

We now represent each G_i as a l-subgroup of $P(T)$.

For each $i \in I$ and each $\gamma \in \Gamma(H)$ let $\psi_{i\gamma}$ and $\xi_{i\gamma}$ be the l-homomorphisms of G_i into $P(H_i/H_\gamma\theta_i \times G_i/H_\gamma\theta_i\sigma_i^{-1} + H_i)$ and $P(G_i/H_\gamma\theta_i\sigma_i^{-1} + H_i)$, respectively, defined by analogy with the definition of ψ_γ and ξ_γ in Section 1. Then we define a mapping $\phi_i : G_i \to P(T)$ as follows: let $p \in T$.

If $p(\alpha) = 1$, then define $p(g\phi_i) = q$ where

$$q(\alpha) = 1 = p(\alpha), \; q(i) = G_{i,\gamma} + g_{i\gamma} + g, \;\; q(j) = p(j) \text{ for } j \neq i.$$

If $p(\alpha) \neq 1$ and $g \in G_i$, suppose that

$$(p(\alpha)\theta_i, G_{i,\gamma} + g_{i\gamma})g\psi_{i\gamma} = (y\theta_i, G_{i,\gamma} + g'_{i\gamma}).$$

Then define $p(g\phi_i) = q$ where

$$q(\alpha) = y, \; q(i) = G_{i,\gamma} + g'_{i\gamma}, \; q(j) = p(j) \text{ for } j \neq i.$$

It is not difficult to see that ϕ_i is faithful and is essentially just a product of Holland representations and so is an l-isomorphism.

Now consider $h\theta_i\phi_i$, $h\theta_j\phi_j$ for $h \in H$, $i, j \in I$. Let $p \in T$.

Case (i), $p(\alpha) = 1$. Then $p(h\theta_i\phi_i) = q$, where

$$q(\alpha) = p(\alpha), \; q(i) = G_{i,\gamma} + g_{i\gamma} + h\theta_i = G_{i,\gamma} + g_{i\gamma} = p(i),$$
$$q(r) = p(r), \text{ for } r \neq i,$$

since $h\theta_i$ is in the centre of G_i and $H_i \subseteq G_{i,\gamma}$. Thus $p(h\theta_i\phi_i) = p$ and, similarly, $p(h\theta_j\phi_j) = p$.

Case (ii), $p(\alpha) = H_\gamma + a$. Then $p(h\theta_i\phi_i) = q$, say, where $q(r) = p(r)$ for $r \neq \alpha, i$. Now $q(i)$ and $q(\alpha)$ are determined as in Lemmas 1.1, 1.2, by

$$(H_\gamma \theta_i + a\theta_i, G_{i,\gamma} + g_{i\gamma})(h\theta_i \psi_{i\gamma})$$

and this is determined by

$$(H_\gamma \theta_i \sigma_i^{-1} + a\theta_i + g_{i\gamma})(h\theta_i)\xi_{i\gamma}.$$

Now,

$$\begin{aligned}(H_\gamma \theta_i \sigma_i^{-1} + a\theta_i + g_{i\gamma})(h\theta_i)\xi_{i\gamma} &= H_\gamma \theta_i \sigma_i^{-1} + a\theta_i + g_{i\gamma} + h\theta_i \\ &= H_\gamma \theta_i \sigma_i^{-1} + (a+h)\theta_i + g_{i\gamma}\end{aligned}$$

since $h\theta_i$ is in the centre of G_i. Hence,

$$(H_\gamma \theta_i + a\theta_i, G_{i,\gamma} + g_{i\gamma})h\theta_i \psi_{i\gamma} = (H_\gamma \theta_i + (a+h)\theta_i, G_{i,\gamma} + g_{i\gamma}).$$

Thus

$$q(\alpha) = H_\gamma + a + h \text{ and } q(r) = p(r), \text{ for } r \neq \alpha$$

Similarly, we will find that $p(h\theta_j\varphi_j) = q$.

Thus $\theta_i\varphi_i = \theta_j\varphi_j$ for all i, j while clearly, for any i, j, k,

$$G_i\varphi_i \cap G_j\varphi_j = \bigcap_i G_i\varphi_i = H\theta_k\varphi_k.$$

We call the 1-subgroup of $P(T)$ generated by the 1-subgroups $G_i\varphi_i$ the *1-permutational product of the 1-groups G_i with the 1-subgroups H_i amalgamated.* Thus we have shown

THEOREM 2.1. *Let $\{G_i : i \in I\}$ be a set of 1-groups, $\{H_i : i \in I\}$ a set of convex 1-subgroups such that H_i is in the centre of G_i, let H be an 1-group and, for each i, θ_i be an 1-isomorphism of H onto H_i. Then there exists an 1-group P, the 1-permutational product of the 1-groups G_i with the 1-subgroups H_i amalgamated, and 1-isomorphisms φ_i of G_i into P such that $\theta_i\varphi_i = \theta_j\varphi_j$ for all i, j, $G_i\varphi_i \cap G_j\varphi_j = \bigcap_i G_i\varphi_i = H\theta_k$, for all i, j, k, and P is generated as an 1-group by $\{G_i\varphi_i\}$. Moreover, if all the G_i are abelian, then so is P.*

The last remark of the theorem follows since it can easily be shown that the elements of $G_i\varphi_i$ commute with the elements of $G_j\varphi_j$ (provided $i \neq j$). Thus, if the G_i are all abelian then $\{G_i\varphi_i\}$ generates an abelian subgroup of P and hence an abelian 1-subgroup of P.

Note: We point out that the class of 1-groups with non-trivial convex central 1-subgroups does contain non-abelian members. Suppose that G is a torsion free nilpotent group with lower central series

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \cdots \supset \gamma_{n+1}(G) = 1$$

and torsion free factors $\gamma_i(G)/\gamma_{i+1}(G)$. Let the torsion free abelian group $\gamma_n(G)$ be endowed with some lattice order: provided the rank of $\gamma_n(G)$ is not one, one could

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ensure that this is not a total order. Let each torsion free abelian factor $\gamma_i(G)/\gamma_{i+1}(G)$, $i \neq n$, be endowed with some total order. Then the order defined on G as follows is a lattice order, with respect to which G is an 1-group: for $a \in \gamma_i(G) \setminus \gamma_{i+1}(G)$ define a to be positive if $\gamma_{i+1}(G)+a$ is positive in $\gamma_i(G)/\gamma_{i+1}(G)$. Then $\gamma_n(G)$ is a convex central 1-subgroup.

3. Permutational products of 0-groups

To form the permutational product of totally ordered groups with certain *normal* convex subgroups amalgamated we need not resort to the Holland representation of an 1-group. We may proceed directly, as for groups.

Let $\{G_i : i \in I\}$ be a set of 0-groups, $\{H_i : i \in I\}$ a set of normal convex subgroups, H an 0-group and, for each i, let θ_i be an 0-isomorphism of H onto H_i. For each i, select a set of representatives $C(H_i) = \{h_i, \alpha\}$ of the right cosets of H_i in G_i. Let T be the cartesian product of H with the $C(H_i)$, considered as the set of choice functions p from $I \cup \{\alpha\}$ such that $p(\alpha) \in H$, $p(i) \in C(H_i)$. We order T by first well ordering I and then defining $p > q$ if $p(i) > q(i)$, where i is the first element of I for which $p(i) \neq q(i)$, or $p(i) = q(i)$ for all $i \in I$ and $p(\alpha) > q(\alpha)$.

For any element g_i of G_i we define an element ρ_{g_i} of $P(T)$ as follows: $p\rho_{g_i} = q$ where $p(i)+(p(\alpha))\theta_i+g_i = q(i)+(q(\alpha))\theta_i$, and $q(j) = p(j)$, for $i \neq j \in I$.

Now define $\rho_i : G_i \to P(T)$ by $g_i\rho_i = \rho_{g_i}$. Then, for each i, ρ_i is an 0-isomorphism of G_i into $P(T)$ such that $\theta_i\rho_i = \theta_j\rho_j$, for all i, j and $G_i\rho_i \cap G_j\rho_j = \bigcap_k G_k\rho_k = H\theta_i\rho_i$.

We call the 1-subgroup P_0 of $P(T)$ generated by the set of $G_i\theta_i$ *the* 1*-permutational product of the* 0*-groups* G_i *with amalgamated subgroups* H_i. In general, P_0 is not an 0-group. However,

THEOREM 3.1. *If, for each i, H_i is in the centre of G_i then P_0 is an* 0*-subgroup of $P(T)$. If each G_i is abelian so is P_0.*

PROOF. It is straightforward to show that if H_i is in the centre of G_i, for all i, then elements from distinct $G_i\rho_i$ commute. Hence, if the G_i are themselves abelian, so will the 1-subgroup generated by the $G_i\rho_i$ be abelian.

It then follows that the group generated by the $G_i\rho_i$ is just the generalized direct product D of the $G_i\rho_i$ and the order induced on D can be defined directly as follows: Let $\rho = \rho_{g_1}\rho_{g_2}\cdots\rho_{g_n} \in D$, where $g_\alpha \in G_{i_\alpha}$; we may assume that $i_1 < i_2 < \cdots < i_n$ in the order on I; then $\rho > 1$ if and only if either $g_\alpha > 0$, where i_α is the first index such that $g_\alpha \notin H_{i_\alpha}$ or $g_\alpha \in H_{i_\alpha}$, for i_α, and $g_1\theta_{i_1}^{-1}+g_2\theta_{i_2}^{-1}+\cdots+g_n\theta_{i_n}^{-1} > 0$ in H. This is easily seen to endow D with a total order. Hence, P_0 is just the subgroup generated by the $G_i\rho_i$ and is totally ordered.

We give the following example to illustrate that even if the H_i are normal abelian convex subgroups of the 0-groups G_i, then the 1-permutational product

P_0 of the G_i with the H_i amalgamated need not be an 0-group. It suffices to produce an element not comparable to 1.

EXAMPLE. Let Z^R denote the group of functions $f: R \to Z$ of finite support (where R = real numbers, Z = additive group of integers), ordered by: $f > 0$ if $f(a) > 0$ where a is the largest number in the support of f. Let $\alpha, \beta \in P(R)^+$ be such that $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$ is not comparable to 1; that is, such that for some $a, b \in R$

$$a[\alpha, \beta] > a \text{ and } b[\alpha, \beta] < b.$$

The graphs of such a pair are given in Figure 1.

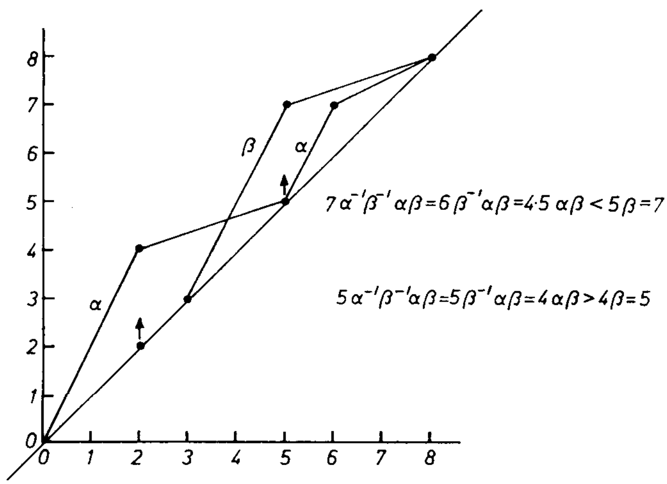

Figure 1

These induce automorphisms $f \to f^\alpha, f \to f^\beta$ of Z^R where

$$f^\alpha(x) = f(x\alpha^{-1}), f^\beta(x) = f(x\beta^{-1}).$$

Let $G_1 = Z^R \times_\varphi Z$, the semidirect product of Z^R with Z via the isomorphism $\varphi : Z \to \text{Aut } Z^R$ where $1\varphi = \alpha$. Similarly, let $G_2 = Z^R \times_\theta Z$ where $1\theta = \beta$. Then G_1 and G_2 are subgroups of the wreath product $G = (Z, Z)wr(R, P(R))$ of the permutation groups (Z, Z), $(R, P(R))$ where Z acts on Z by translation and $P(R)$ acts on R naturally.

Now form the 1-permutational product P_0 of G_1 and G_2 amalgamating Z^R, with $\rho_1 : G_1 \to P_0$ and $\rho_2 : G_2 \to P_0$. We take the obvious representatives of the cosets of Z^R and order $I = \{1, 2\}$ in the natural manner. Let $f_a, f_b \in Z^R$, be such that

$$f_a(x) \begin{cases} 1 \text{ if } x = a, \\ 0 \text{ otherwise}; \end{cases} \quad f_b(x) \begin{cases} 1 \text{ if } x = b \\ 0 \text{ otherwise.} \end{cases}$$

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Then $f_a^{[\alpha,\beta]} \in Z^R \subseteq G$ is such that

$$f_a^{[\alpha,\beta]}(x) = f_a(x[\alpha,\beta]^{-1}) = \begin{cases} 1 & \text{if } x = a[\alpha,\beta] \\ 0 & \text{otherwise} \end{cases}$$

and so $f_a^{[\alpha,\beta]} > f_a$. Similarly $f_b^{[\alpha,\beta]} < f_b$.

Now consider the element $[x\rho_1, y\rho_2]$ of P_0, where $x = (1,0) \in G_1$ and $y = (1,0) \in G_2$. Then, for the element $(f_a, 0, 0)$ of T, where T is defined as in Theorem 3.1,

$$\begin{aligned} &(f_a, 0, 0)[x\rho_1, y\rho_2] \\ &\quad = (f_a, 0, 0)(x\rho_1)^{-1}(y\rho_2)^{-1}(x\rho_1)(y\rho_1) \\ &\quad = (f_a\alpha^{-1}, (1,0), 0)(y\rho_2)^{-1}(x\rho_1)(y\rho_1) \\ &\quad = \cdots \\ &\quad = (f_a^{[\alpha,\beta]}, 0, 0) > (f_a, 0, 0). \end{aligned}$$

On the other hand $(f_b, 0, 0)\,[x\rho_1, y\rho_2] = (f_b^{[\alpha,\beta]}, 0, 0) > (f_b, 0, 0)$. Thus the element $[x\rho_1, y\rho_2]$ in P_0 is not comparable to 1 and P_0 is not an 0-group.

Note: In the case of groups, the permutational product of two groups A and B amalgamating a subgroup H, central in each, is just the central product of the two groups. However, there is some difficulty in defining the central product if A and B are 1-groups. If H is in the centres of A and B then the central product of A and B amalgamating H is $C = A \times B/N$ where N is the subgroup of $A \times B$ generated by $\{(h, -h) : h \in H\}$. However, if we wish to endow C, or some similar group, with a lattice order by first endowing $A \times B$ with the cardinal order and then factoring, then the kernel $\bar{N}$ has to be an 1-ideal and contain N. Thus if $h > 0$ we must have $(h, 0) = (h, -h) \vee 0$ and $(0, -h) = (h, -h) \wedge 0$ in $\bar{N}$ and so the 'embeddings' of A and B in $A \times B/\bar{N}$ would not be monomorphisms.

4. Amalgams of representable 1-groups

As remarked in Section 2, Conrad has shown that if H is a convex 1-subgroup of an 1-group G then the mapping $\sigma : M \to M \cap H$ is a bijection of the set of prime subgroups of G not containing H onto the set of prime subgroups of H. Clearly, if M is an 1-ideal of G then $M\sigma$ is an 1-ideal of H, although the converse need not be true. For want of a more appropriate name, if σ^{-1} maps 1-ideals of H onto 1-ideals of G then we shall call H an *l.e. convex 1-subgroup* of G (short for: 1-ideals are extendable from H to G). For instance, if all the prime subgroups of G are normal then any convex 1-subgroup of G would be an 1.e. convex 1-subgroup of G.

An 1-group G is called *representable* if there exists an 1-isomorphism of G onto a subdirect sum of a cardinal sum of 0-groups. Conrad [2] has shown that G is representable if and only if every regular 1-ideal is a prime subgroup. A *regular*

1-*ideal* of an 1-group is an 1- ideal which is maximal in the lattice of 1-ideals with respect to not containing some element. If H is an 1.e. convex 1-subgroup of a representable G then it is not difficult to show that σ is also a bijection of regular 1-ideals of G not containing H onto the regular 1-ideals of H.

THEOREM 4.1. *Let* $[G_i; H; \theta_i; H_i]$ *be an amalgam of* 1-*groups where each* G_i *is representable and each* H_i *is a normal l.e. convex* 1-*subgroup of* G_i. *Then the amalgam is embeddable in a* 1-*group.*

PROOF. For each i, let σ_i be the bijection, described above, of the prime subgroups of G_i not containing H_i onto the prime subgroups of H_i.

Let $\{L_\gamma : \gamma \in \Gamma\}$ be the set of regular 1-ideals of H; then $\{L_\gamma \theta_i : \gamma \in \Gamma\}$ is the set of regular 1-ideals of H_i, and $\{L_\gamma \theta_i \sigma_i^{-1} : \gamma \in \Gamma\}$ is the set of regular 1-ideals of G_i not containing H_i; let $\{L_{i,\delta} : \delta \in \Gamma_{i,1}\}$ be the set of regular 1-ideals of G_i containing H_i.

We denote by φ_i the natural 1-isomorphism of G_i into

$$\bar{G}_i = (\sum_{\gamma \in \Gamma} \boxplus G_i/L_\gamma \theta_i \sigma_i^{-1}) \boxplus (\sum_{\delta \in \Gamma_{i,1}} \boxplus G_i/L_{i,\delta}),$$

where, for 1-groups M_α, $\sum_{\alpha \in A} \boxplus M_\alpha$ denotes the (unrestricted) cardinal sum of the 1-groups M_α. We denote by π_γ, $\pi_{i,\delta}$ the projections of $\bar{G}_i$ onto $G_i/L_\gamma \theta_i \sigma_i^{-1}$ and $G_i/L_{i,\delta}$, respectively.

Now, for each $\gamma \in \Gamma$, we have the amalgam of 0-groups $[G_i/L_\gamma \theta_i \sigma_i^{-1}; H/L_\gamma; \theta_i \pi_\gamma; H_i/L_\gamma \theta_i]$ where, by $\theta_i \pi_\gamma$, we mean the naturally induced 1-isomorphism of H/L_γ onto $H_i/L_\gamma \theta_i$. By Section 3, we can always embed such an amalgam in a 1-group. So, for each γ, choose such an embedding $\{\rho_{i,\gamma}\}$ into an 1-group K_γ.

We write

$$K = (\sum_{\gamma \in \Gamma} \boxplus K_\gamma) + \sum_{i \in I} \boxplus (\sum_{\delta \in \Gamma_{i,1}} \boxplus G_i/L_{i,\delta}).$$

We denote by ι_γ, $\iota_{i,\delta}$ the injection of K_γ and $G_i/L_{i,\delta}$ into K. Then, for each i,

$$\Psi_i = (\prod_{\gamma \in \Gamma} \varphi_i \pi_\gamma \rho_{i,\gamma} \iota_\gamma) \times (\prod_{\delta \in \Gamma_{i,1}} \varphi_i \pi_{i,\delta} \iota_{i,\delta})$$

is an 1-isomorphism of G_i into K and $\{\Psi_i\}$ is an embedding of $[G_i; H; \theta_i; H_i]$ into K.

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