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ON SUPERSOLUBLE GROUPS OF WIELANDT LENGTH TWO

ASIF ALI

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Abstract

This paper gives a characterisation of finite supersoluble groups of Wielandt length two of order coprime to six.

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1. Introduction

This paper treats finite supersoluble groups of Wielandt length two and provides a characterisation of such groups whose order is coprime to six.

Recall that the Wielandt subgroup $\omega(G)$ of a group G is the subgroup of elements of G normalising each subnormal subgroup of G. It is non-trivial in every finite, non-trivial group ([9]). A group has Wielandt length one if $\omega(G) = G$. But if $\omega(G) \neq G$ and $\omega(G/\omega(G)) = G/\omega(G)$ then G is said to have Wielandt length two. We will denote by $\mathscr{W}_2$ the class of all finite groups of Wielandt length at most two.

The main results of this paper can be thought of as a generalisation of results of [6] for p-groups of Wielandt length two and what we need from [6] is summarised and extended in section two. One of our main results is that a non-nilpotent supersoluble group of odd order and Wielandt length 2 splits over its nilpotent residual (Theorem 3.6). This result is a consequence of the technical result (Theorem 3.5) which is also crucial in the characterisation of these groups in Section 4. The characterisation of supersoluble groups of Wielandt length two and order coprime to six essentially comes from analysing the properties that the splitting theorem gives us and can be summarised by saying that we can find sufficient information about the structure of the nilpotent residual and a complement, as well as the action of a complement on the residual to ensure that a group with those properties will be a supersoluble group of Wielandt length two and exponent coprime to six. However the details are rather technical and we show how the group can be built up one prime at a time. Definition 4.1 extracts the necessary features of a normal Sylow subgroup, while Definition 4.3 gives the way in which a complement of a normal Sylow subgroup must act on the normal Sylow subgroup. Then Theorem 4.6 and Theorem 4.7 show that supersoluble groups of Wielandt length two and order coprime to six are characterised as groups with the structure given by these definitions. The restriction to groups of order coprime to six comes from the fact that p-groups of Wielandt length two are more difficult to classify for the primes two and three. Indeed 2-groups of Wielandt length two have not yet been classified.

2. Preliminary results

For convenience, we state some results we will use frequently in what follows and will use them without further reference.

THEOREM 2.1 ([3, A.1.3]). *Let U, V and W be subgroups of a group G with $V \subseteq U$. Then $U \cap (VW) = V(U \cap W)$.*

THEOREM 2.2 ([3, Proposition A.12.5]). *If Q is a π'-group of operators of a π-group P, then*

(1) $P = [P, Q]C_P(Q)$.
(2) $[P, Q] = [P, Q, nQ]$ *for all* $n \geq 1$.
(3) *If P is abelian, then* $P = [P, Q] \times C_P(Q)$.

THEOREM 2.3 ([1, Theorem 2.4]). *Let $G = BA$ be a semidirect product of subgroups A and B of coprime order with A nilpotent and normal. If P is the set of those elements of $\omega(B)$ which act by conjugation as power automorphisms on A, then $\omega(G) = P\omega(A)$.*

For the applications we make in later sections we have found it useful to complete some of the detail omitted in Ormerod's Theorem [6]. Our account will be directed to the application we make in the later sections, where the groups will have order coprime to six: the 3-groups in Ormerod's work will therefore be omitted here.

We begin with the presentation of the following group

$$H = \left\langle x, y : [x, y, x] = x^{p^r}, [x, y, y] = y^{p^r}, x^{p^{2r}} = y^{p^{2r}} = [x, y]^{p^r} = 1 \right\rangle. \tag{2.1}$$

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It is easy to see, by Dyck's Theorem, that there is a homomorphism from Ormerod's group $H(p,r)$ onto H. Since by its construction $|H| = p^{5r} = |H(p,r)|$, H is isomorphic to $H(p,r)$.

The properties we require of $H(p,r)$ are summed up in the following theorem.

THEOREM 2.4. *$H = H(p,r)$ is a regular group of order p^{5r} and nilpotency class three satisfying*

(1) $H/H' \cong C_{p^r} \times C_{p^r}$;
(2) $Z_2(H)$ *has exponent* p^r;
(3) $H/\gamma_3(H)$ *has exponent* p^r.

The proofs follow easily from the relations in (2.1).

Now let $L_n(p^r)$ be the free group of rank n in the variety of all groups of nilpotency class at most two and exponent dividing p^r. Next set $G_n(p^r) = H(p,r) *_{N_2} L_n(p^r)$ for $n \geq 1$, $r \geq 1$, the second nilpotent product of $H(p,r)$ and $L_n(p^r)$ (the second nilpotent product of groups A and B is defined to be E/N, where $E = A * B$ is the free product of A and B and $N = [[A,B]^G, G, G]$; see for example [5, Section 6.4].

For a p-group A, define $e(A)$ to be the positive number such that $p^{e(A)}$ is the exponent of $A/\gamma_3(A)$ and note that $e(H(p,r)) = r$.

The next result gives the classification of $\mathscr{W}_2$-groups of [6] and is essential in our classification.

THEOREM 2.5 ([6, Theorem A]). *Let $p > 3$ be a prime. For all $n \geq 1$, $r \geq 1$, $G_n(p^r) \in \mathscr{W}_2$.*

Conversely, if $G \in \mathscr{W}_2$ is a p-group with $e(G) = r$ and if G can be generated by $n+2$ elements, then G is a homomorphic image of $G_n(p^r)$.

For convenience, we write $G = G_n(p^r)$, $H = H(p,r)$ and $L = L_n(p^r)$ in what follows, p, n and r being understood.

Now we state results, which give connections between two numerical invariants of p-groups in $\mathscr{W}_2$ and are used later in the article.

LEMMA 2.6. (1) $Z_2(G) = Z_2(H)L[L,H]$.
(2) $e(G) = e(H)$.
(3) *Let A be a p-group, with $p > 3$, and Wielandt length two. Then $Z_2(A)$ has exponent dividing $p^{e(A)}$.*

PROOF. (1) By the definition of second nilpotent product, we have $G = HL[L,H]$. Also $Z_2(H)L[L,H] \subseteq Z_2(G)$; and $Z_2(G) \cap H \subseteq Z_2(H)$. Therefore,

$$Z_2(H)L[L,H] \subseteq Z_2(G) \subseteq L[L,H](Z_2(G) \cap H) \subseteq L[L,H]Z_2(H),$$

which gives the result claimed.

(2) This is because every commutator of weight three in G is a power of one of the forms $[h_1, h_2, h_3]$, $[h_1, l_1, h_2]$, $[h_1, l_1, l_2]$ or $[l_1, l_2, l_3]$, where $h_i \in H$, $l_i \in L$ $(1 \leq i \leq 3)$. Here we use the Jacobi identity which holds in a metabelian group. However, all but the first of these are necessarily trivial in G. Hence $\gamma_3(G) = \gamma_3(H)$. It follows that $G/\gamma_3(G) \cong H/\gamma_3(H) *_{N_2} L$. Both factors on the right have exponent dividing $p^{e(H)}$ and $H/\gamma_3(H)$ has exactly this exponent, so, by regularity, $G/\gamma_3(G)$ has the exponent exactly $p^{e(H)}$. That is $e(G) = e(H)$, as required.

(3) If A has nilpotency class at most two, there is nothing to prove. So suppose that A has nilpotency class three, and that it can be generated by $n+2$ elements. Then, for some $N \lhd G = G_n(p^{e(A)})$, $G/N \cong A$.

Suppose that $g \in G$ and $gN \in Z_2(G/N)$. Since $Z_2(G)N/N \subseteq Z_2(G/N)$ and $Z_2(G)$ has exponent $p^{e(A)}$ by Theorem 2.4 and (1) above, we may suppose that $g \notin Z_2(G)$ and therefore that $g \in H$ but $g \notin H'$, since G is regular.

Moreover we may suppose that $g = x^m y^n$ for some integers m, n. Then for $r = e(A)$:

$$x^{mp^r} = [x, y, x]^m = [g, y, x] \in N$$

and

$$y^{np^r} = [x, y, y]^n = [y, x, y]^{-n} = [g, x, y]^{-1} \in N.$$

From this we see, using [4, Satz 3.9.4], that $g^{p^r} = (x^m y^n)^{p^r} = x^{mp^r} y^{np^r} \in N$.

Hence $Z_2(A)$ has exponent dividing $p^r = p^{e(A)}$. □

LEMMA 2.7. *Let $p > 3$ be a prime, let G_1 be a p-group of Wielandt length two and nilpotency class three and G_2 a p-group of nilpotency class at most two. Let $W = G_1 *_{N_2} G_2$. If $N \subseteq [G_1, G_2]$ is a normal subgroup of W, then $W/N \in \mathscr{W}_2$ if and only if the exponent of G_2 divides $p^{e(G_1)}$.*

PROOF. First suppose that G_2 has exponent dividing p^r, where $r = e(G_1)$. Also suppose that G_2 is generated by m elements. By Theorem 2.5, for some positive integer n, there is an onto homomorphism $\theta : G_n(p^r) \to G_1$. It follows that θ may be extended to an onto homomorphism $G_{m+n}(p^r) \to G_1 *_{N_2} G_2$, so $W \in \mathscr{W}_2$ by Theorem 2.5 and hence $W/N \in \mathscr{W}_2$.

Conversely suppose that $W/N \in \mathscr{W}_2$. Then

$$G_2 \cong G_2N/N \subseteq Z_2(W)N/N \subseteq Z_2(W/N)$$

and so the exponent of G_2 divides $p^{e(W/N)}$ by Lemma 2.6'(3). Therefore

$$(W/N)/\gamma_3(W/N) \cong W/\gamma_3(W)N \cong (W/\gamma_3(W))/(\gamma_3(W)N/\gamma_3(W))$$

so $e(W/N) \leq e(W) = e(G_1)$. Hence the exponent of G_2 divides $p^{e(G_1)}$ as required. □

3. Some basic results

The main result of this section is Theorem 3.6 which says that in a supersoluble group of odd order and Wielandt length two, the nilpotent residual is complemented. To prove this fact we need the following results.

LEMMA 3.1. *Let A be a normal Sylow p-subgroup of a non-nilpotent group G and B be a Hall p'-subgroup of G.*

If N is the nilpotent residual of G and H is the nilpotent residual of B, then $N = H[B, A]$.

PROOF. Since A is a normal Sylow p-subgroup of G, we have $[B, A] = [B, A, jB]$ for all $j \geq 1$ and so $[B, A] \subseteq N$. Since $[B, A]$ is normal in G, we have

$$G/[B, A] = (B[B, A]/[B, A]) \times (A/[B, A])$$

and so $N/[B, A]$ is the nilpotent residual of $B[B, A]/[B, A]$.

Clearly, $B[B, A]/H[B, A]$ is nilpotent and so $N \subseteq H[B, A]$. On the other hand, the nilpotent residual of $B[B, A]/[B, A]$ is isomorphic to the nilpotent residual of $B/(B \cap [B, A]) \cong B$. Therefore $N/[B, A] \cong H$ and hence $N = H[B, A]$. □

LEMMA 3.2. *Let A be a normal Sylow p-subgroup of a non-nilpotent soluble group G of odd order and Wielandt length two and B be a Hall p'-subgroup of G. If B acts non-trivially on $A/\omega(A)$, then A has nilpotency class at most two.*

PROOF. Since A is normal, $A \in \mathscr{W}_2$ and therefore $A/\omega(A)$ is abelian. Also, it is easy to see that A has nilpotency class at most three.

Let us suppose, contrary to the claim of the lemma that A has nilpotency class exactly three. It follows from Theorem 2.5 that A has elements a_1, a_2 for which

$$[a_1, a_2, a_1] = a_1^{p^r},$$

where $r = e(A)$. Now since $G/\omega(G)$ is a T-group, $B\omega(G)/\omega(G)$ acts as a power automorphism by conjugation on $A/\omega(A)$ (by [7, 13.4.4 and 13.4.6]) and it is immediate from [2, Theorem 5.3.1] that these power automorphisms are universal (that is, they map each element of $A/\omega(A)$ to the same power). Suppose $b \in B$ induces a non-trivial universal power automorphism by conjugation on $A/\omega(A)$. Then there exists an integer m which is not divisible by p, for which $a_1^b = a_1^m c$ and $a_2^b = a_2^m d$ where $c, d \in \omega(A) \subseteq Z_2(A)$, by [8]. Then, using the regularity of A and Lemma 2.6 (3):

$$\begin{aligned}[a_1, a_2, a_1]^{m^3} &= [a_1^m c, a_2^m d, a_1^m c] = [a_1, a_2, a_3]^b = (a_1^{p^r})^b = (a_1^m c)^{p^r} \\ &= (a_1^m)^{p^r} c^{p^r} [c, a_1^m]^{-p^r(p^r-1)/2} = (a_1^{p^r})^m = [a_1, a_2, a_1]^m.\end{aligned}$$

It follows that $m^3 - m \equiv 0 \pmod{p}$. Hence $m^2 \equiv 1 \pmod{p}$ as $m \not\equiv 0 \pmod{p}$. This means that $b^2 \in C_B((A/\omega(A)/(\Phi(A/\omega(A)))$ whence $b^2 \in C_B(A/\omega(A))$ by [3, Theorem A.9.14]. Therefore $b \in C_B(A/\omega(A))$ because $|B|$ is odd. This contradicts our choice of b. We conclude therefore, that A has nilpotency class at most two, as claimed. □

The following lemmas give useful information about (non-nilpotent) supersoluble groups. The statements involve increasingly more precise hypothesis on the groups involved.

LEMMA 3.3. *Let A be a normal Sylow p-subgroup of a non-nilpotent supersoluble group G of odd order and Wielandt length two and B be a Hall p'-subgroup of G. Then either $[B, A] \subseteq \omega(A)$ or $C_A(B) \subseteq \omega(A)$.*

PROOF. As in Lemma 3.2 B acts as a group of universal power automorphism on $A/\omega(A)$. Therefore, either $[b, A] \subseteq \omega(A)$ for all $b \in B$ or, for some $b \in B$, $C_{A/\omega(A)}(b) = 1$. Therefore, either $[B, A] \subseteq \omega(A)$ or $C_A(B) \subseteq \omega(A)$. □

The next lemma gives more information about the semidirect product of subgroups of coprime order of a group.

LEMMA 3.4. *Let A be a normal Sylow subgroup of a supersoluble group G of odd order and B a Hall p'-subgroup of G. Suppose A has nilpotency class exactly two, B acts as a group of universal power automorphisms on $A/\omega(A)$ and $[B, A] = A$. Then $C_A(B) = 1$.*

PROOF. By [3, Theorem A.11.6], A/A' has a direct decomposition

$$A/A' = A_1/A' \times \cdots \times A_s/A'$$

into B-admissible subgroups A_i/A' with the following properties for each $i = 1, \ldots, s$:

(1) A_i/A' is indecomposable as a B-module.
(2) $(A_i/A')/\Phi(A_i/A')$ is an irreducible B-module.

Since A is supersoluble, $(A_i/A')/\Phi(A_i/A')$ is cyclic of prime order. Hence A_i/A' is cyclic by [7, 5.2.12]. Therefore $A_i/A' = \langle y_i A' \rangle$, for $1 \leq i \leq s$, is cyclic of prime power order. Let $b \in B$. Then we may write $y_i^b = y_i^{m_i^{(b)}} c_i$ for some $c_i \in A'$ and some integers $m_i^{(b)}$, for $1 \leq i \leq s$. Note that $m_i^{(b)} \not\equiv 0 \pmod{p}$, for $1 \leq i \leq s$. Since $A' \subseteq \Phi(A)$, it follows that $A = \langle y_1, y_2, \ldots, y_s \rangle$, by [7, 5.2.12]. Also note that for each i there is at least one $b \in B$ such that $m_i^{(b)} \not\equiv 1 \pmod{p}$: otherwise $[B, A] \neq A$. We aim now to show that

$$A' = \langle [y_i, y_j] : y_i \notin \omega(A), y_j \notin \omega(A) \rangle.$$

To this end suppose that, for some j, $y_j \in \omega(A)$. Then choose $b \in B$ such that $m_j^{(b)} \not\equiv 1 \pmod p$. For simplicity we write $m_i = m_i^{(b)}$, for $1 \le i \le s$. By [6, Corollary 4.3] there is an integer n such that $[y_i, y_j] = y_i^n$ for $1 \le i \le s$. Note that p divides n, otherwise $[y_i, y_i] \ne 1$. Hence, since A has nilpotency class two,

$$\begin{aligned}[y_i, y_j]^{m_i m_j} &= [y_i, y_j]^b = (y_i^n)^b = (y_i^b)^n = (y_i^{m_i} c_i)^n \\ &= (y_i^n)^{m_i} c_i^n [c_i, y_i^{m_i}]^{n(n-1)/2} = [y_i, y_j]^{m_i} c_i^n [c_i, y_i^{m_i}]^{n(n-1)/2},\end{aligned}$$

whence $[y_i, y_j]^{m_i(m_j-1)} \in \Phi(A')$. Now $[y_i, y_j] \notin \Phi(A')$ would mean $m_i(m_j - 1) \equiv 0 \pmod p$ leading to $m_j \equiv 1 \pmod p$, a contradiction. Hence $[y_i, y_j] \in \Phi(A')$. It follows that A' is generated by the commutators $[y_i, y_j]$ where neither y_i nor y_j belongs to $\omega(A)$.

Finally, we are given that each $b \in B$ induces, by conjugation, a universal power automorphism on $A/\omega(A)$. That is, in particular, for some integer m, $m \equiv m_i^{(b)} \pmod p$ if $y_i \notin \omega(A)$. It follows that, for all such pairs i, j (when $y_i, y_j \notin \omega(A)$),

$$[y_i, y_j]^b = [y_i, y_j]^{m_i m_j} = [y_i, y_j]^{m^2} \pmod{\Phi(A')}.$$

Hence $[y_i, y_j]^b = [y_i, y_j] \pmod{\Phi(A')}$ if and only if $m^2 \equiv 1 \pmod p$ and that is if and only if $y_i^{b^2} = y_i \pmod{A'}$ whence if and only if $y_i^b = y_i \pmod{A'}$ since $(2, |b|) = 1$. This is a contradiction to our choice of b. Hence at least one $b \in B$ acts fixed point freely on $A'/\Phi(A')$ and so $[A', B] = A'$. Thus $C_{A'}(B) = 1$. Finally note that $C_{A/A'}(B) = 1$ since $A/A' = [B, A/A']$. Therefore $C_A(B) \subseteq C_A(B) \cap A' = C_{A'}(B) = 1$ as required. □

THEOREM 3.5. *Let A be a normal Sylow p-subgroup of a non-nilpotent supersoluble group G of odd order and Wielandt length two and B be a Hall p'-subgroup of G. Then* $[B, A] \cap C_A(B) = 1$.

PROOF. If A is abelian, the result is immediate. Therefore suppose that A is non-abelian. As in Lemma 3.2, we see that B acts as a group of universal power automorphisms on $A/\omega(A)$. By Lemma 3.3, either $[B, A] \subseteq \omega(A)$ or $C_A(B) \subseteq \omega(A)$. First suppose that $[B, A] \subseteq \omega(A)$. As $\omega(A)$ is abelian, we have that

$$\omega(A) = [\omega(A), B] \times C_{\omega(A)}(B).$$

But $[B, A, B] = [B, A]$ and $[B, \omega(A), B] = [B, \omega(A)]$. Now

$$[B, A] = [B, A, B] \subseteq [\omega(A), B] \subseteq [B, A],$$

so $[B, A] = [B, \omega(A)]$. As we know from above that $[B, \omega(A)] \cap C_{\omega(A)}(B) = 1$, therefore $[B, A] \cap C_{\omega(A)}(B) = 1$. But $C_{\omega(A)}(B) = \omega(A) \cap C_A(B)$. Therefore, $[B, A] \cap C_A(B) = 1$ (as $[B, A] \subseteq \omega(A)$), as required.

Now suppose that $C_A(B) \subseteq \omega(A)$ so that $[B, A/\omega(A)] = A/\omega(A)$. If $[B, A]$ is abelian, we have $C_{[B,A]}(B) = 1$ and hence $C_A(B) \cap [B, A] = 1$ as required.

We get the same result when $[B, A]$ is non-abelian because $[B, A]$ and B satisfy the hypothesis of Lemma 3.4: here we have relied on Lemmas 3.2 and 3.3. □

Now we use these results to prove the following theorem.

THEOREM 3.6. *Let N be the nilpotent residual of a supersoluble group G of odd order and Wielandt length two. Then N is complemented in G.*

PROOF. If G is nilpotent, there is nothing to prove. Therefore we suppose that G is non-nilpotent. Let A be the normal Sylow p-subgroup of G, where p is the largest prime dividing $|G|$ and B a Hall p'-subgroup of G so that $G = BA$. We can also write $G = B(C_A(B)[B, A])$. By induction on the order of G, if B is non-nilpotent then the nilpotent residual (say H) of B must be complemented in B. Let X be a complement of H so that $B = XH$. If B is nilpotent then $H = 1$ and $X = B$. By Lemma 3.1, we know that $N = H[B, A] \lhd G$. Let $Y = XC_A(B)$. Then $G = NY$ and

$$N \cap Y = H[B, A] \cap XC_A(B) = (H \cap X)([B, A] \cap C_A(B)).$$

But by Theorem 3.5 we have $C_A(B) \cap [B, A] = 1$. Therefore $N \cap Y = 1$. Thus we conclude that N is complemented in G. □

The following result gives a necessary and sufficient condition for the direct product of an abelian group and a T-group to be a T-group.

LEMMA 3.7. *Let G_1 be a T-group and B_1 be a complement of $\gamma_3(G_1)$ in G_1. If G_2 is abelian, then $G_1 \times G_2$ is a T-group if and only if $(|\gamma_3(G_1)|, |G_2|) = 1$ and $B_1 \times G_2$ is a Dedekind group.*

PROOF. The existence of B_1 is ensured by [7, 13.4.4]. The proof is a routine application of [7, 13.4.6 and 13.4.4]. □

The following lemma gives conditions for an abelian p-group G_1 (for a prime $p > 3$) acting as a group of power automorphisms on a p-group G_2 of nilpotency class at most two, to lie in the Wielandt subgroup of the semidirect product G_1G_2.

We use the standard notation $\Omega_r(G_2)$ to denote the subgroup of G_2 generated by elements of order p^r.

LEMMA 3.8. *Let G_1 be an abelian p-group (for $p > 3$) of exponent p^r and G_2 be a p-group of nilpotency class at most two. Let $\theta : G_1 \to \mathrm{Paut}(G_2)$ be a homomorphism and write $G = G_1G_2$ for the semidirect product of G_2 by G_1 under θ. Then $G_1 \subseteq \omega(G)$ if and only if G_1 centralises $\Omega_r(G_2)$.*

PROOF. We start by observing that since G_1 is abelian, we have $G' = G_2'[G_2, G_1]$. Also since $G_1 \subseteq \mathrm{Paut}(G_2)$, [2, Theorem 2.2.1] gives the following

$$[G_2, G_1] \subseteq Z(G_2)$$

and hence $G' \subseteq Z(G_2)$ (since G_2' is also contained in $Z(G_2)$).

Now suppose that $G_1 \subseteq \omega(G)$ then by [8], we have $G_1 \subseteq \omega(G) \subseteq Z_2(G)$ and we know from [7, 5.1.11(3)] that $Z_2(G)$ commutes with G'. Therefore G' centralises G_1 and hence $G' \subseteq Z(G)$ and so G has nilpotency class at most two.

Since G_1 is abelian of exponent p^r, it is generated by elements of order p^r. So it is sufficient to show that elements of order p^r of G_1 centralise $\Omega_r(G_2)$. Let g_1 be an element of G_1 such that $|g_1| = p^r$ and let $g_2 \in \Omega_r(G_2)$. Note that the exponent of $\Omega_r(G_2)$ divides p^r, since G_2 is regular. Now put $g = g_1 g_2$. For $x \in G_1$, there exists an integer m such that $g^x = g^m$ and so $(g_1 g_2)^x = g_1 g_2^x = (g_1 g_2)^m$. But as G is a regular p-group, G_1 acts as a group of universal power automorphisms on G by [2, Theorem 5.3.1]. So we have $g_2^x = g_2^m$. But using [4, Satz 3.9.4], we have

$$g_1 g_2^m = (g_1 g_2)^m = g_1^m g_2^m [g_2, g_1]^{-m(m-1)/2}.$$

As $G_1 \cap G_2 = 1$, we have $g_1^{m-1} = [g_2, g_1]^{m(m-1)/2} = 1$. This means that $m \equiv 1 \pmod{|g_1|}$ and therefore $m \equiv 1 \pmod{|g_2|}$ (as $|g_2|$ divides $|g_1|$). Thus we conclude that $g_2^x = g_2$ and so G_1 acts trivially on $\Omega_r(G_2)$.

Conversely suppose that G_1 centralises $\Omega_r(G_2)$. For any $g \in G$, there exist $g_1 \in G_1$ and $g_2 \in G_2$ such that $g = g_1 g_2$. Let $x \in G_1$. By hypothesis if $g_2 \in \Omega_r(G_2)$, then $g^x = g$. Suppose that g_2 does not belong to $\Omega_r(G_2)$ and let $|g_2| = p^s$ for $s > r$. But $g_2^{p^{s-r}}$ is an element of G_2 such that $|g_2^{p^{s-r}}| = p^r$ and hence belongs to $\Omega_r(G_2)$. This means that $(g_2^{p^{s-r}})^x = g_2^{p^{s-r}}$. But as G_2 is a regular p-group, G_1 acts as a group of universal power automorphisms on G_2 by [2, Theorem 5.3.1]. Therefore there exists a positive integer m such that $g_3^x = g_3^m$ for all $g_3 \in G_2$ and so $(g_2^{p^{s-r}})^x = g_2^{mp^{s-r}} = g_2^{p^{s-r}}$. Therefore $g_2^{p^{s-r}(m-1)} = 1$ and hence $(m-1)p^{s-r} = tp^s$ for some positive integer t. This means $m - 1 = tp^r$ and so $m = 1 + tp^r$. Hence $g^x = g_1^x g_2^x = g_1 g_2^x = g_1^m g_2^m$.

We claim that $g_1^m g_2^m = (g_1 g_2)^m$. By [4, Satz 3.9.4], we have

$$g_1^m g_2^m = (g_1 g_2)^m c_2^{\binom{m}{2}} c_3^{\binom{m}{3}} y,$$

where c_2 and c_3 are products of commutators with entries g_1 and g_2 of weight two and three respectively, and $y \in \gamma_4(G)$. But since $g_1^{m-1} = 1$, we immediately see from [4, Satz 3.10.6] that $c_2^{\binom{m}{2}} c_3^{\binom{m}{3}} = 1$, since $p \geq 5$. This means that $g_1^m g_2^m = (g_1 g_2)^m y$ and hence $(g\gamma_4(G))^x = g^m \gamma_4(G)$. Thus

$$G_1 \gamma_4(G)/\gamma_4(G) \subseteq \omega(G/\gamma_4(G)).$$

Since $G/\gamma_4(G)$ has a factorisation satisfying the hypothesis of the theorem we have, from the first paragraph of the proof, that $G/\gamma_4(G)$ has nilpotency class two. In other words, $\gamma_3(G) \subseteq \gamma_4(G)$ and since G is nilpotent, we immediately see that $\gamma_3(G) = 1$ and hence G has nilpotency class at most two. This proves our claim that

$$g^x = g_1^m g_2^m = (g_1 g_2)^m = g^m$$

and thus $G_1 \subseteq \omega(G)$. □

We explicitly record one of the main features of the above theorem.

COROLLARY 3.9. *Let G_1 be an abelian p-group of exponent p^r and G_2 be a p-group of nilpotency class at most two on which G_1 acts as a group of power automorphisms. If G_1 centralises $\Omega_r(G_2)$, then the semidirect product of G_2 by G_1 has nilpotency class at most two.*

We will need the following result: it has a routine proof.

THEOREM 3.10. *Let $G = BA$ where A is normal in G and $A \cap B = 1$ with A nilpotent and B supersoluble. Then G is supersoluble if and only if for every prime q dividing $|A|$, $B_{q'}/C_{B_{q'}}(A_q)$ is abelian of exponent dividing $q-1$, where $B_{q'}$ is a Hall q'-subgroup of B.*

4. A structure theorem

We now have enough information in hand to construct all finite supersoluble groups of Wielandt length two and order coprime to six.

To begin we introduce a definition which abstracts the properties elucidated in Theorem 3.5 and in Lemmas 3.2, 2.7 and 3.8. In this section all groups will have order coprime to six.

DEFINITION 4.1. We say that a p-group A has a *special factorisation* Y_0N_0 if the following properties hold:

(1) N_0 is of nilpotency class at most two and $Y_0 \in \mathscr{W}_2$.
(2) $N_0 \lhd A$, $N_0 \cap Y_0 = 1$ and $A = Y_0N_0$.
(3) Conjugation by the elements of Y_0 induces power automorphisms on N_0.
(4) If Y_0 is non-abelian, then
    (a) $A' \subseteq Y_0$;
    (b) $e(N_0) \leq e(Y_0)$ if Y_0 has nilpotency class 3.
(5) If Y_0 is abelian then $[Y_0, \Omega_r(N_0)] = 1$, where p^r is the exponent of Y_0.

LEMMA 4.2. (a) *Let A be a p-group having a special factorisation. Then $A \in \mathscr{W}_2$.*
(b) *If p is the largest prime dividing the order of a supersoluble $\mathscr{W}_2$-group G and if A_0 is a normal Sylow p-subgroup of G, and B_0 a Hall p'-subgroup of G, then $N_0 = [B_0, A_0]$ and $Y_0 = C_{A_0}(B_0)$ afford a special factorisation of A_0.*

PROOF. (a) If Y_0 is abelian, then by (5) of the definition, and Corollary 3.9, A has nilpotency class at most two and so $A \in \mathscr{W}_2$. Therefore suppose that Y_0 is non-abelian. Then $A' \subseteq Y_0$. This means that N_0 is abelian and $[N_0, Y_0] = 1$. It follows from Lemma 2.7 and (4) of the definition that $A \in \mathscr{W}_2$.

(b) By Theorem 3.5 we know that $A_0 = Y_0 N_0$ with $N_0 \lhd A_0$ and $N_0 \cap Y_0 = 1$. Hence, by (a), $Y_0 \in \mathscr{W}_2$ since $Y_0 \cong A_0/N_0$. From Lemma 3.3 either $N_0 \subseteq \omega(A_0)$ or $Y_0 \subseteq \omega(A_0)$. In the first case $[N_0, Y_0] \subseteq N_0 \cap Y_0 = 1$, so $A_0 = N_0 \times Y_0$. Since also $N_0' = 1$, we have $A_0' = N_0' Y_0' [N_0, Y_0] = Y_0' \subseteq Y_0$. In particular $Y_0' \neq 1$ ensures $A_0' \subseteq Y_0$. What is more, if Y_0 has nilpotency class three then we conclude from Lemma 2.7 that the exponent of N_0 divides $p^{e(Y_0)}$ so that (4) holds. Clearly in the case when $Y_0' \neq 1$, (3) is satisfied too.

Finally if $Y_0 \subseteq \omega(A_0)$ then Lemma 3.8 ensures that (5) holds, and also that (3) is satisfied in this case. □

DEFINITION 4.3. $G = B_0 A_0$ is a *matched extension* of A_0 by B_0 if, for some prime p,

(1) A_0 is a normal p-subgroup having special factorisation $Y_0 N_0$ with $N_0 = [A_0, B_0]$ and $Y_0 = C_{A_0}(B_0)$;
(2) B_0 is a supersoluble p'-group in $\mathscr{W}_2$;
(3) $B_0/C_{B_0}(N_0)$ is abelian of exponent dividing $p - 1$;
(4) if Y_0 is abelian then the elements of B_0 induce, by conjugation, power automorphisms in $N_0/N_0 \cap \omega(A_0)$;
(5) $(|\gamma_3(B_0/\omega(B_0))|, |B_0/C_{B_0}(A_0)|) = 1$.

LEMMA 4.4. *If $G = B_0 A_0$ is a matched extension, then G is a supersoluble $\mathscr{W}_2$-group.*

PROOF. The aim of the proof is to calculate $\omega(G)$ and to show that $G/\omega(G)$ is a T-group. First of all we have, $\omega(G) = P_0 \omega(A_0)$, where P_0 is the subgroup of $\omega(B_0)$ inducing power automorphisms in A_0, by conjugation.

Note that $C_{\omega(B_0)}(A_0) \subseteq P_0$. Also $C_{\omega(B_0)}(A_0) = C_{B_0}(A_0) \cap \omega(B_0)$.

Now $B_0/\omega(B_0)$ is a T-group, by hypothesis, and $B_0/C_{B_0}(A_0)$ is abelian by (3) of the definition of a matched extension. It follows from (5) of the definition of matched extension and Lemma 3.7 that $B_0/\omega(B_0) \times B_0/C_{B_0}(A_0)$ is a T-group. Hence, by [7, 13.4.7], $C_{\omega(B_0)}(A_0)$ is a T-group. It then follows that B_0/P_0 is a T-group since B_0/P_0 is isomorphic to a homomorphic image of $B_0/C_{\omega(B_0)}(A_0)$.

If A_0 is abelian, then $\omega(G) = P_0A_0$ and so $G/\omega(G) \cong B_0/P_0$. This implies that $G \in \mathscr{W}_2$.

Now suppose that A_0 is non-abelian. In this case $P_0 = C_{\omega(B_0)}(A_0)$, by [2, Theorem 5.3.2]. There are two cases to consider: either $Y_0' \neq 1$ or $Y_0' = 1$.

In the first case it follows from property (4) of the definition of special factorisation, that $N_0 \subseteq \omega(A_0)$. Hence $G/\omega(G) \cong Y_0/Y_0 \cap \omega(A_0) \times B_0/P_0$, a direct product of T-groups of relatively coprime orders and so $G/\omega(G)$ is also a T-group.

In the second case, when Y_0 is abelian, we have from property (5) of the definition of special factorisation and Lemma 3.8 that $Y_0 \subseteq \omega(A_0)$. Hence

$$A_0\omega(G)/\omega(G) \cong A_0/\omega(A_0) \cong N_0/N_0 \cap \omega(A_0),$$

an abelian p-group. However

$$G/\omega(G) = (B_0\omega(G)/\omega(G))(A_0\omega(G)/\omega(G),$$

and $B_0\omega(G)/\omega(G) \cong B_0/P_0$ is a T-group of p'-order acting by conjugation on $A_0\omega(G)/\omega(G)$ as power automorphisms: property (4) of matched extension. Hence $G/\omega(G)$ is a T-group.

This completes the proof that $G \in \mathscr{W}_2$. To see that G is supersoluble, use Theorem 3.10. □

We now generalise the concept of a matched extension.

DEFINITION 4.5. Let $G = YN$, with $N \lhd G$, $N \cap Y = 1$ and both N and Y nilpotent. Suppose that $p_1, p_2, \ldots, p_r \geq 5$ are the primes in decreasing order which divide $|G|$ and put N_i, Y_i for the Sylow p_i-subgroups of N and Y respectively, $1 \leq i \leq r$.

We define a *generalised matched extension* inductively as follows:

(i) If $r = 1$ and $G = Y_1N_1$, then Y_1N_1 is a special factorization of G.

(ii) If $r > 1$ and B_1 is a generalised matched extension, G is a matched extension of Y_1N_1 by B_1.

The last lemma now enables us to prove the following theorem.

THEOREM 4.6. *Every generalised matched extension of nilpotent groups whose orders are coprime to 6 is a supersoluble group in* $\mathscr{W}_2$.

PROOF. We use induction on the number r of primes dividing the order of the generalised matched extension G. If $r = 1$ then, in the notation of the definition above, G is a p_1-group in $\mathscr{W}_2$, so we are done.

Suppose $r > 1$ and a generalised matched extension involving at most $r - 1$ primes is a supersoluble $\mathscr{W}_2$-group. Then if G is a generalised matched extension involving

r primes, we may write $G = B_1A_1$, where B_1 is a generalised matched extension involving the $r-1$ primes $p_2, p_3, \dots, p_r$, where A_1 is a p_1-group, and where the extension B_1A_1 is matched. By induction B_1 is supersoluble p_1'-group in $\mathscr{W}_2$. Hence by Lemma 4.4, G is a supersoluble $\mathscr{W}_2$-group. This completes the induction. $\square$

Conversely we show that generalised matched extensions give rise to all supersoluble $\mathscr{W}_2$-groups.

THEOREM 4.7. *Let G be a supersoluble $\mathscr{W}_2$-group of order coprime to* 6. *Then G has an expression as a generalised matched extension.*

PROOF. We use induction on the number r of primes dividing $|G|$. If G is a p_1-group then, with $Y_1 = G$ and $N_1 = 1$ we see that G is a generalised matched extension.

Suppose $r > 1$ and that supersoluble $\mathscr{W}_2$-groups with fewer than r prime divisors of their orders are generalised matched extensions.

If p_1 is the largest prime dividing $|G|$, let A_1 be the normal Sylow p_1-subgroup of G and let B_1 be a Hall p_1'-subgroup of G, so that $G = B_1A_1$. Write $N_1 = [B_1, A_1]$ and $Y_1 = C_{A_1}(B_1)$. By induction B_1 has an expression as a generalised matched extension.

By Lemma 4.2 (b) $A_1 = Y_1N_1$ is a special factorisation of A_1 and B_1 is a supersoluble p_1'-group, so (1), (2) of the definition of matched extension hold. We need to verify that the remaining axioms (3)–(5) of matched extension hold for $G = B_1A_1$.

First of all (3) holds because, by Theorem 3.10, N_1B_1 is supersoluble.

To prove (4) we observe first that if A_1 is abelian, then (4) is automatically satisfied as $N_1 \cap \omega(A_1) = N_1 \cap A_1 = N_1$ and so $N_1/N_1 \cap \omega(A_1)$ is trivial. So suppose that Y_1 is abelian but A_1 is non-abelian. Then $Y_1 \subseteq \omega(A_1)$ is immediate from Lemma 3.3 if N_1 is non-abelian; and if N_1 is abelian and $N_1 \subseteq \omega(A_1)$ then $[N_1, Y_1] \subseteq N_1 \cap Y_1 = 1$, so A_1 is abelian, a contradiction. Hence, in this case, $Y_1 \subseteq \omega(A_1)$. Therefore

$$A_1\omega(G)/\omega(G) \cong A_1/\omega(G) \cap A_1 \cong A_1/\omega(A_1) \cong N_1/N_1 \cap \omega(A_1).$$

Next observe that $\omega(G) = P_1\omega(A_1)$ where P_1 is the subgroup of $\omega(B_1)$ inducing power automorphisms on A_1. Since A_1 is non-abelian, $P_1 = C_{\omega(B_1)}(A_1)$ by [2, Theorem 5.3.2]. Hence, since $G/\omega(G)$ is a T-group and

$$G/\omega(G) = (A_1\omega(G)/\omega(G))(B_1\omega(G)/\omega(G)),$$

it follows that B_1 acts as a group of power automorphisms on $N_1/N_1 \cap \omega(A_1)$, as required to confirm (4).

Finally, to prove (5) note that $C_{\omega(B_1)}(A_1) = C_{B_1}(A_1) \cap \omega(B_1)$, so the T-group $B_1/P_1 = B_1/C_{\omega(B_1)}(A_1)$ is a subdirect product of $B_1/\omega(B_1)$ and $B_1/C_{B_1}(A_1)$. Since A_1, B_1 have coprime orders, $C_{B_1}(A_1)$ is just the intersection of the centralisers of the chief factors of G contained in A_1. But since G is supersoluble, each of these centralisers contains B_1. It then follows that $B_1/C_{B_1}(A_1)$ is abelian. Now (5) follows from the following lemma. □

LEMMA 4.8. *If H is a T-group and M_1, M_2 are normal subgroups of H for which $M_1 \cap M_2 = 1$ and H/M_2 is abelian, then $(|H/M_2|, |\gamma_3(H/M_1)|) = 1$.*

PROOF. Firstly $\gamma_3(H) \subseteq M_2$ and so $M_1 \cap \gamma_3(H) = 1$. Therefore

$$\gamma_3(H/M_1) = \gamma_3(H)M_1/M_1 \cong \gamma_3(H),$$

whereas H/M_1 is isomorphic to a factor group of $H/\gamma_3(H)$. The result follows since

$$(|\gamma_3(H)|, |H/\gamma_3(H)|) = 1,$$

by [7, 13.4.4]. □

With this lemma we have concluded our characterisation of supersoluble $\mathscr{W}_2$-groups of order coprime to 6. Theorems 4.6–4.7 show that they are precisely the groups with a generalised matched extension.

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Department of Mathematics
Quaid-I-Azam University
Islamabad 45320
Pakistan
e-mail: asif.ali@anu.edu.au