ON MAXIMAL NILPOTENT SUBRINGS OF RIGHT NOETHERIAN RINGS

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1. Introduction. Applying Hopkins's Theorem asserting that each unitary right Artinian ring is right Noetherian, G. Köthe and K. Shoda proved the following theorem (cf. Köthe [**7**], p. 360, Theorem 1 and p. 363, Theorem 5): If R is a unitary right Artinian ring, then the following statements hold:

(i) Each nilpotent subring of R is contained in a maximal nilpotent subring of R.

(ii) The intersection of all maximal nilpotent subrings of R is the maximal nilpotent two-sided ideal of R.

(iii) All maximal nilpotent subrings of R are conjugate.

Our problem is to decide which of these statements remain valid in right Noetherian rings. It is an immediate consequence of the Theorem of Levitzki (cf. Jacobson [**6**, p. 199, Theorem 1]), and Theorem 1 of Herstein and Small [**5**, p. 775], that each nilpotent subring of a right Noetherian ring R is contained in a maximal nilpotent subring of R (Proposition 1).

In [**1**] D. W. Barnes proved statement (ii) for all rings with minimum condition for right ideals. Now, if R is any right Artinian ring, then the sum $B(R)$ of all nilpotent ideals of R is nilpotent, and $R/B(R)$ is a unitary right Artinian and right Noetherian ring. Hence Barnes's Theorem is a consequence of the following theorem:

If R is a ring such that the sum $B(R)$ of all nilpotent ideals of R is nilpotent, and that $R/B(R)$ is right Noetherian, then statement (ii) *holds in R.*

This theorem is an easy consequence of our Theorem 1. It is perhaps remarkable that we do not make full use of the maximum condition for right ideals of $R/B(R)$. In particular, it follows that the intersection of all maximal nilpotent subrings of a right Noetherian ring R is the maximal nilpotent ideal $B(R)$ of R (Corollary 4).

In rings without an identity element the customary concept of conjugacy is not applicable. Consequently we term the subrings X and Y of R *quasi-conjugate* if there exists a pair of elements u, v in R satisfying $u+v = uv = vu$ such that Y is the totality of elements $x-ux-xv+uxv$ with x in X. Then our Theorem 2 asserts: If R is a right Artinian ring without additive subgroups of type p^{∞}, then its maximal nilpotent subrings are quasi-conjugate.

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2. Notations and definitions. Throughout this paper, every ring R is associative. The existence of an identity in R is not assumed.

R^+ = additive group of the ring R.

$A \oplus B$ = ring-theoretical direct sum of the ideals A, B of R.

$A \dotplus B$ = direct sum of the right ideals A, B of R.

$\sum_{\mu \in I}^{d} R_\mu$ = discrete direct sum of the rings R_μ.

$\sum_{\mu \in I}^{c} R_\mu$ = complete direct sum [or Cartesian sum] of the rings R_μ.

If M is a subset of R, then we denote by $(M)_R$ the right ideal of R generated by M.

Let X be a subset of the ring R. Then

$$X_l(R) = \{x \in R \mid xX = 0\}, \quad X_r(R) = \{x \in R \mid Xx = 0\}.$$

$$Z^r(R) = \{x \in R \mid x_r(R) \text{ is an essential right ideal of } R\} = \text{right singular ideal of } R.$$

The subrings X and Y of the ring R are *quasi-conjugate* if there is a quasi-regular element $u \in R$ such that

$$Y = \{x - ux - xv + uxv \mid x \in X, \text{ where } v \text{ is the quasi-inverse of } u \text{ in } R.\}$$

Right Goldie ring = ring with ascending chain condition on right annihilators and on direct sums of right ideals.

The right ideal $U \neq 0$ of the ring R is called *uniform* if $X \cap Y \neq 0$ for all right ideals $X \neq 0 \neq Y$ of R with $X, Y \leqq U$.

If R is a semi-prime right Goldie ring, then, by Goldie [**4**, p. 202, Theorem, 1.1], there is a positive integer n such that

(i) every direct sum of uniform right ideals of M contains at most n terms,

(ii) a right ideal X of R is essential if and only if X contains a sum of n uniform right ideals.

The integer n is called the *dimension* of R and is denoted by dim R.

The right ideals U, V of R are *subisomorphic* if there is a right R-module monomorphism θ of U into V, and a right R-module monomorphism μ of V into U.

3. Maximal nilpotent subrings of rings with certain chain conditions. Let $\mathfrak{n}$ be a non-empty set of subrings S of the ring R such that the union of every tower $\mathfrak{t}$ of $\mathfrak{n}$ belongs to $\mathfrak{n}$. Then by Zorn's Lemma we obtain that each $\mathfrak{n}$-subring of the ring R is contained in a maximal $\mathfrak{n}$-subring of R. If $\mathfrak{n}$ is the set of all locally nilpotent (resp. nil) subrings of R, $\mathfrak{n}$ satisfies our transfinite induction hypothesis. Thus we have

LEMMA 1. (*a*) *Each locally nilpotent subring of an arbitrary ring R is contained in a maximal locally nilpotent subring of R.*

(*b*) *Each nil subring of an arbitrary ring R is contained in a maximal nil subring.*

By Herstein and Small [**5**, p. 775, Theorem 1], every nil subring of a ring R satisfying the ascending chain conditions on right and left annihilators is nilpotent. This implies

COROLLARY 1. *If R is a ring with maximum conditions on right and left annihilators, then each nilpotent subring is contained in a maximal nilpotent subring.*

Now let R be a right Noetherian ring. Then the sum $B(R)$ of all nilpotent ideals of R is nilpotent by Levitzki's Theorem (cf. Jacobson [**6**, p. 199, Theorem 1]). Since $R/B(R)$ is a semi-prime right Noetherian ring, it satisfies the maximum conditions on right and left annihilators by Procesi and Small [**12**, p. 81, Lemma 2]. Hence from Corollary 1 and Lemma 1 we deduce

PROPOSITION 1. *Each nilpotent subring of a right Noetherian ring R is contained in a maximal nilpotent subring of R.*

In the following it will be proved that the intersection of all maximal nilpotent subrings of a right Noetherian ring R is just the sum $B(R)$ of all nilpotent ideals of R.

LEMMA 2. *Let $M \neq 0$ be a nilpotent subring of the prime right Goldie ring R. Let r be the exponent of M and $n = \dim R$. Then*

(*a*) $r \leqq n$,

(*b*) $\dim (M)_R \leqq n-1$,

(*c*) *there exists a set of r idempotents $e_0 = 0$, $e_k \neq 0$ $(k = 1, 2, \ldots, r-1)$ of the ring Q of right quotients of R satisfying $e_kQ = M^{r-k}Q$ such that*

$$M \leqq R \cap \sum_{k=1}^{r-1} (e_k - e_{k-1})Q(1-e_k).$$

Proof. $(M)_R$ is a right ideal of R; so $\dim (M)_R \leqq \dim R = n$. Assume that $\dim (M)_R = n$; then $MQ = Q$ by Goldie [**4**, p. 212, Lemma 4.3]. Hence $0 = M^rQ = Q \neq 0$. This contradiction proves (*b*).

Since M is a nilpotent subring of the unitary Artinian ring Q, which is a ring of $n \times n$ matrices over a division ring, the exponent r of M satisfies $r \leqq n$ by Levitzki [**8**, p. 625, Zusatz], and (*a*) has been proved.

For each integer $1 \leqq m \leqq r-1$ we have $M^mQ > M^{m+1}Q$; for equality implies that $M = 0$, because Q has an identity element. If $d(r-i) = \dim (M^{r-i})_R$ for $i = 1, 2, \ldots, r-1$, then there are primitive orthogonal idempotents $e_{i,j_i} \neq 0$ $(i = 1, 2, \ldots, r-1, j_i = 1, 2, \ldots, d(r-i))$ of Q such that

$$M^{r-i}Q = e_{1,1}Q \dotplus \cdots \dotplus e_{1,d(r-1)}Q \dotplus \cdots \dotplus e_{i,1}Q \dotplus \cdots \dotplus e_{i,d(r-i)}Q.$$

If $e_k = e_{1,1} \dotplus \cdots \dotplus e_{1,d(r-1)} \dotplus e_{2,1} \dotplus \cdots \dotplus e_{k,1} \dotplus \cdots \dotplus e_{k,d(r-k)}$ for $k = 1, 2, \ldots, r-1$, then we obtain $e_ke_{k+1} = e_k = e_{k+1}e_k$, because the e_{i,j_i} are orthogonal idempotents. Furthermore we have $M^{r-k}Q = e_kQ$. From $e_{k+1}Q = M^{r-k-1}Q > M^{r-k}Q = e_kQ$ for all $k = 1, 2, \ldots, r-1$ we deduce for each $a \in M$ that

$$a(e_{k+1}Q) = aM^{r-k-1}Q \leqq M^{r-k}Q = e_kQ.$$

Hence $ae_{k+1} = e_k q \in e_k Q$ for some $q \in Q$. Since

$$(e_{k+1}-e_k)ae_{k+1} = (e_{k+1}-e_k)e_k q = e_{k+1}e_k q - e_k q = e_k q - e_k q = 0,$$

it follows that $(e_{k+1}-e_k)a \in (e_{k+1})_1(Q) = Q(1-e_{k+1})$. Therefore

$$(e_{k+1}-e_k)a \;\in\; (e_{k+1}-e_k)Q(1-e_{k+1})$$

is a consequence of $e_{k+1}e_k = e_k = e_k e_{k+1}$. Now $M \leqq MQ = e_{r-1}Q$ implies that $a = e_{r-1}a$. Hence

$$a = \sum_{k=1}^{r-1}(e_k - e_{k-1})a \;\in\; \sum_{k=1}^{r-1}(e_k - e_{k-1})Q(1-e_k),$$

where $e_0 = 0$. This completes the proof.

The nilpotent ring M has *exponent h*, if h is the least positive integer r with $M^r = 0$.

LEMMA 3. *Let R be a prime right Goldie ring, and Q its ring of right quotients. Then the following statements hold:*

(*a*) *There are* $n = \dim R$ *primitive orthogonal idempotents* $g_i \neq 0$ *of Q such that* $g_k Q \cap R \neq 0 \neq R \cap Qg_k$ *for all* $k = 1, 2, \ldots, n$.

(*b*) *If the* $f_i \neq 0$ $(i = 1, 2, \ldots, n)$ *are n primitive orthogonal idempotents of Q such that* $f_i Q \cap R \neq 0 \neq R \cap Qf_i$ *for all i, then*

(i) $T = R \cap \left[\sum_{j=1}^{n-1} f_j Q\left(\sum_{h=j+1}^{n} f_h\right)\right]$ *and* $U = R \cap \left[\sum_{j=1}^{n-1}\left(\sum_{h=j+1}^{n} f_h\right)Qf_j\right]$

are maximal nilpotent subrings of R with exponent n satisfying $U \cap T = 0$, *and*

(ii) *if N is a nilpotent subring of R containing an element t of the form*

$$t = \sum_{i=1}^{n-1} f_i q_i f_{i+1},$$

where $0 \neq f_i q_i f_{i+1} \in R$ *and* $q_i \in Q$ *for all* $i = 1, 2, \ldots, n-1$, *then* $N \leqq T$, *and N has exponent n.*

(*c*) *For each nilpotent subring M of R there exists a regular element* $c \in R$ *such that* $c^{-1}Mc \cap R$ *is contained in a maximal nilpotent subring S of R with exponent* $n = \dim R$. [c^{-1} denotes the inverse of c in Q.]

Proof. By Goldie's Theorem for prime rings, Q is a ring of $n \times n$ matrices over a division ring D. Hence there are n orthogonal primitive idempotents $e_i \in Q$. Since R is a classical right order of Q, each e_i has the form $e_i = a_i c_i^{-1}$, where $a_i, c_i \in R$, and c_i is regular. By Jacobson [6, p. 263, Lemma 1], there exist regular elements $b_1, b_2, \ldots, b_n, c \in R$ such that $c_i^{-1} = b_i c^{-1}$ for $i = 1, 2, \ldots, n$. If $g_i = c^{-1}e_i c$ for all i, then the g_i form a maximal set of orthogonal primitive idempotents of Q. Clearly $R \cap g_i Q \neq 0$ for all i. This together with $0 \neq a_i b_i = cg_i$ proves (*a*).

Now let $f_i \neq 0$ $(i = 1, 2, \ldots, n)$ be n orthogonal primitive idempotents of Q such that $f_iQ \cap R \neq 0 \neq R \cap Qf_i$ for all i. Since R is a prime ring we have

$$0 \neq (f_iQ \cap R)(Qf_h \cap R) \leqq f_iQf_h \cap R \quad \text{for } i, h = 1, 2, \ldots, n. \tag{3.1}$$

Let x_i $(i = 1, 2, \ldots, n-1)$ be $n-1$ elements of Q such that $0 \neq f_ix_if_{i+1} \in R$. Let N be a nilpotent subring of R containing the element

$$x = \sum_{i=1}^{n-1} f_ix_if_{i+1}.$$

Clearly $x \neq 0$, because the f_i are orthogonal primitive idempotents of Q. Now x satisfies $x^n = 0$. But $x^{n-1} \neq 0$, because

$$0 = x^{n-1} = (f_1x_1f_2)(f_2x_2f_3)\ldots(f_{n-2}x_{n-2}f_{n-1})(f_{n-1}x_{n-1}f_n)$$

would imply that

$$\begin{aligned} 0 = x^{n-1}Q &= (f_1x_1f_2)(f_2x_2f_3)\ldots(f_{n-2}x_{n-2}f_{n-1})(f_{n-1}x_{n-1}f_n)Q \\ &= (f_1x_1f_2)(f_2x_2f_3)\ldots(f_{n-2}x_{n-2}f_{n-1})Q = \ldots \\ &= (f_1x_1f_2)Q = f_1Q \neq 0, \end{aligned}$$

since Q is an associative ring and the right ideals f_kQ $(k = 1, 2, \ldots, n)$ are minimal right ideals of Q. From $x^{n-1} \neq 0$ we obtain by Lemma 2 that $n = \dim R$ is the exponent of N. Therefore

$$0 < N^{n-1}Q < N^{n-2}Q < \ldots < N^2Q < NQ < Q$$

is a (right) composition series of the unitary simple Artinian ring Q. Hence from

$$0 \neq x^{n-s} = \sum_{j=1}^{s} [(f_jx_jf_{j+1})(f_{j+1}x_{j+1}f_{j+2})\ldots(f_{j+n-s-1}x_{j+n-s-1}f_{j+n-s})] \in N^{n-s}$$

for $s = 1, 2, \ldots, n-1$ we obtain $N^{n-s}Q = (f_1 + f_2 + \ldots + f_s)Q$ for all s. Therefore by application of Lemma 2 we have

$$N \leqq R \cap \left[\sum_{j=1}^{n-1} f_jQ\left(1 - \sum_{h=1}^{j} f_h\right)\right] = R \cap \left[\sum_{j=1}^{n-1} f_jQ\left(\sum_{h=j+1}^{n} f_h\right)\right] = T.$$

This completes the proof of Lemma 3(*b*) (ii).

Now we have to show that T is a maximal nilpotent subring of R. Since the f_i are orthogonal idempotents, T is a nilpotent subring of R. By Lemma 2 of Procesi and Small [**12**, p. 81] and Corollary 1 we know that T is contained in a maximal nilpotent subring S of R. From (3.1) we deduce the existence of $n-1$ elements $q_i \in Q$ such that $0 \neq f_iq_if_{i+1} \in R$ for $i = 1, 2, \ldots, n-1$. Clearly $f_iq_if_{i+1} \in T$ for all i. Thus

$$t = \sum_{i=1}^{n-1} f_iq_if_{i+1} \in T \leqq S.$$

Hence, by Lemma 3(*b*) (ii), $S \leqq T$, and T is a maximal nilpotent subring of R.

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If
$$U = R \cap \left[\sum_{j=1}^{n-1}\left(\sum_{h=j+1}^{n} f_h\right)Qf_j\right],$$

then U is a nilpotent subring of R, and, by the right–left symmetry of the given proof for T being a maximal nilpotent subring of R, it can be shown that U is a maximal nilpotent subring of R. Since the elements f_k $(k = 1, 2, \ldots, n)$ are orthogonal idempotents, it is easy to see that $U \cap T = 0$. Thus Lemma 3(b) (i) holds.

Now let M be a nilpotent subring of R with exponent r. By Lemma 2 there exists a set of $r-1$ idempotents $e_k \neq 0$ $(k = 1, 2, \ldots, r-1)$ of the ring Q such that

$$M \leqq R \cap \left(\sum_{k=1}^{r-1} (e_k - e_{k-1})Q(1-e_k)\right),$$

where $e_0 = 0$. By the proof of Lemma 2, each e_k has the form

$$e_k = e_{1,1} + \ldots + e_{1,d(r-1)} + e_{2,1} + \ldots + e_{k,d(r-k)},$$

where $d(r-i) = \dim(M^{r-i})_R$, and where the $e_{i,j_i} \neq 0$ $[i = 1, 2, \ldots, r-1;\ j_i = 1, 2, \ldots, d(r-i)]$ are orthogonal primitive idempotents of Q. Clearly this set of orthogonal primitive idempotents of Q can be extended to a maximal set of n orthogonal primitive idempotents $f_j \neq 0$ $(j = 1, 2, \ldots, n)$ of Q such that

$$e_{i,j_i} = f_{j_i} + \sum_{s=1}^{i} d(r-s-1)$$

for $i = 1, 2, \ldots, r-1$. Hence

$$M \leqq R \cap \left[\sum_{j=1}^{n-1} f_j Q\left(\sum_{h=j+1}^{n} f_h\right)\right].$$

Each $f_j = x_j c_j^{-1}$, where $x_j, c_j \in R$, and c_j is regular. By Jacobson [6, p. 263, Lemma 1] there are regular elements $b_1, b_2, \ldots, b_n, c \in R$ with $c_j^{-1} = b_j c^{-1}$. Let $g_j = c^{-1} f_j c$ for $j = 1, 2, \ldots, n$. By the proof of Lemma 3(a) the primitive orthogonal idempotents g_j satisfy $g_j Q \cap R \neq 0 \neq Qg_j \cap R$ for all j. Hence

$$c^{-1}Mc \cap R \leqq R \cap \left[\sum_{j=1}^{n-1} g_j Q\left(\sum_{h=j+1}^{n} g_h\right)\right] = T.$$

By Lemma 3(b)(i), T is a maximal nilpotent subring of R with exponent n.

This completes the proof of Lemma 3.

Remark 1. Let R be a prime right Goldie ring, and M be a maximal nilpotent subring of R. If $c \in R$ is a regular element of R, then in general the nilpotent subring $c^{-1}Mc \cap R$ of R is not a maximal nilpotent subring of R, as can be seen by the following

Example. Let K be a unitary principal right ideal domain which is not a left Ore domain (e.g. Goldie [**4**, p. 219]). Hence there are $x, y \in K$ such that $Kx \cap Ky = 0$. Let R be the ring of all 2×2 matrices (a_{ij}) $(i, j = 1, 2)$ with $a_{1j} \in Kx$ and $a_{i2} \in Ky$. If $\hat{K}$ is the division ring of right quotients of K, then $Q = \hat{K}_2$ is the classical ring of right quotients of R, by Faith and Utumi [**2**, p. 59]. Hence R is a prime right Goldie ring (cf. Jacobson [**6**, p. 268], Goldie's Theorem for prime rings). Let M be the ring of all matrices (a_{ij}) of R with $a_{11} = a_{12} = a_{22} = 0$ and $a_{21} \in Kx$. We shall prove that M is a maximal nilpotent subring of R. If this were not true, then there would be a $v \in R$, $v \notin M$ such that $\{M, v\}$ is a nilpotent subring of R. By Lemma 2 we have $\{M, v\}^2 = 0$. Let

$$v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},$$

where $a, c \in Kx$ and $b, d \in Ky$. Then $v - g \in \{M, v\}$. Thus $(v-g)^2 = 0$. Hence $a = 0 = d$. From $v^2 = 0$ we deduce that $c = 0$ or $b = 0$. If we had $b \neq 0$, then we would obtain $v + z \in \{M, v\}$ for

$$z = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}.$$

Therefore $(v+z)^2 = 0$ which implies that $bx = 0$. Hence $b = 0$, and we get $v = g \in M$, a contradiction. Therefore M is a maximal nilpotent subring of R. If N is the ring of all matrices $(a_{ij}) \in R$ with $a_{11} = a_{21} = a_{22} = 0$ and $a_{12} \in Ky$, then by a similar argument N also is a maximal nilpotent subring of R. Clearly

$$c = \begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix}$$

is a regular element of R, and $c^{-1}Mc \cap R \leqq N$. If $c^{-1}Mc \cap R$ were a maximal nilpotent subring of R, then we would have $c^{-1}Mc \cap R = N$. Hence there would be an element $w \in Kx$ with $y^{-1}wy = y$, and thus we would have $0 \neq y = w = kx \in Kx \cap Ky = 0$. This contradiction shows that $c^{-1}Mc \cap R < N$.

It is well known (cf. Goldie [**4**], p. 206, Theorem 3.2) that each semi-prime right Goldie ring has the following properties:

(*a*) $Z_r(R) = 0$.

(*b*) Each set of independent uniform right ideals of R which are subisomorphic in pairs has a finite number of elements.

(*c*) Each right ideal $X \neq 0$ of R contains a uniform right ideal $U \neq 0$ of R.

We therefore call a ring R with the properties (*a*), (*b*) and (*c*) a *generalized right Goldie ring*.

LEMMA 4. *Let $\mathfrak{n}$ be a set of nil subrings of the semi-prime generalized right Goldie ring R such that the union of each tower of $\mathfrak{n}$-rings is an $\mathfrak{n}$-ring. If each locally nilpotent subring X of R belongs to $\mathfrak{n}$, then there exist two maximal $\mathfrak{n}$-subrings H and G of R with $H \cap G = 0$.*

Proof. Since R is a semi-prime ring with the properties (a), (b) and (c), by Lemma 4.2 and Lemma 4.5 of [**10**] R is an irredundant subdirect sum of the prime right Goldie rings $R_\mu = R/P_\mu$ ($\mu \in I$, I a well ordered index set), where the ideals P_μ are the maximal two-sided annihilator ideals of R. If $A_\mu = (P_\mu)_r(R)$, then A_μ is a minimal two-sided annihilator ideal of R by L. Levy [**9**, p. 65, Lemma 3.1]. Since R is semi-prime, $A_\mu \cap P_\mu = 0$ for all $\mu \in I$. Hence A_μ is isomorphic to a two-sided ideal $\bar{A}_\mu$ of R_μ. If we identify A_μ with $\bar{A}_\mu$, then A_μ is an essential right ideal of the prime right Goldie ring R_μ. Hence A_μ is a prime ring with $Z^r(A_\mu) = 0$, and $\dim_R(A_\mu) = \dim_{A_\mu}(A_\mu) = \dim_{R_\mu}(A_\mu) = \dim R_\mu$. Thus A_μ has a ring of right quotients $Q(A_\mu)$ which coincides with the ring of right quotients Q_μ of R_μ, because A_μ is a two-sided ideal of R. Application of Lemma 3[(a) and (b) (i)] shows the existence of a maximal set of primitive orthogonal idempotents $g_i^{(\mu)} \in Q_\mu$ ($i = 1, 2, \ldots, n_\mu$; $\mu \in I$) such that

$$T_\mu = A_\mu \cap \left[\sum_{j=1}^{n_\mu - 1} g_j^{(\mu)} Q_\mu \left(\sum_{h=j+1}^{n_\mu} g_h^{(\mu)}\right)\right],$$

and

$$S_\mu = A_\mu \cap \left[\sum_{j=1}^{n_\mu - 1} \left(\sum_{h=j+1}^{n_\mu} g_h^{(\mu)}\right) Q_\mu g_j^{(\mu)}\right]$$

are maximal nilpotent subrings of A_μ with exponent n_μ. Furthermore $T_\mu \cap S_\mu = 0$.

Let $T = \sum_{\mu \in I}^d T_\mu$. Then T is a nil subring of R contained in $\sum_{\mu \in I}^d A_\mu \leqq R$. If $t_i \in T$ ($i = 1, 2, \ldots, m$), where m is any positive integer, then there is a finite number z (depending on the $t_i \in T$) of minimal two-sided annihilators A_k ($k = 1, 2, \ldots, z$) such that the subring $\{t_1, t_2, \ldots, t_m\}$ of R generated by the t_i is contained in $V = A_1 \oplus A_2 \oplus \ldots \oplus A_z$. Since V is a complete direct sum of a finite number of prime right Goldie rings, V is a semi-prime right Goldie ring. Hence the nil subring $\{t_1, t_2, \ldots, t_m\}$ of V is nilpotent by application of Procesi and Small [**12**, p. 81, Lemma 2] and Herstein and Small [**5**, p. 775, Theorem 1]. Thus T is a locally nilpotent subring of R. Hence T is an $\mathfrak{n}$-subring of R. Since the union of each tower of $\mathfrak{n}$-rings is an $\mathfrak{n}$-ring, T is contained in a maximal $\mathfrak{n}$-subring G of R. If $S = \sum_{\mu \in I}^d S_\mu$, by the same argument we get that S is contained in a maximal $\mathfrak{n}$-subring H of R. From $G \geqq T$ and $H \geqq S$ we obtain $G_\mu \geqq T_\mu$ and $H_\mu \geqq S_\mu$ for all $\mu \in I$. For each $\mu \in I$ we define

$$F_\mu = R_\mu \cap \left[\sum_{j=1}^{n_\mu - 1} g_j^{(\mu)} Q_\mu \left(\sum_{h=j+1}^{n_\mu} g_h^{(\mu)}\right)\right],$$

and

$$E_\mu = R_\mu \cap \left[\sum_{j=1}^{n_\mu - 1} \left(\sum_{h=j+1}^{n_\mu} g_h^{(\mu)}\right) Q_\mu g_j^{(\mu)}\right].$$

Clearly $F_\mu \cap E_\mu = 0$ for all $\mu \in I$.

We now want to show that $G_\mu \leqq F_\mu$ and $H_\mu \leqq E_\mu$ for all $\mu \in I$, By Lemma 3(a) applied to the prime right Goldie ring A_μ we have $g_k^{(\mu)} Q_\mu \cap A_\mu \neq 0 \neq Q_\mu g_k^{(\mu)} \cap A_\mu$ for all $k = 1, 2, \ldots, n_\mu$ and all $\mu \in I$. Since each A_μ is a prime ring we get

$$0 \neq g_j^{(\mu)} Q_\mu g_h^{(\mu)} \cap A_\mu \leqq g_j^{(\mu)} Q_\mu g_h^{(\mu)} \cap R_\mu$$

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for all $j, h = 1, 2, \ldots, n_\mu$. Therefore there are $n_\mu - 1$ elements $q_i^{(\mu)} \in Q_\mu$ $(i = 1, 2, \ldots, n_\mu - 1)$ such that for all i we have

$$0 \neq g_i^{(\mu)} q_i^{(\mu)} g_{i+1}^{(\mu)} \in g_i^{(\mu)} Q_\mu g_{i+1}^{(\mu)} \cap A_\mu \leqq T_\mu \leqq G_\mu.$$

Since $G_\mu = (G + P_\mu)/P_\mu$, G_μ is a nil subring of the prime right Goldie ring R_μ, because each $\mathfrak{n}$-ring is a nil ring. Hence G_μ is nilpotent. Clearly

$$t_\mu = \sum_{i=1}^{n_\mu - 1} g_i^{(\mu)} q_i^{(\mu)} g_{i+1}^{(\mu)} \in T_\mu \leqq G_\mu \leqq R_\mu.$$

Hence, by application of Lemma 3(b) (ii) to the prime right Goldie ring R_μ, $G_\mu \leqq F_\mu$. By the right-left symmetry of this argument we get $H_\mu \leqq E_\mu$.

Now

$$G \leqq \sum_{\mu \in I}^{c} G_\mu \leqq \sum_{\mu \in I}^{c} F_\mu, \quad \text{and} \quad H \leqq \sum_{\mu \in I}^{c} H_\mu \leqq \sum_{\mu \in I}^{c} E_\mu.$$

Therefore $G \cap H = 0$, because $G_\mu \cap F_\mu = 0$ for all $\mu \in I$. This completes the proof of Lemma 4.

Before stating our Theorem 1 we restate some definitions of [**11**]. Let $\mathfrak{U}$ be the universal class of all [associative] rings; then a single-valued function f assigning to every ring R a (two-sided) ideal fR of R is called a *preradical* over $\mathfrak{U}$ if it satisfies

$$(fR)^\mu \leqq fR^\mu \text{ for every epimorphism } \mu \text{ of } R.$$

It is well known (cf. Jacobson [6]) that the lower and the upper nil radical, the Levitzki-radical and the Jacobson-radical are preradicals.

Let f and g be preradicals over $\mathfrak{U}$. Then we define $f \leqq g$ if and only if $fR \leqq gR$ for all rings R. If f is a preradical over $\mathfrak{U}$, we term a ring S an *f-ring* if $S = fS$. The ideal X of the ring R is an *f-ideal* of R if X is an f-ring. We denote by $s_f R$ the sum of all f-ideals of the ring R. Using these definitions and notations we now establish the following theorem.

THEOREM 1. *Let f be a preradical over $\mathfrak{U}$ with the following properties*:

A. *If L is the Levitzki-radical and N the upper nil radical over $\mathfrak{U}$, then $L \leqq f \leqq N$.*

B. *Extensions of f-rings by f-rings are f-rings.*

C. *The union of each tower of f-rings is an f-ring.*

If $R/s_f R$ is a generalized right Goldie ring, then the sum $s_f R$ of all f-ideals of the ring R is the intersection of all maximal f-subrings of R.

Proof. Let $\mathfrak{t}$ be a tower of f-ideals X of R. If $T = \sum_{X \in \mathfrak{t}} X$, then T is an f-ideal by C. Hence there exists a maximal f-ideal M of R by Zorn's Lemma. If $M \neq s_f R$, then there would be an f-ideal Y of R with $Y \not\leqq M$. Since M is an f-ring, we would have

$$f[M/M \cap Y] = M/M \cap Y \cong (M + Y)/Y.$$

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Hence $M+Y$ would be an f-ideal of R, by B. Thus M would not be a maximal f-ideal of R. This contradiction proves that $M=s_fR$. Since s_fR is an f-ideal of R, the only f-ideal of R/s_fR is 0, by C. From $L\leqq f$ we obtain that R/s_fR is a semi-prime ring.

Using C, another application of Zorn's Lemma establishes the existence of maximal f-subrings of R. Now, if F is such a subring of R, $s_fR\leqq F$. Hence from B it follows that the subring S of R is a maximal f-subring of R if and only if $S\geqq s_fR$ and S/s_fR is a maximal f-subring of $\bar{R}=R/s_fR$. Thus, if F_α $(\alpha\in\bar{A})$ are the maximal f-subrings of R, then the F_α/s_fR are the maximal f-subrings of $\bar{R}$. Since $\bar{R}$ is a semi-prime generalized right Goldie ring, the intersection of all maximal f-subrings of $\bar{R}$ is zero by Lemma 4. Hence

$$\Big(\bigcap_{\alpha\in\bar{A}} F\Big)\Big/s_fR=\bigcap_{\alpha\in\bar{A}}[F_\alpha/s_fR]=0,$$

which implies that

$$\bigcap_{\alpha\in\bar{A}} F_\alpha=s_fR.$$

This completes the proof of Theorem 1.

By Lemma 1 and Jacobson [6, p. 197, Lemma and Proposition 1, and p. 193, Lemma 1], the Levitzki-radical L and the upper nil radical N over $\mathfrak{U}$ satisfy the conditions A, B and C of Theorem 1. Hence we have

COROLLARY 2. (*a*) *If $R/L(R)$ is a generalized right Goldie ring, then the sum $L(R)$ of all locally nilpotent ideals of R is the intersection of all maximal locally nilpotent subrings of R.*

(*b*) *If $R/N(R)$ is a generalized right Goldie ring, then the sum $N(R)$ of all nil ideals of R is the intersection of all maximal nil subrings of R.*

COROLLARY 3. *Let $B(R)$ be the lower nil radical of the ring R. Let $R/B(R)$ be a generalized right Goldie ring with the following property*:

(*b'*) *Each set of independent, uniform right ideals of R which are subisomorphic in pairs has at most n elements, where n is a fixed positive integer.*

Then the following properties of the ring R are equivalent:

(i) *$B(R)$ is nilpotent.*

(ii) *$B(R)$ is the intersection of all maximal nilpotent subrings of R.*

Proof. Clearly (i) is a consequence of (ii). Assume that R satisfies condition (i). Since $R/B(R)$ is a semi-prime generalized right Goldie ring the Levitzki-radical $L(R)$ coincides with $B(R)$, by [**10**, Zusatz 5.4]. Hence $B(R)$ is the intersection of all maximal locally nilpotent subrings of R, by Corollary 2(*a*). Let M be a maximal locally nilpotent subring of R. Then $M/B(R)=\bar{M}$ is a maximal locally nilpotent subring of $R/B(R)=\bar{R}$. By [**10**, Theorem 4.8] $\bar{R}$ is a subring of a complete direct sum $\bar{Q}$ of complete rings of $n_\mu\times n_\mu$ matrices over division rings $K_\mu(\mu\in I)$. From (*b'*) one easily deduces that $n_\mu\leqq n$ for all $\mu\in I$. Hence $\bar{M}$ is nilpotent by application of Lemma 2(*a*). This completes the proof of Corollary 3.

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COROLLARY 4. *The lower nil radical $B(R)$ of a right Noetherian ring is the intersection of all maximal nilpotent subrings of R.*

This follows at once from Corollary 3.

THEOREM 2. *If R is a right Artinian ring without additive subgroups of type p^∞, then the maximal nilpotent subrings of R are quasi-conjugate.*

Proof. Since R^+ does not contain any subgroup of type p^∞, it follows from Fuchs [**3**, p. 283, Theorem 73.1] that

$$R = C_0 \oplus C_1 \oplus C_2 \oplus \ldots \oplus C_r,$$

where C_0 is a torsion-free right Artinian ring, and the C_i $(i = 1, 2, \ldots, r)$ are uniquely determined ($p_i \neq p_j$, if $i \neq j$) right Artinian p_i-rings whose elements are of bounded order.

Let $U_0 = \{(c, \mu) \mid c \in C_0, \mu \in Q$, where Q is the field of rationals$\}$, where the addition is defined componentwise and the multiplication by

$$(c_1, \mu_1)(c_2, \mu_2) = (c_1c_2 + \mu_1c_2 + \mu_2c_1, \mu_1\mu_2). \tag{3.2}$$

Then, by Fuchs [**3**, p. 284], U_0 is a unitary right Artinian ring such that C_0 is an ideal of U_0.

For $i = 1, 2, \ldots, r$ let $p_i^{k_i}$ be the least upper bound of the orders of the elements of C_i. Let Z be the ring of rational integers, and let $Z(p_i^{k_i})$ be the factor ring of Z mod $p_i^{k_i}$. Let

$$U_i = \{(c, \mu) \mid c \in C_i, \mu \in Z(p_i^{k_i})\},$$

where addition is defined componentwise and the multiplication by (3.2). Then, by Fuchs [**3**, p. 285], each U_i $(i = 1, 2, \ldots, r)$ is a unitary right Artinian ring such that C_i is a two-sided ideal of U_i. Hence

$$U = U_0 \oplus U_1 \oplus U_2 \oplus \ldots \oplus U_r$$

is a right Artinian ring. Since each right ideal of R is a right ideal of U, R is right Noetherian. Thus R has maximal nilpotent subrings by Proposition 1. If N is such a subring of R, then the C_j-component $(j = 0, 1, 2, \ldots, r)$

$$N_j = \{x \in R \mid x = n - y \in C_j \text{ for some } n \in N \text{ and } y \in \sum_{k \neq j} U_k\}$$

of N is a maximal nilpotent subring of C_j, and

$$N = N_0 \oplus N_1 \oplus \ldots \oplus N_r$$

by Barnes [**1**, p. 234, Lemma 1], because R is right Artinian. Let M be another maximal nilpotent subring of R. Then

$$M = M_0 \oplus M_1 \oplus \ldots \oplus M_r.$$

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Since $U_0/C \cong Q$, all nilpotent subrings of U_0 are contained in C_0. Hence N_0 and M_0 are maximal nilpotent subrings of the unitary right Artinian ring U_0. Thus, by Köthe [7, p. 363, Theorem 5], there is a unit $u_0 \in U_0$ satisfying

$$M_0 = u_0^{-1} N_0 u_0.$$

Let $u_0 = (c_0, \mu_0)$ and $u_0^{-1} = (d_0, \mu_0^{-1})$.

If $q_0 = (-c_0, 0)\ (0, \mu_0^{-1})$ and $p_0 = (-d_0, 0)(0, \mu_0)$, then p_0 is the quasi-inverse of q_0 in C_0, and we have

$$M_0 = (1-p_0)N_0(1-q_0).$$

For $j = 1, 2, \ldots, r$ we know that N_j (resp. M_j) is a maximal nilpotent subring of C_j. Let J_j be the radical of $Z(p_j^{k_j})$. Then

$$N_j^* = (N_j, 0) + (0, J_j)$$

is a nilpotent subring of U_j. If N_j^* were not a maximal nilpotent subring of U_j, then there would be an element $v \in U_j$ such that $\{N_j^*, v\}$ is nilpotent.

Hence $v = (c_j, z_j)$ $[c_j \in C_j, z_j \in Z(p_j^{k_j})]$ is nilpotent. This implies that $z_j \in J_j$, and

$$c_j = (c_j, 0) = v - (0, z_j) \in \{N_j^*, v\} \cap C_j.$$

Now $\{N_j, c_j\} \leqq \{N_j^*, v\}$ and the fact that N_j is a maximal nilpotent subring of C_j imply that $c_j \in N_j$, which implies that $v \in N_j^*$, a contradiction. Hence N_j^* is a maximal nilpotent subring of U_j. By the same argument we obtain that $M_j^* = (M_j, 0) + (0, J_j)$ is a maximal nilpotent subring of U_j. Thus, by Köthe [7, p. 363, Theorem 5], there is a unit $u_j \in U_j$ satisfying

$$M_j^* = u_j^{-1} N_j^* u_j \, .$$

Now identify M_j with $(M_j, 0)$ and N_j with $(N_j, 0)$. Then it is obvious that $u_j^{-1} N_j u_j \leqq M_j$. Conversely, for each m_j there is an $n_j^* = n_j + z_j$, where $n_j \in N_j, z_j \in J_j$, such that

$$m_j = u_j^{-1}(n_j + z_j)u_j = u_j^{-1} n_j u_j + u_j^{-1} z_j u_j \, .$$

Let $u_j = d_j + r_j$, $u_j^{-1} = e_j + s_j$, where $d_j, e_j \in N_j$, $r_j, s_j \in Z(p_j^{k_j})$ and $r_j s_j = 1 = s_j r_j$. Then it follows that

$$m_j - u_j^{-1} n_j u_j - z_j e_j d_j - s_j z_j d_j - r_j z_j e_j = s_j z_j r_j \in C_j \cap Z(p_j^{k_j}) = 0.$$

Hence $z_j = 0$. This means that $M_j = u_j^{-1} N_j u_j$.

For each $j = 1, 2, \ldots, r$, let $q_j = (-d_j, 0)(0, r_j^{-1})$ and $p_j = (-e_j, 0)(0, r_j)$. Then p_j is the quasi-inverse of q_j in C_j, and we have $M_j = (1-p_j)N_j(1-q_j)$. If $p = \sum_{k=0}^{r} p_k$ and $q = \sum_{k=0}^{r} q_k$, then p is the quasi-inverse of q in R, and it follows that $M = (1-p)N(1-q)$. This completes the proof of Theorem 2.

REFERENCES

1. D. W. Barnes, On the radical of a ring with minimum condition, *J. Australian Math. Soc.* **5** (1965), 234–236.

2. C. Faith and Y. Utumi, On Noetherian prime rings, *Trans. Amer. Math. Soc.* **114**, (1965), 53–60.

3. L. Fuchs, *Abelian groups* (Oxford, London, New York, Paris, 1960).

4. A. W. Goldie, Semi-prime rings with maximum condition, *Proc. London Math. Soc. Ser.* 3 **10** (1960), 201–220.

5. I. N. Herstein and L. Small, Nil rings satisfying certain chain conditions, *Canad. J. Math.* **16** (1964), 771–776.

6. N. Jacobson, *Structure of rings*, Amer Math. Soc. Colloquium Publications **37** (Providence, 1964).

7. G. Köthe, Über maximale nilpotente Unterringe und Nilringe, *Math. Ann.* **103** (1930), 359–363.

8. L. Levitzki, Über nilpotente Unterringe, *Math. Ann.* **105** (1931), 620–627.

9. L. Levy, Unique subdirect sums of prime rings, *Trans. Amer. Math. Soc.* **106** (1963), 64–76.

10. G. Michler, Klassische Quotientenringe von nicht notwendig endlich dimensionalen, halbprimen Ringen, *Math. Z.* (to appear).

11. G. Michler, Radikale und Sockel, accepted for publication by *Math. Ann.*

12. C. Procesi and L. Small, On a theorem of Goldie, *J. Algebra* **2** (1965), 80–84.

Added in proof. We now have an example of a prime right and left Goldie ring with identity having two maximal milpotent subrings which are not isomorphic (cf Michler, *Math. Z.* **100** (1967), p. 180).

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