Bull. Austral. Math. Soc. 58G25, 35P15, 58G30
Vol. 47 (1993) [127–144]

EIGENVALUE ESTIMATES AND ISOPERIMETRIC INEQUALITIES FOR CONE-MANIFOLDS

Craig Hodgson and Johan Tysk

This paper studies eigenvalue bounds and isoperimetric inequalities for Riemannian spaces with cone type singularities along a codimension-2 subcomplex. These "cone-manifolds" include orientable orbifolds, and singular geometric structures on 3-manifolds studied by W. Thurston and others.

We first give a precise definition of "cone-manifold" and prove some basic results on the geometry of these spaces. We then generalise results of S.-Y. Cheng on upper bounds of eigenvalues of the Laplacian for disks in manifolds with Ricci curvature bounded from below to cone-manifolds, and characterise the case of equality in these estimates.

We also establish a version of the Lévy-Gromov isoperimetric inequality for cone-manifolds. This is used to find lower bounds for eigenvalues of domains in cone-manifolds and to establish the Lichnerowicz inequality for cone-manifolds. These results enable us to characterise cone-manifolds with Ricci curvature bounded from below of maximal diameter.

1. Introduction

In this paper, we study eigenvalue bounds and isoperimetric inequalities for Riemannian spaces with cone type singularities. These "cone-manifolds" will be precisely defined in section two, where we also prove some relevant geometric results. Orientable orbifolds and branched coverings of Riemannian manifolds, for instance, are cone-manifolds. Other interesting examples of cone-manifolds arise as singular geometric structures on 3-manifolds, studied, for example, by Thurston [**18, 19**].

In section three, we generalise the results of Cheng [4], on upper bounds of eigenvalues of disks in manifolds with Ricci curvature bounded from below, to cone-manifolds and characterise the case of equality in these estimates.

In section four, we establish a version of the Lévy-Gromov isoperimetric inequality [**10**] for cone-manifolds. Via a Faber-Krahn type argument, we are therefore able to find lower bounds for eigenvalues of domains in cone-manifolds in terms of the eigenvalues

Received 20th January, 1992
The first author was partially supported by NSF grant DMS 88-05672. The second author was partially supported by NSF grant DMS 87-04209, and the Swedish Natural Science Research Council (NFR).

of disks of related volume in space forms. This enables us to establish the Lichnerowicz inequality for cone-manifolds and to characterise cone-manifolds with Ricci curvature bounded from below of maximal diameter.

We would like to thank Sidney Frankel for several helpful conversations.

2. Cone-manifolds

DEFINITIONS: An *n-dimensional cone-manifold* is a simplicial complex M which is a rational homology n-manifold (that is, the link of each vertex has the rational homology of an $(n-1)$-dimensional sphere). In addition, M is a complete metric space with a *smooth* Riemannian metric defined on the complement of the codimension two skeleton of M and on each closed simplex.

The *singular locus* Σ consists of points in M with no neighbourhood isometric to a ball in a Riemannian manifold. Then Σ is a subcomplex of the codimension two skeleton of M. At each point of Σ in the interior of an $(n-2)$-simplex there is a *cone angle* which is the sum of dihedral angles of n-simplices containing the point. In general, the cone angle may vary from point to point within a simplex. The *regular set* $M_r = M - \Sigma$ is a dense open subset of M and is a smooth Riemannian manifold with a metric which is *incomplete* whenever $\Sigma \neq \emptyset$.

Throughout this paper, we make the additional assumption that the singular locus is a union of *totally geodesic* simplices (of dimension $\leqslant n-2$). This condition is probably not essential for our main results, but rules out certain pathological kinds of behaviour of geodesics (compare Example (d) below).

EXAMPLES.

(a) Riemannian orbifolds are spaces locally modelled on Riemannian manifolds modulo finite groups of isometries. These spaces have singular locus modelled on the fixed point sets of elements in these finite isometry groups and are cone-manifolds whenever these fixed point sets have codimension at least two. Here, the singular locus consists of totally geodesic strata with cone angles of the form $2\pi/n$, where n is an integer.

(b) Branched coverings of Riemannian manifolds over codimension two submanifolds are cone-manifolds, with cone angles of the form $2\pi n$, where n is an integer. In this case the singular locus will be totally geodesic if and only if the branching set is totally geodesic.

(c) Any rational homology manifold triangulated by totally geodesic simplices from a space of constant curvature K gives a cone-manifold whose singular locus is a union of totally geodesic simplices. We call these spherical, Euclidean or hyperbolic cone-manifolds when $K > 0$, $K = 0$ or $K < 0$ respectively. Many 3-dimensional examples of such cone-manifolds arise in Thurston's theory of hyperbolic Dehn surgery [**18**, Chapter 5]. Such cone-manifolds are also used extensively in Thurston's proof of the existence of geometric structures on many 3-dimensional orbifolds [**19**]; here, they give a way to interpolate between geometric structures on different orbifolds. In [**7**], Cheeger considers piecewise constant curvature spaces somewhat more general than those considered here. Cheeger proves, however, that if the curvature is non-negative and cone angles are less than 2π, then the two notions coincide.

Given a spherical cone-manifold S of constant curvature 1, let $\mathrm{Cone}_K(S;R)$ denote the (open) cone of constant curvature K with base S and radius R. Topologically, $\mathrm{Cone}_K(S;R)$ is obtained from the space $S\times[0,R)$ by identifying $S\times\{0\}$ to a point. The metric has the form $ds^2=dr^2+s_K(r)^2d\theta^2$ where $d\theta^2$ denotes the metric on S, $r\in[0,R)$ and

$$s_K(r)=\begin{cases}\dfrac{1}{\sqrt{K}}\sin\left(\sqrt{K}r\right) & \text{if } K>0,\\ r & \text{if } K=0,\\ \dfrac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}r\right) & \text{if } K<0.\end{cases}$$

If $K\leqslant 0$ then $\mathrm{Cone}_K(S;R)$ is defined for all $0<R\leqslant\infty$. If $K>0$ then $\mathrm{Cone}_K(S;R)$ is defined for $0<R\leqslant\pi/\sqrt{K}$ and we define the *suspension* $\mathrm{Susp}_K(S)$ of S to be the completion of $\mathrm{Cone}_K(S;R)$ where $R=\pi/\sqrt{K}$. This suspension is obtained by gluing together two closed cones of radius $\pi/\left(2\sqrt{K}\right)$; the centres of these cones are called the *suspension points*. These cones and suspensions are analogues of standard balls and spheres for constant curvature cone-manifolds.

(d) Examples of cone-manifolds with singular locus which is not totally geodesic can also be easily constructed by gluing together simplices. For example, begin with the boundary of a 4-simplex in Euclidean space. This gives a Euclidean cone-manifold structure on the 3-sphere made from five Euclidean 3-simplices such that the cone angles are $<2\pi$. Now construct five new 3-simplices by deforming the 1-skeleton slightly and adding in new 2-dimensional faces. By filling in new faces consistently we can pair these faces together by isometries and obtain new Euclidean cone-manifold structures on the 3-sphere with singular locus no longer totally geodesic and cone angles $<2\pi$. We remind the reader that such examples will not be considered in the remainder of this paper.

At each point p of a cone-manifold M there is a *tangent cone* TM_p isometric to a Euclidean cone $\mathrm{Cone}_0(S_p;\infty)$. Any sequence of enlargements $t_i(M,p)$ of the based metric space (M,p) with scale factors $t_i\to\infty$ converges to TM_p in the sense of Hausdorff convergence (compare Gromov [**11**]) and the convergence is smooth away from Σ. In fact, TM_p is the union of the Euclidean tangent cones to all the n-dimensional simplices containing p. The spherical cone-manifold S_p is the called the *unit tangent cone* at p, and has constant curvature 1.

REMARKS.

(1) A cone-manifold is a path space in the sense of Gromov [**11**]: the distance between two points is the infimum of lengths of paths joining the points. (Compare Lemma 1 below.)

(2) It would be possible to define a cone-manifold in a more intrinsic way as a space modelled locally on Euclidean cones. A "Riemannian structure" could be introduced by equipping each tangent cone with a Euclidean metric (varying continuously from point to point) and defining the length of curves by integrating this infinitesimal metric.

A *geodesic* in a cone-manifold is a curve which is locally length minimising.

LEMMA 1. *Let M be a complete, connected cone-manifold.*

(i) *Then any two points in M can be joined by a geodesic of length equal to the distance between the points.*

(ii) *Given any vector v in TM_p, there is a geodesic g_v in M with initial tangent vector v, and g_v is uniquely defined in some neighbourhood of p.*

PROOF: Part (i) follows by a standard compactness argument as in Gromov [**11**]. Part (ii) is clear unless v is tangent to the singular locus Σ. In this case the result also follows immediately since we are assuming that Σ consists of totally geodesic strata. $\square$

Let $\mathcal{D}_p \subset TM_p$ be the "fundamental domain" at p, consisting of all v in TM_p such that geodesic g_v is defined and minimising up to time 1. Then there is a well-defined, continuous exponential map $exp : \mathcal{D}_p \to M$, defined by $exp(v) = g_v(1)$.

LEMMA 2. *A connected spherical cone-manifold of curvature 1 with all cone angles $< 2\pi$ has diameter $\leqslant \pi$, with equality if and only if the cone-manifold is a suspension. Further, the only two points at distance π apart are the suspension points.*

LEMMA 3. *In a cone-manifold with all cone angles $< 2\pi$, length minimising geodesics do not pass through the singular locus. Such a geodesic may intersect the singular locus at an endpoint or be entirely contained in one stratum of the singular locus.*

REMARKS.

(1) These results both fail if cone angles larger than 2π are allowed. In this case length minimising geodesics may pass through cone points. In fact, there is a pencil of extensions of any geodesic at any cone point with angle $> 2\pi$, consisting of all outgoing geodesic arcs making an angle $\geqslant \pi$ with the incoming arc.

If the singular locus is not totally geodesic then geodesics may intersect the singular locus tangentially.

Lemma 3 implies that if a vector v is not tangent to Σ then the geodesic g_v can be extended until it meets the singular locus. If v is tangent to Σ then g_v is contained in Σ and can be extended until it meets a different strata of the singular locus.

PROOFS: We prove Lemmas 2 and 3 simultaneously by induction on dimension, following an argument of Thurston. Let (2_n), (3_n) be the statements of Lemmas 2 and 3 for n-dimensional cone-manifolds. For $n = 1$ both statements are trivially true.

Assume (2_{n-1}) and (3_{n-1}) are true for $n \geqslant 2$. Let g be a length minimising geodesic in an n-dimensional cone-manifold. Suppose p is a singular point in the interior of g, and let v_- and v_+ be the unit tangent vectors to g at p, directed away from p. Then v_- and v_+ give two points in the unit tangent cone S_p, which is a spherical cone-manifold of dimension $(n-1)$. The angle between these tangent vectors is the distance between v_- and v_+ measured in S_p so is $\leqslant \pi$, by the hypothesis on cone angles if $n = 2$ and by the induction hypothesis (2_{n-1}) if $n > 2$. If this angle is $< \pi$ then the length of g could be reduced by smoothing the corner at p, so g would not be locally length minimising. If the angle is equal to π then it follows from (2_{n-1}) that g is tangent to Σ at p, hence contained in Σ. This proves (3_n).

To prove (2_n), we consider a length minimising geodesic g in an n-dimensional spherical cone-manifold S of curvature 1. By (3_n), g does not pass through the singular locus. Elementary spherical geometry therefore shows that g has length $\leqslant \pi$; hence S has diameter $\leqslant \pi$. Further, if g has length π then its interior has a neighbourhood which is a suspension. Now suppose that S contains two points p, q at distance π apart. Let U be the set of unit tangent vectors $v \in S_p$ such that the geodesic g_v with initial tangent vector v is length minimising on $[0, \pi]$ and joins p to q. Then U is an open subset of S_p by our previous remark. Suppose that v is a vector in the closure of U. We claim that the geodesic g_v is defined on $[0, \pi]$. If not, then g_v ends at some singular point $p_0 \in \Sigma$. Let g' be a shortest geodesic from p_0 to q. Then $g_v \cup g'$ has length $\leqslant \pi$ since v lies in $\overline{U}$. But this curve has a corner at p_0, so can be shortened to give a path from p to q of length less than π, contradicting the choice of p and q. Since the exponential map is continuous, it now follows that $v \in U$. Hence, $U = S_p$ and M is the suspension of S_p, with p and q as suspension points. ▯

The following result is an important consequence of the previous lemmas.

LEMMA 4. *If M is a complete, connected cone-manifold with all cone angles $< 2\pi$ then the exponential map $exp : \mathcal{D}_p \to M$ is onto.*

Next we study $\partial\mathcal{D}_p$ and the cut locus $exp(\partial\mathcal{D}_p)$. If $f : M \to N$ is a continuous map between cone-manifolds, then it is possible to consider directional derivatives at each point of M. If all such directional derivatives exist, then one obtains a map $df : TM_p \to TN_{f(p)}$. We say that q is a *conjugate point* of p in M if $dexp(v) = 0$ for some non-zero $v \in TM_p$ such that $exp(v) = q$.

Lemma 5. *A point $q \in M$ lies in the cut locus $exp(\partial \mathcal{D}_p)$ if and only if at least one of the following cases occurs:*

(1) *There are two minimising geodesics from p to q in M.*
(2) *q is a conjugate point of p.*
(3) *q is in the singular locus Σ and there is a length minimising geodesic from p to q which doesn't extend past the point q.*

Remark. Case (3) always occurs when q is a singular point, unless S_q is a suspension and there is a unique shortest geodesic from p to q which is tangent to the direction of the suspension points in S_q.

Proof: This follows from the usual arguments for Riemannian manifolds together with Lemma 3. □

We now make some comments on the local structure of $exp(\partial \mathcal{D}_p)$ near a point q_0 in the singular locus Σ.

In case (1), there are at least two minimising geodesics from p to q_0. If q_0 is not conjugate to p, then there exists $\delta > 0$ such that only finitely many geodesics of length $< d(p, q_0) + \delta$ join p to q_0. In particular, there are finitely many shortest geodesics from p to q for all q near q_0 and these are obtained by small perturbations of shortest geodesics from p to q_0. Let γ be a shortest geodesic joining p to q_0, with length d_0. Then if γ' is a geodesic sufficiently close to γ joining p to a point q, then

$$\text{length}(\gamma') = d_0 - \varepsilon \cos\theta + O(\varepsilon^2)$$

where ε is the distance from q_0 to q, and θ is the angle pq_0q between γ and q_0q. From this one can easily obtain an explicit description of the tangent cone to $exp(\partial \mathcal{D}_p)$ at q_0 as the cone on a totally geodesic, codimension one subcomplex K of S_{q_0}. (In fact, if $x_1, x_2, \dots, x_k \in S_{q_0}$ are initial vectors of the shortest geodesics from q_0 to p, then K consists of the boundaries of Voronoi regions for the set of points $\{x_1, \dots, x_k\}$ on S_{q_0}. In particular, each point in K is equidistant from at least two of the points $x_1, \dots, x_k$.) For our purposes, it is enough to observe that a wedge of positive measure is excluded from $\mathcal{D}_p$ for each minimising geodesic from p to q_0. Similar arguments apply when case (3) occurs at a non-conjugate point. The local structure of the cut locus near conjugate points can be much more complicated, just as in the Riemannian manifold case.

Given a point $p \in M$, define $c : S_p \to [0, \infty]$ by $c(v) = \sup\{t \mid tv \in \mathcal{D}_p\}$, for $v \in S_p$. In other words, $c(v)$ is the distance from p to its cut locus along the geodesic in direction v.

Lemma 6. *The function $c : S_p \to [0,\infty]$ is continuous.*

Proof: The argument is largely the same as in the case of a Riemannian manifold. Let $\{v_k\}$ be a sequence in S_p approaching v. By taking a subsequence we can assume that $\lim c(v_k)$ exists in $[0,\infty]$. Let $a_k = c(v_k)$ and $a = \lim c(v_k)$.

First we show that $\lim c(v_k) \leqslant c(v)$. For each k,

$$d(p, exp\ tv_k) = t, \text{ for } 0 \leqslant t \leqslant c(v_k).$$

If $exp\ tv$ is defined for $0 \leqslant t \leqslant \lim c(v_k)$, then since d and exp are continuous it follows that

$$d(p, exp\ tv) = t, \text{ for } 0 \leqslant t \leqslant \lim c(v_k),$$

hence $c(v) \geqslant \lim c(v_k)$. However, it is also possible that the geodesic in the direction v meets the singular locus at some point $exp\ tv$ where $t < \lim c(v_k)$ and cannot be extended past this point. In this case, the local analysis given above shows that $\partial\mathcal{D}$ forms a wedge with vertex at tv, and c is continuous at v.

Now assume that $c(v) > \lim c(v_k) = a$. By Lemma 5, $exp\ av$ is not conjugate to p along the geodesic in direction v. It follows that exp is a homeomorphism in a neighbourhood U of av, hence $exp\ a_k v_k$ is not conjugate to p along the geodesic in direction v_k for all k sufficiently large. From the remarks in the previous paragraph and Lemma 5, it follows that there are two minimising geodesics $exp\ tv_k$ and $exp\ tw_k$ from p to $exp\ a_k v_k$ for all large k, with w_k outside the open set U. By choosing a subsequence, we can assume that the w_k converge to $w \in S_p$. Then $w \neq v$ and

$$exp\ aw = \lim exp\ a_k w_k = \lim exp\ a_k v_k = exp\ av.$$

(Note that $exp\ aw$ is necessarily defined, by the same local analysis near the singular locus as in the previous paragraph.) Hence there are two minimising geodesics from p to $exp\ av$ and $c(v) \leqslant a$, contradicting our assumption. $\square$

Corollary. *The sets $\partial\mathcal{D}_p$ and $exp(\partial\mathcal{D}_p)$ have measure zero in the tangent cone TM_p and M respectively.*

For each p in M, let $\mathcal{B}(p;r)$ denote the ball of radius r centred at the origin (that is, cone point) in the tangent cone TM_p and let $B(p;r)$ denote its image under exp, that is, the ball of radius r about p in M.

Lemma 7. *Let M be a cone-manifold of constant curvature K with all cone angles $< 2\pi$. If $\mathcal{D}_p \cap \mathcal{B}(p;r)$ has full measure in $\mathcal{B}(p;r)$ then $B(p;r)$ is isometric to the open cone* $\mathrm{Cone}_K(S_p;r)$ *where S_p is the unit tangent cone at p.*

Proof: From Lemma 6 it follows that if $\mathcal{D}_p \cap \mathcal{B}(p;r)$ has full measure then $\mathcal{B}(p;r) \subset \mathcal{D}_p$. Hence, using Lemma 4, the exponential map exp is a homeomorphism

https://doi.org/10.1017/S0004972700012326 Published online by Cambridge University Press

from $\mathcal{B}(p;r)$ to $B(p;r)$. If M has constant curvature K it follows easily that $B(p;r)$ is isometric to the (open) cone $\mathrm{Cone}_K(S_p;r)$. $\square$

Many standard results in Riemannian geometry can be generalised to cone-manifolds by applying the usual results within closed simplices and piecing together the results. For example, there are versions of Gauss's Lemma, geodesic polar coordinates and Fermi coordinates giving nice local representations of the "Riemannian metric" on a cone-manifold.

We can define sectional and Ricci curvatures as usual in the smooth part of M. On the singular locus the curvature should be regarded as a measure. There is concentrated positive curvature at points where the cone angle is $< 2\pi$ and concentrated negative curvature where the cone angle is $> 2\pi$.

DEFINITION: Given a real number c, we say that a cone-manifold has *(Ricci) curvature* $\geqslant c$ if the (Ricci) curvature is $\geqslant c$ at all smooth points *and* all cone angles at singular points are $< 2\pi$.

Then the usual comparison theorems based on growth rates of Jacobi fields, for example the volume comparison theorems of Bishop [2], all generalise to cone-manifolds. For instance, the volume form dV of a cone-manifold M can be written in geodesic polar coordinates $(r,v) \in [0,\infty) \times S_p$ (with $rv \in \mathcal{D}_p$) as

$$dV = \sqrt{g}(r,v)dr\, d\mu_p(v),$$

where $d\mu_p$ is the volume form of S_p and $\sqrt{g}$ is a function determined by the Jacobi fields along geodesics through p. If M is an n-dimensional cone-manifold with Ricci curvature bounded below by $(n-1)K$, the factor $\sqrt{g}$ satisfies the inequality of Bishop's comparison theorem:

$$\frac{\sqrt{g}\,'(r,v)}{\sqrt{g}(r,v)} \leqslant (n-1)\frac{c_K}{s_K},$$

where $'$ denotes differentiation with respect to r, s_K is defined as in Example (c) above and $c_K = s_K'$. Further, if equality holds for all v then M_r has constant sectional curvature K. By the continuity of the curvature of M_r, the same conclusion can be drawn if equality holds for almost all v.

3. EIGENVALUE BOUNDS FOR DISKS IN CONE-MANIFOLDS

Before stating our eigenvalue bound we shall briefly discuss the Laplace operator on cone-manifolds. Consider first the usual Laplace-Beltrami operator $\triangle$ on the domain

$$D(\triangle) = C_0^\infty(M_r),$$

where M is a compact cone-manifold. The corresponding quadratic form

$$I(\phi,\psi) = -\int_M \phi\Delta\psi = \int_M \nabla\phi\cdot\nabla\psi,$$

is defined for $\phi,\psi \in D(I) = C_0^\infty(M_r)$. The domain $D(I)$ can be completed using the inner product

$$(\phi,\psi)_1 = \int_M \phi\psi + I(\phi,\psi),$$

to a Hilbert space denoted $H^1(M)$ and I extends to a quadratic form $\widehat{I}$ on $H^1(M)$. Since $-\Delta$ is positive and symmetric, $\widehat{I}$ is the quadratic form of a unique self-adjoint extension $\widehat{\Delta}$ of Δ with domain $D\left(\widehat{\Delta}\right) \subset H^1(M)$. In fact,

$$D\left(\widehat{\Delta}\right) = \{u \in H^1(M) : (\Delta u)_{distr} \in L^2(M)\},$$

where $(\Delta u)_{distr}$ is the distribution

$$\phi \mapsto \int_M u\Delta\phi,\ \phi \in C_0^\infty(M_r).$$

The operator $\widehat{\Delta}$ is the so-called Friedrichs extension of Δ (compare [**17**, Theorem X.23]), and is defined for $\phi \in D\left(\widehat{\Delta}\right)$ by

$$\int_M \left(\widehat{\Delta}\psi\right)\phi = -I(\psi,\phi),$$

for all $\phi \in H^1(M)$. Since M is assumed compact one gets the usual direct sum decomposition of $L^2(M)$ into eigenspaces of $\widehat{\Delta}$. The eigenfunctions are furthermore, by standard elliptic theory, smooth on M_r and, as we have seen above, lie in $H^1(M)$. At the end of this paper, we sketch a proof that the eigenfunctions are continuous across Σ.

For compact cone-manifolds the first eigenvalue is, of course, zero and we shall denote the sequence of eigenvalues by

$$0 = \lambda_0(M) < \lambda_1(M) \leqslant \lambda_2(M) \leqslant \dots$$

We will sometimes use the notation $\lambda(M)$ instead of $\lambda_1(M)$.

We will also consider the Dirichlet problem for the Laplace operator on bounded domains Ω in cone-manifolds. We will follow the same construction as above, completing the space $C_0^\infty(\Omega - \Sigma)$ using the inner product

$$(\phi,\psi)_1 = \int_M \phi\psi + \int_M \nabla\phi\cdot\nabla\psi,$$

to a Hilbert space denoted $H_0^1(\Omega)$. The eigenvalues of a domain Ω will be denoted

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leqslant \dots ,$$

and the notation $\lambda(\Omega)$ will often be used in place of $\lambda_1(\Omega)$.

By the variational characterisation of eigenvalues,

$$\lambda(\Omega) = \inf_{\substack{f \in H_0^1(\Omega) \\ f \neq 0}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}.$$

The lemmas from section two then allow us to use the techniques of Cheng in [4] to prove the following generalisation of his Theorem 1.1.

THEOREM 1. *Let M be an n-dimensional cone manifold with Ricci curvature bounded below by $K(n-1)$, for some real number K. Then for any $\delta > 0$, $p \in M$, we have*

$$\lambda(B(p;\delta)) \leqslant \lambda(B_K(\delta)),$$

where $\lambda(B_K(\delta))$ is the first Dirichlet eigenvalue of a disk of radius δ in a simply connected space form of curvature K. Equality occurs if and only if $B(p;\delta)$ has constant curvature K and is isometric to the cone $\mathrm{Cone}_K(S_p;\delta)$.

PROOF: Let T be a (radial) eigenfunction of $\lambda(B_K(\delta))$ chosen so that

$$T|_{[0,\delta)} > 0, \text{ and } T'|_{(0,\delta]} < 0.$$

Define the function F by

$$F(q) = T(d(p,q))$$

for $q \in \overline{B(p;\delta)}$. Then F is clearly continuous and for $(r,v) \in [0,\infty) \times S_p$, satisfying $rv \in \mathcal{D}_p$,

$$|(\nabla F)(exp(rv))| = |T'(r)|,$$

so $|\nabla F|$ is bounded on $B(p;\delta)$. Now, ∇F is continuous except possibly on

$$\partial D_p \cap B(p;\delta),$$

which is a set of measure zero. Since $F|_{\partial B(p;\delta)} = 0$ by construction and Σ is of codimension 2, a straightforward argument (compare [8, p.75]) shows that $F \in H_0^1(B(p;\delta))$.

Now let

$$b(v) = \min(c(v), \delta),$$

where $c(v) = \sup\{r \mid rv \in \mathcal{D}_p\}$ as in Lemma 6. Then, using the notation from the end of section two,

$$\|\nabla F\|^2 = \int_{S_p} d\mu_p(v) \int_0^{b(v)} (T')^2 \sqrt{g}(r,v)\, dr,$$

and

$$\|F\|^2 = \int_{S_p} d\mu_p(v) \int_0^{b(v)} T^2 \sqrt{g}(r,v)\, dr.$$

To obtain the desired eigenvalue bound, it is therefore enough to show

$$\int_0^{b(v)} (T')^2 \sqrt{g}(r,v)\, dr \leqslant \lambda(B_K(\delta)) \int_0^{b(v)} T^2 \sqrt{g}(r,v)\, dr,$$

for almost all $v \in S_p$. Now, with $'$ denoting differentiation with respect to r,

$$\begin{aligned}
\int_0^{b(v)} (T')^2 \sqrt{g}(r,v)\, dr &= TT' \sqrt{g}(r,v)\Big|_0^{b(v)} - \int_0^{b(v)} T(T' \sqrt{g}(r,v))'\, dr \\
&= TT'(b(v))\sqrt{g}(b(v),v) - \int_0^{b(v)} T(T' \sqrt{g}(r,v))'\, dr \\
&\leqslant - \int_0^{b(v)} T\left(T'' + T' \frac{\sqrt{g}\,'(r,v)}{\sqrt{g}(r,v)}\right) \sqrt{g}(r,v)\, dr \\
&\leqslant - \int_0^{b(v)} T\left(T'' + (n-1)\frac{c_K}{s_K} T'\right) \sqrt{g}(r,v)\, dr \\
&= \lambda(B_K(\delta)) \int_0^{b(v)} T^2 \sqrt{g}(r,v)\, dr,
\end{aligned}$$

where the first inequality used the properties of T stated in the beginning of the proof. The second inequality follows from Bishop's comparison theorem [2] for cone-manifolds, as discussed in section two of this paper.

In the case of equality we must therefore have $b(v) = \delta$ almost everywhere and equality in Bishop's theorem for almost all v. It follows that the sectional curvatures are all equal to K in $M_r \cap B(p;\delta)$, and from Lemma 7 of section two we conclude that $B(p;\delta)$ is the cone $\mathrm{Cone}_K(S_p;\delta)$. ∎

REMARKS. Following Cheng in [4, Theorem 2.1], we can, using the max-min principle, bound the jth non-zero eigenvalue, $\lambda_j(M)$, of a compact cone-manifold M with Ricci curvature as in Theorem 1 by

$$\lambda_j(M) \leqslant \lambda(B_K(d(M)/2j)), \tag{1}$$

where $d(M)$ is the diameter of M. From (1) one then obtains bounds for the jth eigenvalue of a compact n-dimensional cone-manifold with non-negative Ricci curvature (compare with [4, Corollary 2.2]),

$$\lambda_j(M) \leqslant \frac{2j^2 n(n+4)}{(d(M))^2}.$$

If the Ricci curvature is bounded below by $(n-1)(-K)$ and the dimension n satisfies $n = 2(m+1)$ for some non-negative integer m, we have

$$\lambda_j(M) \leqslant \frac{(2m+1)^2 K}{4} + \frac{4j^2(1+2^m)^2\pi^2}{(d(M))^2},$$

and if $n = 2m+3$, for some non-negative integer m,

$$\lambda_j(M) \leqslant \frac{(2m+2)^2 K}{4} + \frac{4j^2(1+2^{2m})^2(1+\pi^2)}{(d(M))^2}.$$

Estimate (1) does not, as we have seen above, give bounds that are asymptotically like Weyl's formula. Instead (compare Gromov [10]) we can argue as follows. Let M be an compact n-dimensional cone-manifold with Ricci curvature bounded below by $(n-1)K$, and let $N(\varepsilon)$ be the maximal number of disjoint geodesic disks in M having radius $\varepsilon > 0$. From the discussion in section two, we can apply Bishop's comparison theorem to cone-manifolds with Ricci curvature bounded from below to conclude that

$$N(\varepsilon) \geqslant \frac{V(M)}{V(B_K(2\varepsilon))},$$

where $B_K(2\varepsilon)$ is a ball of radius 2ε in a space-form of curvature K and $V()$ denotes volume. Applying Theorem 1 and using a max-min argument, we have

$$\lambda_{[V(M)/V(B_K(2\varepsilon))]} \leqslant \lambda(B_K(\varepsilon)),$$

where [ ] denotes taking the integer part. Using the fact that for small ε,

$$\lambda(B_K(\varepsilon)) \sim c_D \varepsilon^2,$$

where c_D is the first eigenvalue of the unit disk in $\mathbf{R}^n$, we can conclude that for j sufficiently large

$$\lambda_j(M) \leqslant c(n,K)\left(\frac{j}{V(M)}\right)^{2/n},$$

in asymptotic agreement with Weyl's formula. These estimates can be refined further as in [16] to obtain bounds of the form

$$\lambda_j(M) \leqslant c_1\left(\frac{j+1}{V(M)}\right)^{2/n} + c_2,$$

where c_1 depends on K, $d(M)$ and n, and c_2 depends on K and n.

4. The Lévy-Gromov isoperimetric inequality and lower bounds for eigenvalues

We have the following generalisation of the Lévy-Gromov inequality [**10**] to cone-manifolds.

Theorem 2. *Let M be a compact n-dimensional cone-manifold whose Ricci curvature satisfies*

$$Ric \geqslant (n-1)K,$$

where $K > 0$ is a constant. Let

$$\beta = \frac{V(M)}{V(M_K)}$$

where $V()$ denotes volume and M_K denotes the simply-connected space form of constant curvature K. Given any $\Omega \subset M$ which is a finite disjoint union of normal domains in M, let D be the disk in M_K for which

$$V(\Omega) = \beta V(D).$$

Then, with $A()$ denoting $(n-1)$-dimensional area,

$$A(\partial\Omega) \geqslant \beta A(\partial D),$$

with equality if and only if Ω is isometric to a constant curvature cone $\mathrm{Cone}_K(S;\delta)$.

Proof: As in [**10**], (compare also [**8**, pp.322–325]), we consider hypersurfaces H which divide M into two open sets V_1 and V_2 of equal volume. More precisely, we consider disjoint open sets V_1 and V_2 with Lipschitz boundary $H = \partial V_1 = \partial V_2$. Let $\widehat{H}$ be the (possibly singular) hypersurface for which the $(n-1)$-dimensional Hausdorff measure $A(H)$ achieves its minimum. It follows from Almgren [**1**] that $\widehat{H}$ exists and has a tangent cone at each point.

Remark. It is probably possible to show that $\widehat{H}$ has singularities of codimension at least 7 as in the manifold case, but we don't need this result.

Consider a tangent cone $C = T\widehat{H}_p$ to $\widehat{H}$ at a point $p \in M$ (possibly $p \in \Sigma$). Then C has mean curvature zero at all non-singular points and is area minimising in the Euclidean cone TM_p.

Claim. *Let C be a codimension-one cone contained in a Euclidean cone E. If all points of C are at an angle $< \pi/2$ from a ray ρ in E then C is not an area minimising hypersurface in the cone E.*

Proof: Let S be the set of points at distance 1 from the conepoint x of E. Let pB denote the cone with base $B = C \cap S$ and cone point p on ρ. In particular, xB

is the part of C inside S. Since every point of B lies within an angle of $\pi/2$ from ρ, each distance $d(p,y)$, with $y \in B$ is decreased if p is moved along ρ away from x. Then, $A(pB) < A(xB)$, hence C is not area minimising. □

We will say that $\widehat{H}$ is non-singular at x if there are exactly two (opposite) normal directions to $\widehat{H}$ at x. By Lemma 2 of section two, this is equivalent to requiring that the pair $\left(TM_x, T\widehat{H}_x\right)$ is isometric to the cone on a pair $(\mathrm{Susp}(S), S)$.

Let ν denote the (non-singular) normal bundle to $\widehat{H}$ in M consisting of normal vectors v to $\widehat{H}$ at non-singular points of $\widehat{H}$ and let $\nu_0 \subset \nu$ consist of normal vectors v such that the geodesic g_v is length minimising on $[0,1]$. Then there is a well defined exponential map $exp : \nu_0 \to M$.

LEMMA. *The exponential map $exp : \nu_0 \to M$ is onto.*

PROOF: This follows by essentially the same argument as in Gromov [10]. Given any point $x \in M - \widehat{H}$, suppose a shortest path γ from x to $\widehat{H}$ meets $\widehat{H}$ at y. Then every tangent vector to $\widehat{H}$ at y makes an angle at least $\pi/2$ with γ. Since the unit tangent sphere S_x has diameter $\leqslant \pi$ by Lemma 2, it follows that every tangent vector to $\widehat{H}$ makes an angle $\leqslant \pi/2$ with a ray in the direction furthest from γ. By the claim above and Lemma 2, it follows that that S_y is a suspension (with the direction of γ as a suspension point) and γ is perpendicular to $T\widehat{H}$ at y. Hence y is a non-singular point of $\widehat{H}$, and if x is in Σ then γ lies in Σ. □

It also follows easily that the comparison theorems of E. Heintze and H. Karcher [12], apply to $exp : \nu_0 \to M$. The rest of the proof is now as in [10], compare also [8, pp.322–325]. The argument for the case of equality is similar to that given in the proof of Theorem 1. □

COROLLARY 1. *Let M, K, Ω and D be as in the statement of Theorem 2. Then*

$$\lambda(\Omega) \geqslant \lambda(D),$$

with equality if and only if Ω is isometric to a cone of constant curvature K.

PROOF: Use the modification of the classical arguments of Faber [9] and Krahn [13], given by Berard and Meyer for their [3, Theorem 5]. □

COROLLARY 2. *Let M be a cone-manifold with*

$$Ric \geqslant (n-1)K.$$

Then

$$\lambda(M) \geqslant nK,$$

and $\lambda(M) = nK$ if and only if M has constant curvature K and is the suspension $\mathrm{Susp}_K(S)$ *of a $(n-1)$-dimensional spherical cone-manifold S.*

REMARK. This generalises to cone-manifolds the classical theorem of Lichnerowicz [**14**], and the characterisation of equality due to Obata [**16**].

PROOF: Let ϕ be an eigenfunction of the first non-zero eigenvalue of M. It is shown in the appendix that ϕ is continuous on all of M. Let Ω_1 be the smaller of the two nodal domains of ϕ and let D be the disk in M_K for which

$$V(\Omega_1) = \beta V(D),$$

where $\beta = V(M)/V(M_K)$. Then D is contained in a hemisphere of M_K, and

$$\lambda(M) = \lambda(\Omega_1) \geqslant \lambda(D) \geqslant nK.$$

The case of equality then follows as in the proof of Theorem 1. □

We also have the following result, generalising the classical Bonnet-Myers comparison theorem, and the characterisation of equality, due to Cheng [**4**, Theorem 3.1].

THEOREM 3. *Let M be a cone-manifold with*

$$Ric \geqslant (n-1)K,$$

where $K > 0$. Then the diameter of M satisfies

$$d(M) \leqslant \frac{\pi}{\sqrt{K}},$$

and $d(M) = \pi/\sqrt{K}$ if and only if M has constant curvature K and is the suspension of a $(n-1)$-dimensional spherical cone-manifold.

PROOF: The diameter bound follows from the proof of the classical Bonnet-Myers theorem together with Lemma 3 from section two. The characterisation of the case of equality follows from the characterisation of equality in Theorem 1 and the max-min argument used by Cheng in [**4**]. □

REMARK. The isoperimetric inequality of Theorem 2 can also be utilised to give lower bounds for the higher eigenvalues of cone-manifolds with positive Ricci curvature. Let M_K be as in the statement of the theorem, and divide M into two open subsets Ω_1 and Ω_2 with $\partial\Omega_1 = \partial\Omega_2$. This gives a division of M_K into two disks D_1 and D_2 with $\partial D_1 = \partial D_2$ and $V(\Omega_i) = \beta V(D_i)$ for $i = 1, 2$. We then note that

$$\begin{aligned}\frac{A(\partial\Omega_1)^n}{\min(V(\Omega_1), V(\Omega_2))^{n-1}} &= \frac{A(\partial\Omega_1)^n}{\min(\beta V(D_1), \beta V(D_2))^{n-1}} \\ &\geqslant \frac{(\beta A(\partial D_1))^n}{(\beta \min(V(D_1), V(D_2)))^{n-1}} = \frac{\beta A(\partial D_1)^n}{\min(V(D_1), V(D_2))^{n-1}}.\end{aligned}$$

Hence if $i(M)$ and $i(M_K)$ are the isoperimetric constants of M and M_K respectively, then

$$i(M) \geqslant \beta i(M_K).$$

Following arguments of Cheng and Li in [5] one can then show that

$$\begin{aligned}\lambda_j(M) &\geqslant c(n)\left(\frac{i(M)j}{V(M)}\right)^{2/n} \\ &\geqslant c(n)\left(\frac{\beta i(M_K)j}{\beta V(M_K)}\right)^{2/n} \\ &= c(n)\left(\frac{i(M_K)j}{V(M_K)}\right)^{2/n} = \frac{c'(n)}{K}\cdot j^{2/n},\end{aligned}$$

where $c(n)$ and $c'(n)$ are constants depending only on n.

Appendix. Regularity of the eigenfunctions

As we noted in section three, the eigenfunctions for a cone-manifold M are smooth on M_r and belong to $H^1(M)$. We now outline an argument showing that the eigenfunctions also are continuous at the singular locus Σ. By blowing up the metric at a point $p \in \Sigma$ by a sequence of rescaling factors $\{r_i\}$ tending to infinity, we obtain a sequence of cone-manifolds which converge smoothly outside Σ to the Euclidean cone $\mathrm{Cone}_0(S_p;\infty)$. An eigenfunction u on M with eigenvalue λ naturally gives rise to a sequence of functions u_i defined, for i large enough, on compact subsets of $\mathrm{Cone}_0(S_p;\infty)$. A gradient estimate of Cheng and Yau [6, Theorem 6] assures us that $\{u_i\}$ has a subsequence converging to a harmonic function on the regular part of $\mathrm{Cone}_0(S_p;\infty)$. By a separation of variables argument, one shows that if no such limit extends continuously across the cone point $p \in \mathrm{Cone}_0(S_p;\infty)$, then the eigenfunction u would not lie in $H^1(M)$, contradicting earlier remarks on the eigenfunctions. Thus, some subsequential limit extends continuously across p. If this limit is unique and constant, we are done. If not, then for every large constant L, we can find an $\varepsilon > 0$ and a $\delta > 0$ so that if we replace u inside $B(p;\delta)$ by this continuous extension, $\int_M |\nabla u|^2$ is reduced by $L\varepsilon$ and $\int_M u^2$ is changed by at most ε. Let u_δ be this modification of u. We can then smoothly change u_δ outside $B(p;\delta)\cup\Sigma$ to obtain a function orthogonal to all eigenfunctions with eigenvalue strictly less than λ, changing $\int_M u_\delta^2$ by at most $C_1\varepsilon$ and $\int_M |\nabla u_\delta|^2$ by at most $C_2\varepsilon$, where C_1 and C_2 do not depend on ε. Choosing L large enough, we therefore obtain a contradiction to the fact that u minimises the Rayleigh-Ritz quotient among all functions orthogonal to the eigenfunctions with eigenvalue strictly less than λ.

References

[1] F. Almgren, 'Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints', *Mem. Amer. Math. Soc.* **165** (1976).

[2] R. Bishop and R. Crittenden, *Geometry of manifolds* (Academic Press, New York, 1964).

[3] P. Bérard and D. Meyer, 'Inegalités isopérimetrique et applications', *Ann. Sci. École. Norm. Sup. Paris* **15** (1982), 513–542.

[4] S.-Y. Cheng, 'Eigenvalue comparison theorems and its geometric applications', *Math. Z.* **143** (1975), 289–297.

[5] S.-Y. Cheng and P. Li, 'Heat kernel estimates and lower bounds of eigenvalues', *Comment. Math. Helv.* **56** (1981), 327–338.

[6] S.-Y. Cheng and S.-T. Yau, 'Differential equations on Riemannian manifolds and their geometric applications', *Comm. Pure Appl. Math* **28** (1975), 333–354.

[7] J. Cheeger, 'A vanishing theorem for piecewise constant curvature spaces', in *Lecture Notes in Math.* **1201** (Springer-Verlag, Berlin, Heidelberg, New York, 1986), pp. 33–40,.

[8] I. Chavel, *Eigenvalues in Riemannian geometry* (Academic Press, New York, 1984).

[9] C. Faber, 'Beweis, dass unter allen homogenen Membrane von gleiche Fläche und gleicher Spannung die kreisförmige die tiefsten Grundton gibt', *Sitzungsber.-Bayer Akad. Wiss., Math.-Phys. Munich* (1923), 169–172.

[10] M. Gromov, 'Paul Levy's isoperimetric inequality', (preprint).

[11] M. Gromov, *Structures métriques sur les variétés riemanniennes* (Cedic/Fernand Nathan, Paris, 1981).

[12] E. Heintze and R. Karcher, 'A general comparison theorem with applications to volume estimates for submanifolds', *Ann. Sci. École. Norm. Super. Paris* **11** (1978), 451–470.

[13] E. Krahn, 'Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises', *Math. Ann.* **94** (1925), 97–100.

[14] A. Lichnerowicz, *Geometrie des Groupes des Transformations* (Dunod, Paris, 1958).

[15] P. Li and S.-T. Yau, 'Estimates of eigenvalues of a compact Riemannian manifold', in *Proc. Symp. Pure Math.* **36** (Amer. Math. Soc., Providence, Rhode Island, 1980), pp. 205–240.

[16] M. Obata, 'Certain conditions for a Riemannian manifold to be isometric with a sphere', *J. Math. Soc. Japan* **14** (1962), 333–340.

[17] M. Reed and B. Simon, *Methods of modern mathematical physics II* (Academic Press, 1975).

[18] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes (Princeton University Math. Dept., 1978).

[19] W. Thurston, 'Geometric structures on 3–manifolds with symmetry', (preprint, Princeton University, 1982).

Department of Mathematics
University of Melbourne
Parkville Vic 3052
Australia
cdh@mundoe.maths.mu.oz.au

Department of Mathematics
Uppsala University
Thunbergsvagen 3
S-75238 Uppsala
Sweden
johant@bellatrix.tdb.uu.se