POINTWISE APPROXIMATION BY
BERNSTEIN POLYNOMIALS

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(Received 22 April 2011)

Abstract
We improve the degree of pointwise approximation of continuous functions \( f(x) \) by Bernstein operators, when \( x \) is close to the endpoints of \([0, 1]\). We apply the new estimate to establish upper and lower pointwise estimates for the test function \( g(x) = x \log(x) + (1 - x) \log(1 - x) \). At the end we prove a general statement for pointwise approximation by Bernstein operators.


Keywords and phrases: Bernstein polynomials, Direct theorems, Ditzian–Totik moduli of smoothness.

1. Introduction
In 1994 Ditzian showed in [4] that for the Bernstein polynomials
\[
B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}, \quad x \in [0, 1],
\]
the pointwise approximation
\[
|B_n(f; x) - f(x)| \leq C \omega_2^{\varphi^1}(f, n^{-1/2} \varphi(x)^{1-\lambda}), \quad x \in [0, 1],
\]
holds true for \( \lambda \in [0, 1] \), \( \varphi(x) := \sqrt{x(1-x)} \) and \( f \in C[0, 1] \), where the Ditzian–Totik modulus of second order is given by
\[
\omega_2^{\varphi^1}(f, t) := \sup_{0 < h \leq t} \sup_{x \neq h \varphi^1(x) \in [0,1]} |f(x - h \varphi^1(x)) - 2f(x) + f(x + h \varphi^1(x))|.
\]
We recall that this modulus is equivalent to the \( K \)-functional
\[
K_{\varphi^1}(f, t^2) = \inf(||f - h||_{C[0,1]} + t^2||\varphi^{2h}||_{C[0,1]}).
\]
The infimum is taken on functions satisfying \( h \in AC \), \( h' \in AC_{loc} \) where \( AC \) is the set of all absolutely continuous functions on \([0, 1]\) and \( AC_{loc} \) is the set of absolutely continuous functions on compact subsets of \((0, 1)\). (See [5].)
In 1998 Felten proved in [6] the more general inequality

\[ |B_n(f, x) - f(x)| \leq C \omega_{2}^{\phi} \left( f, n^{-1/2} \frac{\phi(x)}{\phi(x)} \right), \quad x \in [0, 1], \]

where \( \phi : [0, 1] \rightarrow \mathbb{R} \) is an admissible step-weight function of the Ditzian–Totik modulus and \( \phi^2 \) is a concave function. The aim of this note is to improve the estimate (1.1) for \( \lambda = 1 \), when \( x \) is close to the endpoints of \([0, 1]\).

Let us define

\[ \delta(n, x) := \min \left\{ n^{-1/2}, \left( \frac{x(1-x)}{n} \right)^{1/4} \right\}. \]

The following theorem is our main result.

**Theorem 1.1.** The pointwise estimate

\[ |B_n(f, x) - f(x)| \leq C \omega_{2}^{\phi}(f, \delta(n, x)), \quad x \in [0, 1], \quad (1.4) \]

holds true for all \( f \in C[0, 1], \ n \in \mathbb{N} \).

In Section 2 we give the proof of Theorem 1.1. In Section 3 we establish upper and lower bounds for approximation of the function \( g(x) \), defined in (2.1), by Bernstein operators.

### 2. Proof of Theorem 1.1

Let us define \( g : [0, 1] \rightarrow \mathbb{R} \) as

\[ g(x) = x \log(x) + (1 - x) \log(1 - x), \quad x \in (0, 1), \quad (2.1) \]

and \( g(0) = g(1) := 0 \). The problem of evaluating the remainder term

\[ R_n(g, x) = B_n(g, x) - g(x), \quad x \in [0, 1], \]

was formulated by the author in [14] during the fifth Romanian–German Seminar on Approximation Theory, held in Sibiu, Romania, in 2002. More precisely, we proposed to find (best) bounds of the type

\[ k_{1} \cdot \frac{x^{a_1}(1-x)^{a_2}}{n^{b}} \leq R_n(g, x) \leq K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{n^{b}}, \quad x \in [0, 1], \]

where \( k_1, K_2 \) are positive numbers, independent of \( x \) and \( n \). Some days after the conference, Lupaș showed that the above holds with \( a_1 = a_2 = \beta = 1, \ k_1 = \frac{1}{2} \) and \( a_1 = a_2 = b = \frac{1}{2}, \ K_2 = \sqrt{2} \) (see [8, 9]), that is,

\[ \frac{x(1-x)}{2n} \leq R_n(g, x) \leq \sqrt{2} \cdot \sqrt{\frac{x(1-x)}{n}}. \quad (2.2) \]

The function \( g \) was applied in the following direct estimate, proved by Parvanov and Popov in [12].
If $L : C[0, 1] \to C[0, 1]$ is a linear positive operator, preserving linear functions, then
\[
|L(f, x) - f(x)| \leq 2\|f - h\|_{C[0,1]} + |L(g, x) - g(x)| \cdot \|\varphi^2 h''\|_{C[0,1]}
\]
holds for arbitrary $h \in AC, h' \in AC_{loc}, \|\varphi^2 h''\|_{C[0,1]} < \infty$. Instead of $L$ we write $B_n$ and apply the right-hand side of (2.2). Hence
\[
|B_n(f, x) - f(x)| \leq 2\|f - h\|_{C[0,1]} + \sqrt{2} \cdot \|\varphi^2 h''\|_{C[0,1]} \cdot \left(\frac{x(1-x)}{n}\right)^{1/2}.
\]
Therefore
\[
|B_n(f, x) - f(x)| \leq 2K_\varphi\left(f, \left(\frac{x(1-x)}{n}\right)^{1/2}\right).
\]
From the equivalence between $K_\varphi(f, t^2)$ and $\omega_2^2(f, t)$, it follows that
\[
|B_n(f, x) - f(x)| \leq C\omega_2^2\left(f, \left(\frac{x(1-x)}{n}\right)^{1/4}\right). \tag{2.3}
\]
The estimates (2.3) and (1.1) with $\lambda = 1$ complete the proof. □

3. Upper and lower pointwise bounds

The following is a straightforward corollary of Theorem 1.1.

**Corollary 3.1.** The pointwise estimate
\[
|B_n(g, x) - g(x)| \leq C\omega_2^2\left(g, \sqrt{\frac{x(1-x)}{n}}\right), \quad x \in [0, 1], \tag{3.1}
\]
holds true for all $n \in \mathbb{N}$.

**Remark 3.2.** If $x$ is close to the endpoints of $[0, 1]$, then the estimate (3.1) is better than that in (2.1) for $\lambda = 1$, established by Ditzian in [4].

**Remark 3.3.** Other direct pointwise estimates in terms of $K_\varphi$ are proved in [6]. We point out that neither from [6] nor from [4] is it possible to deliver (3.1) as a straightforward corollary.

We continue with lower pointwise bounds. In [1, Theorem 11], using the function $g(x)$ as a ‘universal’ tool, the authors proved that
\[
c(g)\omega_2 \left(g, \sqrt{\frac{x(1-x)}{n}}\right) \leq |B_n(g, x) - g(x)|
\]
does not hold. So the question arises: what kind of modulus is appropriate to serve as a lower pointwise bound for $|B_n(g, x) - g(x)|$? The answer is given in the next theorem.

**Theorem 3.4.** The following inequality holds true:
\[
c \cdot \omega_2^2 \left(g, \sqrt{\frac{x(1-x)}{n}}\right) \leq |B_n(g, x) - g(x)|. \tag{3.2}
\]
PROOF. Using the equivalence between $K_\varphi(g, t^2)$ and $\omega^2_2(g, t)$, we compute

$$c\omega^2_2\left(g, \sqrt{\frac{x(1-x)}{n}} \right) \leq K_\varphi\left(g, \frac{x(1-x)}{n} \right)$$

$$:= \inf_h \left\{ \|g - h\|_{C[0,1]} + \frac{x(1-x)}{n} \cdot \|\varphi' h'\|_{C[0,1]} \right\}$$

$$\leq \|g - g\|_{C[0,1]} + \frac{x(1-x)}{n} \cdot \|\varphi' g'\|_{C[0,1]}$$

$$= \frac{x(1-x)}{n} \leq 2|B_n(g, x) - g(x)|,$$

where the last inequality follows from (2.2). The proof is complete. □

**Remark 3.5.** It was pointed out in [1] that for $f(x) = x^3$, $x \in [0, 1]$, an estimate similar to (3.2) is not possible.

**Remark 3.6.** Theorems 3.4 and 3.7 imply for the function $g(x)$ in (2.1) the two-sided pointwise inequality

$$c\omega^2_2\left(g, \sqrt{\frac{x(1-x)}{n}} \right) \leq |B_n(g, x) - g(x)| \leq C\omega^2_2\left(g, \sqrt{\frac{x(1-x)}{n}} \right).$$  (3.3)

Very recently, motivated by the result of Lupas and considerations set out in [1, 2, 12] we proved in [15] that the values of $\alpha_1 = \alpha_2 = 1$ and $a_1 = a_2 = \frac{1}{2}$ in (1.4) are optimal, that is, we proved the following result.

**Theorem A.** It is not possible to find $a_1 > \frac{1}{2}$, or $a_2 > \frac{1}{2}$, or $\alpha_1 < 1$, or $\alpha_2 < 1$, such that

$$k_1 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n} \leq R_n(g, x) \leq K_2 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{\sqrt{n}}$$

holds true for all $x \in [0, 1]$ with some positive numbers $k_1, K_2$, independent of $x$ and $n$.

Our next statement is the following theorem.

**Theorem 3.7.** In both sides of (3.3) it is not possible to put one and the same modulus: neither $\omega^2_2(g, \sqrt{x(1-x)/n})$ nor $\omega^2_2(g, \sqrt{x(1-x)/n})$.

**Proof.** First we suppose that $\omega^2_2(g, \sqrt{x(1-x)/n})$ could be placed in the left-hand side of (3.3). Setting $x = \frac{1}{2}$ in (1.2), we obtain

$$\Delta^2_{h\varphi}\left(\frac{1}{2}\right) = h^2 \cdot \varphi'\left(\frac{1}{2}\right) \cdot g''(\xi) \geq h^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2(1-\frac{1}{2})} = h^2.$$

Hence by

$$t := \sqrt{\frac{x(1-x)}{n}}, \quad x \in [0, 1] \text{ fixed},$$

we have

$$\omega^2_2(g, t) \geq t^2 = \sqrt{\frac{x(1-x)}{n}}.$$
From our supposition and the last inequality we get
\[ c \sqrt{\frac{x(1-x)}{n}} \leq |B_n(g, x) - g(x)|, \]
which contradicts the statement of Theorem A (left-hand side of the inequality, as \( x \to 0 \)). Also if we suppose that \( \omega_2^x(g, \sqrt{x(1-x)/n}) \) could be placed in the right-hand side of (3.3) due to the fact that (see [3, Theorem 6.1])
\[ \omega_2^x(g, t) \leq C_t^2 \| \varphi^2 g'' \| = C_t^2 \cdot 1, \]
the last inequality would imply that
\[ |B_n(g, x) - g(x)| \leq C \frac{x(1-x)}{n}, \]
which again contradicts Theorem A (right-hand side of the inequality, as \( x \to 0 \)). The proof of Theorem 3.7 is complete. \( \square \)

**Remark 3.8.** The upper pointwise bound in (3.1) in terms of the classical modulus of continuity \( \omega_2(g, \sqrt{x(1-x)/n}) \) was first established in [13]. As already mentioned, this modulus is not appropriate as a lower bound.

It is known that for the ‘test’ function \( f_1(x) = x^2, x \in [0, 1] \),
\[ B_n(f_1, x) - f_1(x) = \frac{x(1-x)}{n} \approx \omega_2(f_1, \sqrt{\frac{x(1-x)}{n}}). \]
What is the situation for all other continuous functions \( f(x) \)? In response to this question, we formulate the following result.

**Theorem 3.9.** There are no constants \( c(f) \) and \( C(f) \) such that
\[ c(f) \Omega_2(f, \sigma(n, x)) \leq |B_n(f, x) - f(x)| \leq C(f) \Omega_2(f, \sigma(n, x)) \]
holds true for all \( f \in C[0, 1], all \ x \in [0, 1] \) and all \( n \in \mathbb{N} \) with appropriate constructive characteristic \( \Omega_2(f, \cdot) \), where \( \Omega_2(f, \cdot) \) satisfies the properties of second-order modulus of smoothness (or related K-functional) and argument \( \sigma(n, x) \).

**Proof.** The proof follows immediately from Theorem A and (2.2) for \( g(x) \). We fix \( n \in \mathbb{N} \) and take \( x \to 0 \). If we suppose that (3.4) holds true, this would imply simultaneously that
\[ \Omega_2(g, \sigma(n, x)) \leq k_1 \frac{x(1-x)}{n} \text{ as } x \to 0, \]
\[ \Omega_2(g, \sigma(n, x)) \geq K_2 \sqrt{\frac{x(1-x)}{n}} \text{ as } x \to 0, \]
with some positive constants \( k_1, K_2 \) independent of \( n, x \), which is not possible. Hence (3.4) fails for \( g(x) \). \( \square \)
Remark 3.10. The case of ‘norm’ estimates is quite different. We mention here the well-known equivalence result of Knoop and Zhou for Bernstein operators, namely

$$c\omega^2(f, \frac{1}{\sqrt{n}}) \leq \|B_n f - f\|_{C[0,1]} \leq C\omega^2(f, \frac{1}{\sqrt{n}})$$

established in 1994 in [7]. Similar strong converse inequalities are valid for many other linear positive operators.

Acknowledgements
This note is dedicated to the memory of late Professor Alexandru Lupăş, one of the organisers of Romanian–German Seminars on Approximation Theory and distinguished Romanian mathematician, teacher and friend.

References

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https://doi.org/10.1017/S0004972711002838 Published online by Cambridge University Press