Canad. J. Math. Vol. **59** (4), 2007 pp. 712–729

Jet Modules

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Abstract. In this paper we classify indecomposable modules for the Lie algebra of vector fields on a torus that admit a compatible action of the algebra of functions. An important family of such modules is given by spaces of jets of tensor fields.

Introduction

In recent years there has been substantial progress in the representation theory of infinite-dimensional Lie algebras of rank $n > 1$, toroidal Lie algebras in particular [EM, L, BB, B, B2]. In this paper we turn our attention to another Lie algebra of rank n, the Lie algebra W_n of vector fields on an n-dimensional torus $\mathbb{T}^n$.

An important class of irreducible representations for W_n has its origin in differential geometry — these are the modules of tensor fields on a torus. In addition to being modules for the Lie algebra of vector fields, tensor fields also admit multiplication by functions. For the torus, which is a flat manifold, the spaces of tensor fields are free modules of a finite rank over the commutative algebra of functions $\mathcal{F}(\mathbb{T}^n)$. We formalize this property in the definition of a category $\mathcal{J}$ of W_n-modules (*cf.* [ER]).

We also discuss another class of W_n-modules of a geometric nature — the modules of jets of tensor fields [S]. Jets of functions are used as a tool for the symmetry analysis for partial differential equations [O]. The action on a space of jets of the Lie algebra of vector fields, known under the term "prolongation of vector fields", plays a key role in that theory.

From the algebraic point of view, jet modules are typically not irreducible, but are often indecomposable. The goal of the present paper is to classify indecomposable modules in the category $\mathcal{J}$.

Let us state our result in case $n = 1$, for the sake of simplicity of notations.

Theorem *There is a one-to-one correspondence between indecomposable W_1-modules J in category $\mathcal{J}$ and pairs (λ, U), where $\lambda \in \mathbb{C}/\mathbb{Z}$ and U is a finite-dimensional indecomposable module for a Lie algebra*

$$W_1^+ = \mathrm{Span}\left\langle z^k \frac{d}{dz} \;\middle|\; k \geq 1 \right\rangle.$$

Such a correspondence is given by the tensor product decomposition

$$J = \mathcal{F}(\mathbb{T}^1) \otimes U,$$

Received by the editors December 6, 2004.
This research is supported by the Natural Sciences and Engineering Research Council of Canada.
AMS subject classification: 17B66, 58A20.

where W_1 acts according to the formula

$$\left(\frac{1}{2\pi i}e^{2\pi isx}\frac{d}{dx}\right)(e^{2\pi imx}\otimes u)=(m+\lambda)e^{2\pi i(s+m)x}\otimes u+\sum_{k\geq 1}\frac{s^k}{k!}e^{2\pi i(s+m)x}\otimes\rho\left(z^k\frac{d}{dz}\right)u.$$

The sum in the right-hand side is finite since for every finite-dimensional representation of W_n^+, $\rho\left(z^k\frac{d}{dz}\right)=0$ for $k\gg 1$.

This result reduces the classification of modules in $\mathcal{J}$ to a problem in a completely finite-dimensional set-up — describing finite-dimensional representations of certain finite-dimensional Lie algebras.

The case of rank one is of interest by itself since in this case we deal with the representations of the Virasoro algebra with a trivial action of the center, and the Virasoro algebra plays a prominent role in applications to physics. Only in the case $n=1$ we have a complete classification of irreducible modules with finite-dimensional weight spaces [M]. Very little is known about indecomposable W_n-modules even in the rank one case.

The concept of a polynomial module, introduced in [BB] (see also [BZ]), has turned out to be extremely useful for the present paper. We prove that all modules in category $\mathcal{J}$ are polynomial modules, and this property allows us to establish the classification result. It is interesting that unlike all previously known examples, the degrees of the structure polynomials for the modules in $\mathcal{J}$ can be arbitrarily high.

The technique developed in this paper also allows us to recover Eswara Rao's classification [ER] of irreducible modules in category $\mathcal{J}$, significantly simplifying his proof.

The structure of the paper is the following. In Section 1 we discuss the module structure on the space of jets of tensor fields. Motivated by the construction of jet modules, we introduce a category $\mathcal{J}$ in Section 2, and state our main theorems at the end of the section. In Sections 3 and 4 we prove the classification results for the modules in category $\mathcal{J}$. In Section 5 we give the analogous result for the semidirect product of W_n with a multi-loop algebra.

1 Jets of Tensor Fields

The group of diffeomorphisms of a manifold and its Lie algebra of vector fields have several natural constructions of modules coming from differential geometry. Examples of such modules include the space of functions on a manifold and more generally, the space of tensor fields. In addition to these, one can also consider the spaces of jets of tensor fields which also admit a natural action of the group of diffeomorphisms and the Lie algebra of vector fields [S].

In this section we will review the construction of the bundle of jets of tensor fields and the module structure on the vector space of its sections.

First, let us discuss the notations that will be used in the paper. We denote the set of non-negative integers by $\mathbb{Z}_+$, and consider a partial order on $\mathbb{Z}_+^n$ where $\alpha\geq\beta$ whenever $\alpha-\beta\in\mathbb{Z}_+^n$. The standard basis of $\mathbb{Z}^n$ will be denoted by $\{\epsilon_1,\dots,\epsilon_n\}$. For a coordinate system $\{x^i\}_{i=1,\dots,n}$ and $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$, we denote by $f^{(\alpha)}(x)$ the

https://doi.org/10.4153/CJM-2007-031-2 Published online by Cambridge University Press

partial derivative $\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n} f(x)$. We will use notations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $(x-p)^\alpha = (x^1 - p^1)^{\alpha_1} \cdots (x^n - p^n)^{\alpha_n}$, etc. We will also use a convention of dropping the summation symbol when we take a sum over matching upper and lower indices: $u^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}$.

Let M be an n-dimensional C^∞ manifold which is allowed to be either real or complex. However the spaces of functions on M, vector fields, etc., will be always taken to be complex spaces.

We begin by recalling the definition of an N-jet of a function. Let f_1, f_2 be two C^∞ functions defined in a neighbourhood of a point p on manifold M. We say that f_1 is equivalent to f_2 at p, $f_1 \sim f_2$, if all partial derivatives of f_1 and f_2 at p of orders up to N are equal:

$$f_1^{(\alpha)}(p) = f_2^{(\alpha)}(p), \quad \text{for all } 0 \le |\alpha| \le N.$$

An equivalence class for this relation is called an N-jet of a function at p. Any N-jet at p has a unique Taylor polynomial representative:

$$\sum_{0 \le |\alpha| \le N} f_{[\alpha]}(p)(x-p)^\alpha.$$

The set of N-jets at p forms a finite-dimensional vector space with coordinates $\{f_{[\alpha]}(p) \mid 0 \le |\alpha| \le N\}$ and a basis $\{(x-p)^\alpha \mid 0 \le |\alpha| \le N\}$. If we now let p vary over M, we get the vector bundle of N-jets of functions. Let us denote by $J_N(M)$ the space of sections of this bundle.

Now we are going to describe the action of the group of diffeomorphisms $\mathrm{Diff}(M)$ on $J_N(M)$. Suppose for a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ we have $\varphi(p) = q$. Set $\psi = \varphi^{-1}$. Let $\{x^i\}$ be a coordinate system near p and $\{y^i\}$ a coordinate system near q.

Let F be a section of an N-jet bundle, $F \in J_N(M)$, with its value at p given by the equivalence class

$$F(p) \sim \sum_{0 \le |\alpha| \le N} f_{[\alpha]}(p)(x-p)^\alpha.$$

Then the value of the section φF at point q is defined by the jet

$$\begin{aligned} \varphi F(q) &\sim \sum_{0 \le |\alpha| \le N} f_{[\alpha]}(p)(\psi(y) - \psi(q))^\alpha \\ &\sim \sum_{0 \le |\alpha| \le N} f_{[\alpha]}(p)\Big(\sum_{\beta > 0} \frac{1}{\beta!}\psi^{(\beta)}(q)(y-q)^\beta\Big)^\alpha. \end{aligned} \tag{1.1}$$

The coordinates of φF at q are then computed by expanding the last expression in powers of $(y-q)$, and dropping terms of degrees greater than N.

If we pass to the infinitesimal action, we will obtain the action of the Lie algebra of vector fields $\mathrm{Vect}(M)$ on $J_N(M)$.

For a vector field $\overline{u} = u^i(x)\frac{\partial}{\partial x^i}$ we consider the corresponding flow $\varphi_\epsilon \in \mathrm{Diff}(M)$, $\varphi_\epsilon^i = x^i - \epsilon u^i(x) + o(\epsilon)$. We now assume that the point p and its image q under φ_ϵ are in the same chart with coordinates $\{x^i\}$. Then for the inverse, $\psi_\epsilon = \varphi_\epsilon^{-1} \in \mathrm{Diff}(M)$, we have $\psi_\epsilon^i = x^i + \epsilon u^i(x) + o(\epsilon)$, and (1.1) becomes

$$\begin{aligned}\varphi_\epsilon F(q) &\sim \sum_{0\le|\alpha|\le N} f_{[\alpha]}(\psi_\epsilon(q))\Big(\sum_{\beta>0}\frac{1}{\beta!}\psi_\epsilon^{(\beta)}(q)(x-q)^\beta\Big)^\alpha \\ &\sim F(q) + \epsilon \sum_{0\le|\alpha|\le N} u^i(q)\frac{\partial f_{[\alpha]}}{\partial x^i}(q)(x-q)^\alpha \\ &\quad + \epsilon \sum_{0\le|\alpha|\le N} f_{[\alpha]}(q)\sum_{j=1}^n \alpha_j \sum_{\beta>0}\frac{1}{\beta!}(u^j)^{(\beta)}(q)(x-q)^{\alpha-\epsilon_j+\beta} + o(\epsilon).\end{aligned}$$

Thus the action of $\mathrm{Vect}(M)$ on $J_N(M)$ is given by the formula

$$\begin{aligned}\overline{u}F(q) &\sim \sum_{0\le|\alpha|\le N} u^i(q)\frac{\partial f_{[\alpha]}}{\partial x^i}(q)(x-q)^\alpha \\ &\quad + \sum_{0\le|\alpha|\le N} f_{[\alpha]}(q)\sum_{j=1}^n \alpha_j \sum_{\beta>0}\frac{1}{\beta!}(u^j)^{(\beta)}(q)(x-q)^{\alpha-\epsilon_j+\beta}.\end{aligned} \tag{1.2}$$

We can see from these formulas that the subspace of sections with derivatives up to order ℓ vanishing everywhere, $\{F \in J_N(M) \mid f_{[\alpha]} \equiv 0 \text{ for all } 0 \le |\alpha| \le \ell\}$, is a submodule for the actions of both $\mathrm{Diff}(M)$ and $\mathrm{Vect}(M)$. The factor-module in this case is isomorphic to the space $J_\ell(M)$ of sections of the bundle of ℓ-jets.

Now let us look at a more general case of tensor fields. A tensor field of type (s,k) is a section of the corresponding tensor bundle. Suppose that a tensor field F is given in a chart with local coordinates $\{x^i\}$ by the expression

$$F(x) = f_{(j_1\cdots j_k)}^{(i_1\cdots i_s)}(x)dx^{j_1}\cdots dx^{j_k}\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_s}}.$$

Then the diffeomorphism $\varphi \in \mathrm{Diff}(M)$ with $\varphi^{-1} = \psi$ acts on F according to the formula

$$\begin{aligned}\varphi F(y) & f_{(j_1\cdots j_k)}^{(i_1\cdots i_s)}(\psi(y))\frac{\partial\psi^{j_1}}{\partial y^{j_1'}}\cdots\frac{\partial\psi^{j_k}}{\partial y^{j_k'}}\frac{\partial\varphi^{i_1'}}{\partial x^{i_1}}(\psi(y))\cdots\frac{\partial\varphi^{i_s'}}{\partial x^{i_s}}(\psi(y)) \\ &\times dy^{j_1'}\cdots dy^{j_k'}\frac{\partial}{\partial y^{i_1'}}\cdots\frac{\partial}{\partial y^{i_s'}}.\end{aligned} \tag{1.3}$$

The corresponding action of the Lie algebra $\mathrm{Vect}(M)$ can be conveniently encoded using representations of the Lie algebra gl_n. Let us explain this construction. The tensor bundle in question is a tensor product of s copies of the tangent bundle and k copies of the cotangent bundle. For a given coordinate system $\{x^i\}$, there is an action of the Lie algebra gl_n on the cotangent space where we set $\{dx^i\}$ as the standard

https://doi.org/10.4153/CJM-2007-031-2 Published online by Cambridge University Press

basis of the natural gl_n-module. The tangent space becomes the dual of the natural module for gl_n. The action of the elementary matrices E^p_q (a matrix that has entry 1 in position (p,q) and zeros elsewhere) on the tangent and cotangent spaces is given by the formulas:

$$E^p_q dx^i = \delta^i_q dx^p, \quad E^p_q \frac{\partial}{\partial x^i} = -\delta^p_i \frac{\partial}{\partial x^q}.$$

The fiber $V = \operatorname{Span}\langle dx^{j_1}\cdots dx^{j_k}\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_s}} \mid 1 \le j_1,\dots,j_k, i_1,\dots i_s \le n\rangle$ of the (s,k) tensor bundle gets the structure of a gl_n-module as a tensor product.

In local coordinates $\{x^i\}$, any tensor field of type (s,k) is a linear combination of $f(x)v$, where $v \in V$. The action of a vector field $\overline{u} = u^i(x)\frac{\partial}{\partial x^i}$ is then given by the (Lie derivative) formula:

$$\overline{u}(f(x)v) = \Big(u^i(x)\frac{\partial f}{\partial x^i}\Big)v + \Big(f(x)\frac{\partial u^i}{\partial x^j}\Big)E^j_i v. \tag{1.4}$$

Just as in the case of functions, we can introduce the bundle of N-jets of tensor fields. The fiber at a point p is spanned by the jets

$$v^{(\alpha)} = (x-p)^\alpha v,\ \ v \in V, 0 \le |\alpha| \le N.$$

Again, let us define the action of $\operatorname{Diff}(M)$ on the space $J_N T^{(s,k)}(M)$ of sections of this bundle. If we have a section F with value at a point p given by an equivalence class of a local tensor field

$$F(p) \sim \sum_{0\le|\alpha|\le N} f^{(i_1\dots i_s)}_{(j_1\dots j_k)[\alpha]}(p)(x-p)^\alpha dx^{j_1}\cdots dx^{j_k}\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_s}},$$

then φF is a section with value at $q = \varphi(p)$ given by the equivalence class of the image of this local tensor field under the action of φ according to (1.3):

$$\begin{aligned}\varphi F(q) \sim \sum_{0\le|\alpha|\le N} & f^{(i_1\dots i_s)}_{(j_1\dots j_k)[\alpha]}(\psi(q))(\psi(y)-\psi(q))^\alpha \\ & \times \frac{\partial\psi^{j_1}}{\partial y^{j_1'}}\cdots\frac{\partial\psi^{j_k}}{\partial y^{j_k'}}\frac{\partial\varphi^{i_1'}}{\partial x^{i_1}}(\psi(y))\cdots\frac{\partial\varphi^{i_s'}}{\partial x^{i_s}}(\psi(y))dy^{j_1'}\cdots dy^{j_k'}\frac{\partial}{\partial y^{i_1'}}\cdots\frac{\partial}{\partial y^{i_s'}}.\end{aligned}$$

The infinitesimal variant of the above formula yields the action of the Lie algebra $\operatorname{Vect}(M)$ on $J_N T^{(s,k)}(M)$:

$$\begin{aligned}\overline{u}(f(x)v^{(\alpha)}) = \Big(u^i(x)\frac{\partial f(x)}{\partial x_i}\Big)v^{(\alpha)} &+ \sum_{j=1}^n \alpha_j \sum_{\beta>0}\frac{1}{\beta!}f(x)(u^j)^{(\beta)}(x)v^{(\alpha-\epsilon_j+\beta)} \\ &+ \sum_{j=1}^n\sum_{\beta\ge0}\frac{1}{\beta!}f(x)(u^i)^{(\beta+\epsilon_j)}(x)(E^j_i v)^{(\alpha+\beta)}.\end{aligned} \tag{1.5}$$

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We would like to point out some properties of the module $J_N T^{(s,k)}(M)$. In addition to being a module for the Lie algebra $\mathrm{Vect}(M)$, it is also a module over a commutative algebra $\mathcal{F}(M)$ of functions on M. Moreover, (1.5) shows that the two structures are compatible in the following way:

$$\overline{u}(f(x)F) = (\overline{u}f(x))F + f(x)(\overline{u}F), \quad \overline{u} \in \mathrm{Vect}(M), f \in \mathcal{F}(M), F \in J_N T^{(s,k)}(M).$$

We see that vector fields act as derivations of the multiplication of the jets of tensor fields by functions. This motivates the definition of a category of modules which will be introduced in the next section.

2 Category $\mathcal{J}$

For the rest of the paper, the manifold will be an n-dimensional torus $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We consider the algebra $\mathcal{F}(\mathbb{T}^n)$ of Fourier polynomials on the torus with the basis $\{e^{2\pi i m x} \mid m \in \mathbb{Z}^n\}$. The Lie algebra $\mathrm{Vect}(\mathbb{T}^n)$, also denoted as W_n, is a free module over $\mathcal{F}(\mathbb{T}^n)$ of rank n with a basis (as an $\mathcal{F}(\mathbb{T}^n)$-module) $\left\{ d_j = \frac{1}{2\pi i}\frac{\partial}{\partial x^j} \mid j = 1, \dots, n \right\}$. A basis of W_n over $\mathbb{C}$ is given by

$$\left\{ d_j(s) = \frac{1}{2\pi i} e^{2\pi i s x} \frac{\partial}{\partial x^j} \;\middle|\; s \in \mathbb{Z}^n, j = 1, \dots, n \right\}.$$

The subspace spanned by $\{d_j\}_{j=1,\dots,n}$ is a Cartan subalgebra in W_n.

Note that because the torus is a flat manifold, all bundles considered in Section 1 are trivial. Equivalently, the spaces of sections of these bundles are free modules over $\mathcal{F}(\mathbb{T}^n)$.

Let us define the following category $\mathcal{J}$ of W_n-modules:

Definition A W_n-module J belongs to category $\mathcal{J}$ if the following properties hold:

(J1) The action of d_j, $j = 1, \dots, n$, on J is diagonalizable.
(J2) J is a free $\mathcal{F}(\mathbb{T}^n)$-module of a finite rank.
(J3) For any $\overline{u} \in W_n$, $f \in \mathcal{F}(\mathbb{T}^n)$, $w \in J$, $\overline{u}(fw) = (\overline{u}f)w + f(\overline{u}w)$.

Strictly speaking, we should denote this category $\mathcal{J}_n$, but we will omit the subscript most of the time. All the $\mathrm{Vect}(\mathbb{T}^n)$-modules discussed in Section 1 belong to category $\mathcal{J}$.

The goal of this paper is to classify indecomposable and irreducible modules in this category. Irreducible modules in $\mathcal{J}$ have been already classified by Eswara Rao [ER], but here we simplify the proof and make it more lucid. Eswra Rao constructed a class of indecomposable W_n-modules.[1]

Remark When we talk about a submodule for a module in category $\mathcal{J}$, we mean a subspace which is invariant under the action of both W_n and $\mathcal{F}(\mathbb{T}^n)$. Also, in this paper an indecomposable module will be understood to be non-zero.

[1] S. Eswara Rao, *A new class of modules for derivations of Laurent polynomial ring in n variables.* Unpublished ms, 2003.

The concept of a polynomial module will play a central role in our proof. The definition of a polynomial module was given in [BB] in a general setup. Here we adapt that definition for the particular case of W_n. We will show below that one can choose a basis $v_1, \dots, v_k$ of J over $\mathcal{F}(\mathbb{T}^n)$ in such a way that the action of W_n is as follows:

$$d_j(s)(e^{2\pi i m x} v_r) = \sum_{\ell=1}^{k} f_{jr\ell}(s, m) e^{2\pi i (m+s) x} v_\ell, \quad s, m \in \mathbb{Z}^n. \tag{2.1}$$

We say that module J is *a polynomial module* if the structure constants $f_{jr\ell}(s, m)$ are polynomials in $s, m \in \mathbb{Z}^n$.

The desired classification of modules in $\mathcal{J}$ will be obtained in three steps by proving the following theorems (see the next section for more detailed statements).

Theorem 2.1 *Let $n = 1$. Every W_1-module in category $\mathcal{J}_1$ is a polynomial module.*

We will see that with very little effort one can derive from Theorem 2.1 its generalization to an arbitrary rank.

Theorem 2.2 *Every W_n-module in category $\mathcal{J}$ is a polynomial module.*

From this theorem we almost immediately deduce the classification of the indecomposable modules in category $\mathcal{J}$. In order to state this result, we need to introduce the Lie algebra W_n^+.

Consider the Lie algebra of derivations of the algebra of polynomials in n variables:

$$\mathrm{Der}(\mathbb{C}[z_1, \dots, z_n]) = \mathrm{Span}\langle z^\alpha \frac{\partial}{\partial z_j} \mid \alpha \in \mathbb{Z}_+^n, j = 1, \dots, n\rangle.$$

The Lie algebra W_n^+ is defined as a subalgebra in $\mathrm{Der}\big(\mathbb{C}[z_1, \dots, z_n]\big)$:

$$W_n^+ = \mathrm{Span}\Big\langle z^\alpha \frac{\partial}{\partial z_j} \;\Big|\; \alpha \in \mathbb{Z}_+^n \backslash \{0\}, j = 1, \dots, n \Big\rangle.$$

Theorem 2.3 *There is a one-to-one correspondence between indecomposable modules in category $\mathcal{J}$ and pairs (λ, U), where $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$ and U is an indecomposable finite-dimensional module for W_n^+.*

3 Structure of Indecomposable Modules

The finiteness condition (J2) implies that every module in the category $\mathcal{J}$ is a finite direct sum of indecomposable submodules. This allows us to restrict our attention to indecomposable modules.

Let us write the weight decomposition of an indecomposable module J with respect to the Cartan subalgebra of W_n:

$$J = \bigoplus_{\mu \in \mathbb{C}^n} J_\mu,$$

where $J_\mu = \{w \in J \mid d_j w = \mu_j w\}$. It is easy to see that $d_j(s)\mathcal{J}_\mu \subseteq J_{\mu+s}$ and $e^{2\pi isx}\mathcal{J}_\mu \subseteq J_{\mu+s}$, thus the weights of J are split into $\mathbb{Z}^n$-cosets in $\mathbb{C}^n$, and the submodules corresponding to distinct cosets form a direct sum. Since we assumed J to be indecomposable, its weight lattice is a single coset $\lambda + \mathbb{Z}^n$, $\lambda \in \mathbb{C}^n$. Let us denote the weight space J_λ by U. It is easy to see that the basis of U is also a basis of J as an $\mathcal{F}(\mathbb{T}^n)$-module. Thus, by (J2), the space U is finite-dimensional and $J = \mathcal{F}(\mathbb{T}^n) \otimes U$.

Since $e^{2\pi isx}U = J_{\lambda+s}$, we may identify each weight space of J with the same finite-dimensional space U. The operator $d_j(s)\colon U \to e^{2\pi isx}U$ induces an endomorphism $D_j(s)\colon U \to U$, such that $d_j(s)|_U = e^{2\pi isx}D_j(s)$. In particular we have $D_j(0) = \lambda_j \operatorname{Id}$.

The finite-dimensional operator $D_j(s) \in \operatorname{End} U$ completely determines the action of $d_j(s)$ on J since by (J3),

$$d_j(s)(e^{2\pi imx}v) = (d_j(s)e^{2\pi imx})v + e^{2\pi imx}d_j(s)v = e^{2\pi i(m+s)x}(m_j \operatorname{Id} + D_j(s))v. \tag{3.1}$$

Now we see that the action of $d_j(s)$ is written in the form (2.1). The structure constants $f_{jr\ell}$ are encoded in the operators $m_j \operatorname{Id} + D_j(s)$. The dependence on m here is clearly polynomial, so the statement that J is a polynomial W_n-module is equivalent to the claim that the dependence of the family of operators $\{D_j(s)\}$ on s is polynomial.

Theorem 3.1 *An indecomposable W_n-module $J = \mathcal{F}(\mathbb{T}^n) \otimes U$ in category $\mathcal{J}$ is a polynomial module. The action of W_n can be written as follows:*

$$d_j(s)\left(e^{2\pi imx}v\right) = e^{2\pi i(m+s)x}\left(m_j \operatorname{Id} + D_j(s)\right)v,$$

where operators $D_j(s) \in \operatorname{End} U$ have a polynomial dependence on $s \in \mathbb{Z}^n$ and $D_j(0) = \lambda_j \operatorname{Id}$.

Proving Theorem 3.1 directly would be rather technical. Instead, we will derive it from its special case of $n = 1$, which is Theorem 2.1. The proof of Theorem 2.1 will be deferred to the next section.

First of all, let us write down the commutator relations between operators $D_j(s)$.

Lemma 3.2

$$[D_j(s), D_k(m)] = m_j(D_k(s+m) - D_k(m)) - s_k(D_j(s+m) - D_j(s)). \tag{3.2}$$

This lemma can be derived in a straightforward way from the commutator relations in W_n

$$[d_j(s), d_k(m)] = m_j d_k(s+m) - s_k d_j(s+m) \tag{3.3}$$

and (3.1). The details of this computation are left as an exercise.

Proof of Theorem 3.1 We assume that the claim of the theorem holds in rank one case, $n = 1$.

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Consider the following subalgebras in W_n, each isomorphic to W_1:

$$W_1^{(j)} = \mathrm{Span}\Big\langle \frac{1}{2\pi i} e^{2\pi i s x^j} \frac{\partial}{\partial x^j} \;\Big|\; s \in \mathbb{Z} \Big\rangle.$$

The subspace $\bigoplus_{m\in\mathbb{Z}} e^{2\pi i m x^j} U \subset J$ is a $W_1^{(j)}$-module which belongs to category $\mathcal{J}_1$. Applying Theorem 2.1, we get that the family of operators $\{D_j(s\epsilon_j)\}$ has a polynomial dependence on $s \in \mathbb{Z}$.

Without loss of generality we may restrict ourselves to proving that $D_1(s_1, \dots, s_n)$ is a polynomial in $s_1, \dots, s_n$. This is of course equivalent to showing that $D_1(s_1, 1 + s_2, \dots, 1 + s_n)$ is a polynomial in $s_1, s_2, \dots, s_n$. We will prove by induction the claim that $D_1(s_1, 1 + s_2, \dots, 1 + s_j, 1, \dots, 1)$ is a polynomial in $s_1, s_2, \dots, s_j$.

Let us establish the basis of induction. From (3.2) we get that

$$[D_1(s_1\epsilon_1), D_1(0, 1, \dots, 1)] = -s_1 D_1(s_1, 1, \dots, 1) + s_1 D_1(s_1\epsilon_1),$$

and so

$$s_1 D_1(s_1, 1, \dots, 1) = s_1 D_1(s_1\epsilon_1) - [D_1(s_1\epsilon_1), D_1(0, 1, \dots, 1)].$$

The right-hand side is manifestly a polynomial in s_1, and so is the left-hand side. Note also that the right-hand side vanishes at $s_1 = 0$ because $D_1(0) = \lambda_1 \,\mathrm{Id}$. Hence this polynomial has a factor of s_1, and thus $D_1(s_1, 1, \dots, 1)$ is a polynomial in s_1, which proves the basis of induction.

Let us now prove the inductive step. Again from (3.2) we get

$$\begin{aligned} &D_1(s_1, 1 + s_2, \dots, 1 + s_{j-1}, 1 + s_j, 1, \dots, 1) \\ &\qquad = \big[D_j(s_j\epsilon_j), D_1(s_1, 1 + s_2, \dots, 1 + s_{j-1}, 1, \dots, 1)\big] \\ &\qquad\quad + D_1(s_1, 1 + s_2, \dots, 1 + s_{j-1}, 1, \dots, 1). \end{aligned}$$

By induction assumption, the right-hand side is a polynomial in $s_1, \dots, s_{j-1}, s_j$, and so is the left-hand side. This completes the induction and Theorem 3.1 is now proved (under assumption of validity of Theorem 2.1).

Now given that $D_j(s)$ is a polynomial function with a constant term $D_j(0) = \lambda_j \,\mathrm{Id}$, we can expand it in a finite sum:

$$D_j(s) = \lambda_j \,\mathrm{Id} + \sum_{\alpha \in \mathbb{Z}_+^n \setminus \{0\}}^{\text{finite}} \frac{s^\alpha}{\alpha!} D_j^{(\alpha)}, \tag{3.4}$$

where the operators $D_j^{(\alpha)} \in \mathrm{End}\, U$ are independent of s and $D_j^{(\alpha)} = 0$ for all but finitely many $\alpha \in \mathbb{Z}_+^n \setminus \{0\}$. ∎

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Theorem 3.3

(i) *The operators* $D_j^{(\alpha)} \in \operatorname{End} U$ *yield a finite-dimensional representation on space* U *of a Lie algebra*

$$W_n^+ = \operatorname{Span}\left\langle z^\alpha \frac{\partial}{\partial z_j} \;\middle|\; \alpha \in \mathbb{Z}_+^n \backslash \{0\},\ j = 1, \dots, n \right\rangle,$$

given by $\rho\left(z^\alpha \frac{\partial}{\partial z_j}\right) = D_j^{(\alpha)}$.

(ii) *There is a one-to-one correspondence between indecomposable modules in category* $\mathcal{J}$ *and pairs* (λ, U), *where* $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$ *and* U *is an indecomposable finite-dimensional module for* W_n^+. *This correspondence is given by the tensor product decomposition* $J = \mathcal{F}(\mathbb{T}^n) \otimes U$, *and the action of* W_n *is*

$$\frac{1}{2\pi i} e^{2\pi i s x} \frac{\partial}{\partial x^j} (e^{2\pi i m x} v) = (m_j + \lambda_j) e^{2\pi i (m+s)x} v + \sum_{\beta > 0} \frac{s^\beta}{\beta!} e^{2\pi i (m+s)x} \rho\left(z^\beta \frac{\partial}{\partial z_j}\right) v. \tag{3.5}$$

(iii) *There is a one-to-one correspondence between irreducible modules in category* $\mathcal{J}$ *and pairs* (λ, V), *where* $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$ *and* V *is an irreducible* $gl_n(\mathbb{C})$*-module. The action of* W_n *on* $J = \mathcal{F}(\mathbb{T}^n) \otimes V$ *is as follows:*

$$\frac{1}{2\pi i} e^{2\pi i s x} \frac{\partial}{\partial x^j} (e^{2\pi i m x} v) = (m_j + \lambda_j) e^{2\pi i (m+s)x} v + \sum_{p=1}^{n} s_p e^{2\pi i (m+s)x} \rho(E_j^p) v. \tag{3.6}$$

Proof Let us determine the commutator relations between operators $D_j^{(\alpha)}$. We have

$$[D_j(s), D_k(m)] = \sum_{\alpha, \beta \in \mathbb{Z}_+^n \backslash \{0\}} \frac{s^\alpha m^\beta}{\alpha! \beta!} \left[D_j^{(\alpha)}, D_k^{(\beta)} \right].$$

On the other hand, by Lemma 3.2,

$$[D_j(s), D_k(m)] = \sum_{\gamma \in \mathbb{Z}_+^n \backslash \{0\}} m_j \frac{(s+m)^\gamma - m^\gamma}{\gamma!} D_k^{(\gamma)} - \sum_{\gamma \in \mathbb{Z}_+^n \backslash \{0\}} s_k \frac{(s+m)^\gamma - s^\gamma}{\gamma!} D_j^{(\gamma)}.$$

Two polynomials have equal values whenever their coefficients coincide. If we equate the coefficients at $\frac{s^\alpha m^\beta}{\alpha! \beta!}$, $\alpha, \beta \in \mathbb{Z}_+^n \backslash \{0\}$, we get

$$\left[D_j^{(\alpha)}, D_k^{(\beta)} \right] = \beta_j D_k^{(\alpha+\beta-\epsilon_j)} - \alpha_k D_j^{(\alpha+\beta-\epsilon_k)}. \tag{3.7}$$

We see that these are precisely the commutator relations in the algebra W_n^+:

$$\left[z^\alpha \frac{\partial}{\partial z_j}, z^\beta \frac{\partial}{\partial z_k} \right] = \beta_j z^{\alpha+\beta-\epsilon_j} \frac{\partial}{\partial z_k} - \alpha_k z^{\alpha+\beta-\epsilon_k} \frac{\partial}{\partial z_j},$$

so the map $\rho\colon W_n^+ \to \operatorname{End} U$, given by $\rho\left(z^\alpha \frac{\partial}{\partial z_j}\right) = D_j^{(\alpha)}$, is a representation of W_n^+. This completes the proof of part (i).

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Let us prove (ii). We have already seen that an indecomposable module J in category $\mathcal{J}$ yields a coset of weights $\lambda + \mathbb{Z}^n \subset \mathbb{C}^n$, and a finite-dimensional representation of W_n^+. It is easy to check that the W_n^+-module U is independent of the choice of the weight λ in the coset. Conversely, the commutator relations (3.7) imply (3.2) and together with (3.1) give (3.3), provided, of course, that the right-hand side in (3.4) is finite. Thus we need to show that for a finite-dimensional representation (U, ρ) of the Lie algebra W_n^+, we will have that $D_j^{(\alpha)} = \rho\big(z^\alpha \frac{\partial}{\partial z_j}\big) = 0$ for all but finitely many $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$. This will follow from the following simple lemma.

Lemma 3.4 *Let $\mathcal{L}$ be a Lie algebra, and (U, ρ) its finite-dimensional representation. Suppose that for $x, y_1, y_2, \ldots \in \mathcal{L}$ we have*

$$[x, y_k] = \nu_k y_k, \quad \nu_k \in \mathbb{C}\, k = 1, 2, \ldots .$$

Then there are at most $(\dim U)^2 - \dim U + 1$ distinct eigenvalues for which $\rho(y_k) \neq 0$.

Proof In representation ρ we have $[\rho(x), \rho(y_k)] = \nu_k \rho(y_k)$. However an element $\rho(x)$ in the Lie algebra $gl(U)$ may have at most $(\dim U)^2 - \dim U + 1$ distinct eigenvalues in the adjoint representation, which implies the claim of the Lemma. ■

To apply this lemma, we consider the element $E = z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n} \in W_n^+$. The following relations hold:

$$\Big[E, z^\alpha \frac{\partial}{\partial z_j}\Big] = (|\alpha| - 1) z^\alpha \frac{\partial}{\partial z_j}.$$

Then by Lemma 3.4, $\rho\big(z^\alpha \frac{\partial}{\partial z_j}\big) = 0$ for all but finitely many α.

To complete the proof of part (ii) of Theorem 3.3, we note that an indecomposable W_n-module in category $\mathcal{J}$ yields an indecomposable W_n^+-module and vice versa.

Now let us prove part (iii) and assume that J is an irreducible module in category $\mathcal{J}$. From part (ii) we know that such a module is determined by a pair (λ, V), where $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$ and V is a finite-dimensional W_n^+-module. The module V has to be irreducible, because we have a correspondence between W_n^+-submodules $S \subset V$ and W_n-submodules $\mathcal{F}(\mathbb{T}^n) \otimes S \subset J$.

Let us show that irreducible finite-dimensional W_n^+-modules V are just irreducible $gl_n(\mathbb{C})$-modules. Let V_ν be an eigenspace of the operator $\rho(E)$ in V corresponding to eigenvalue ν. It is easy to see that

$$\rho\Big(z^\alpha \frac{\partial}{\partial z_j}\Big) V_\nu \subseteq V_{\nu + |\alpha| - 1}.$$

This implies that $\bigoplus_{k=1}^\infty V_{\nu+k}$ is a W_n^+-submodule in V. However V is irreducible, which implies that $V_{\nu+k} = (0)$ for all $k = 1, 2, \ldots$, and hence

$$\rho\Big(z^\alpha \frac{\partial}{\partial z_j}\Big) = 0 \quad \text{for all } |\alpha| > 1.$$

Thus the ideal

$$W_n^{++} = \mathrm{Span}\Big\langle z^\alpha \frac{\partial}{\partial z_j} \;\Big|\; \alpha \in \mathbb{Z}_+^n,\ |\alpha| > 1,\ j = 1, \dots, n \Big\rangle$$

vanishes in every finite-dimensional irreducible W_n^+-module. But $W_n^+/W_n^{++} \cong gl_n(\mathbb{C})$, and the claim of part (iii) of the theorem follows. ■

Next, let us give an example of a family of indecomposable finite-dimensional W_n^+-modules.

Example 1 Let V be an irreducible finite-dimensional $gl_n(\mathbb{C})$-module and let

$$\widetilde{V} = \mathbb{C}[z_1, \dots, z_n] \otimes V.$$

Then we can define on $\widetilde{V}$ the structure of a tensor module for the Lie algebra $\mathrm{Der}\,\mathbb{C}[z_1, \dots, z_n]$ and its subalgebra W_n^+ (*cf.* (1.4), see also [Ru]):

$$z^\beta \frac{\partial}{\partial z_j}(z^\alpha v) = \alpha_j z^{\alpha+\beta-\epsilon_j} v + \sum_{k=1}^n \beta_k z^{\alpha+\beta-\epsilon_k}(E_j^k v).$$

As a W_n^+-module, this tensor module has finite-dimensional factors

$$V_N = \widetilde{V}/\langle z^\alpha \otimes V \mid |\alpha| > N\rangle.$$

Using the correspondence of Theorem 3.3(ii), we construct a representation of the Lie algebra $\mathrm{Vect}(\mathbb{T}^n)$ on space $\mathcal{F}(\mathbb{T}^n) \otimes V_N$ (setting $\lambda = 0$):

$$\begin{aligned}\frac{1}{2\pi i} e^{2\pi i s x} \frac{\partial}{\partial x^j}(e^{2\pi i m x} z^\alpha v) &= m_j e^{2\pi i(m+s)x} z^\alpha v \\ &\quad + \alpha_j \sum_{\beta>0} \frac{s^\beta}{\beta!} e^{2\pi i(m+s)x} z^{\alpha+\beta-\epsilon_j} v \\ &\quad + \sum_{\beta>0} \frac{s^\beta}{\beta!} \sum_{k=1}^n \beta_k e^{2\pi i(m+s)x} z^{\alpha+\beta-\epsilon_k} E_j^k v.\end{aligned} \tag{3.8}$$

Comparing (3.8) with (1.5), we see that the module $\mathcal{F}(\mathbb{T}^n) \otimes V_N$ is in fact isomorphic to a module of sections of N-jets of tensor fields corresponding to the gl_n-module V. The isomorphism is given by $z^\alpha v = (2\pi i)^{|\alpha|} v^{(\alpha)}$.

Remark It is curious to point out that for this family of polynomial modules, the degree of the structure polynomials is equal to $N + 1$ and thus could be arbitrarily high. In the previously known examples (see [BB, BZ]) the degree was at most 3.

Example 2 Let V_N be the W_n^+-module defined in Example 1, corresponding to the trivial 1-dimensional gl_n-module V. Now take its dual W_n^+-module V_N^* and use the correspondence of Theorem 3.3 to construct a W_n-module $\mathcal{F}(\mathbb{T}^n) \otimes V_N^*$. One can show that $\mathcal{F}(\mathbb{T}^n) \otimes V_N^*$ is the module of the differential operators of orders less than or equal to N studied in [GO].

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4 Rank One Case

In this section we will give the proof of Theorem 2.1. Since we will deal exclusively with the case $n = 1$, we may simplify notations denoting x^1 as x, $d_1(s)$ as $d(s)$, etc. Let J be an indecomposable module in category $\mathcal{J}_1$. As in Section 3 we see that $J = \mathcal{F}(\mathbb{T}^1) \otimes U$. Then (3.1) becomes

$$d(s)(e^{2\pi imx}v) = e^{2\pi i(m+s)x}(m \operatorname{Id} + D(s))v, \quad s, m \in \mathbb{Z}. \tag{4.1}$$

The operators $D(s) \in \operatorname{End} U$ satisfy the commutator relations, *cf.* (3.2):

$$[D(s), D(m)] = (m - s)D(s + m) - mD(m) + sD(s). \tag{4.2}$$

Theorem 4.1 *Every W_1-module J in category $\mathcal{J}_1$ is a polynomial module.*

Proof Just as in Section 3, we will assume J to be indecomposable, and so (4.1) holds. We will prove this theorem by showing that a family of operators $\{D(s)\}$ on a finite-dimensional space U satisfying (4.2) must have a polynomial dependence on $s \in \mathbb{Z}$.

The relations (4.2) define an infinite-dimensional Lie algebra $\mathcal{L}$ with basis $\{D(s) \mid s \in \mathbb{Z}\}$, and we are studying a finite-dimensional representation ρ of $\mathcal{L}$ on space U.

Our strategy will be the following. First we will find eigenvectors of $D(-1)$ in the adjoint representation of $\mathcal{L}$. We shall see that these eigenvectors are difference derivatives of $D(s)$ with respect to s. By applying Lemma 3.4, we will conclude that higher order derivatives of $D(s)$ vanish, which means that $D(s)$ is a polynomial in s.

Let us define the operator of a forward difference. Let $f\colon \mathbb{Z} \to A$ be a function of an integer variable with values in an abelian group A (in our case A is the vector space $\mathcal{L}$). The *forward difference* ∂f is a function $\partial f\colon \mathbb{Z} \to A$, defined by $\partial f(s) = f(s+1) - f(s)$. By iteration, we can also define higher order forward differences of f. It is easy to see that

$$\partial^m f(s) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(s + k). \tag{4.3}$$

Let us now additionally assume that A is a module over $\mathbb{Q}$, so that we can define interpolation polynomials. For any $N + 1$ distinct points $r_1, \dots, r_{N+1} \in \mathbb{Z}$ and any set of values $a_1, \dots, a_{N+1} \in A$, there exists a unique polynomial $f(t) \in A[t]$ of degree at most N such that $f(r_j) = a_j$ for $j = 1, \dots, N + 1$ [vdW, §22].

Lemma 4.2 *Fix $s \in \mathbb{Z}$.*

(i) *If $\partial^m f(s) = 0$ for all $m \geq 0$, then $f(r) = 0$ for all $r \geq s$.*

(ii) *Suppose $\partial^m f(s) = 0$ for all $m > N$. Let $g(t)$ be the interpolation polynomial of degree at most N defined by $g(j) = f(j)$ for $j = s, s + 1, \dots, s + N$. Then $f(r) = g(r)$ for all $r \geq s$.*

Proof Part (i) can be easily proved by induction using (4.3). To prove part (ii), we consider the function $h(r) = f(r) - g(r)$. Since $h(s) = \cdots = h(s+N) = 0$, we get from (4.3) that $\partial^m h(s) = 0$ for $m = 0, \dots, N$. For $m > N$, we have $\partial^m f(s) = 0$ by assumption and $\partial^m g(s) = 0$ since it is a polynomial of degree at most N. Thus $\partial^m h(s) = 0$ for all $m \geq 0$, and according to part (i) of the lemma, $f(r) = g(r)$ for all $r \geq s$. This completes the proof of the lemma. ■

Now consider the following elements in the Lie algebra $\mathcal{L}$:

$$y_m = \partial^{m+1} D(-1) = \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} D(-1+k).$$

Lemma 4.3

(i) *For $m \geq 0$, y_m is an eigenvector for* $\operatorname{ad} D(-1)$ *with eigenvalue $\nu_m = -m$.*
(ii) *For $m, k \geq 0$, $[y_k, y_m] = (m-k) y_{m+k}$.*

Proof In the calculation below we will use two elementary properties of binomial coefficients:

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} = 0 \text{ for } m \geq 1, \quad \text{and} \quad (r+1)\binom{m}{r+1} = (m-r)\binom{m}{r}.$$

Now,

$$\begin{aligned}
[D(-1), y_{m-1}] &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} [D(-1), D(-1+k)] \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \big(kD(-2+k) - (k-1)D(-1+k) - D(-1) \big) \\
&= -\sum_{\substack{r=0 \\ r=k-1}}^{m-1} (-1)^{m-r} (r+1) \binom{m}{r+1} D(-1+r) \\
&\quad - \sum_{k=0}^{m} (-1)^{m-k} (k-1) \binom{m}{k} D(-1+k) \\
&\quad - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} D(-1)
\end{aligned}$$

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$$
\begin{aligned}
&= -\sum_{r=0}^{m}(-1)^{m-r}(m-r)\binom{m}{r}D(-1+r) \\
&\quad -\sum_{r=0}^{m}(-1)^{m-r}(r-1)\binom{m}{r}D(-1+r) \\
&= (-m+1)\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}D(-1+r) = (-m+1)y_{m-1}.
\end{aligned}
$$

Part (ii) of the lemma will not be used in this paper and its proof is left as an exercise. ■

Now we are ready to complete the proof of Theorem 2.1. Combining Lemma 4.3(i) with Lemma 3.4, we conclude that there exists N such that

$$\rho(y_m) = \rho(\partial^{m+1}D(-1)) = 0$$

for all $m \geq N$. Then by Lemma 4.2(ii), there exists an End U-valued polynomial $g(s)$ such that $\rho(D(r)) = g(r)$ for all $r \geq -1$. It only remains to prove that $\rho(D(r)) = g(r)$ for all $r \in \mathbb{Z}$.

To achieve this, we take $p = 2, 3, 4, \ldots$ and consider a subalgebra $\mathcal{L}_p \subset \mathcal{L}$:

$$\mathcal{L}_p = \operatorname{Span}\langle D(pk) \mid k \in \mathbb{Z}\rangle.$$

It is easy to see that the map $\theta_p \colon \mathcal{L}_p \to \mathcal{L}$ defined by $\theta_p(D(pk)) = pD(k)$, is an isomorphism. Thus everything we proved for $\mathcal{L}$ is also valid for $\mathcal{L}_p$. This means that there exists a polynomial $g_p(s)$ such that $\rho(D(pr)) = g_p(pr)$ for all $r \geq -1$. Since the values of the polynomials $g(s)$ and $g_p(s)$ coincide at infinitely many points, we conclude that $g_p(s) = g(s)$. Taking now $r = -1$ and letting $p = 2, 3, \ldots$, we get that $\rho(D(-p)) = g(-p)$. Thus $\rho(D(r)) = g(r)$ for all $r \in \mathbb{Z}$. Theorem 1 is now proved. ■

5 Modules for the Semidirect Product of $\mathrm{Vect}(\mathbb{T}^n)$ with a Multi-Loop Algebra

Let $\dot{\mathfrak{g}}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. Consider a multi-loop Lie algebra

$$\widetilde{\mathfrak{g}} = \mathcal{F}(\mathbb{T}^n) \otimes \dot{\mathfrak{g}}.$$

The Lie algebra $W_n = \mathrm{Vect}(\mathbb{T}^n)$ acts in a natural way on the multi-loop algebra, so we can form the semidirect product $\mathfrak{g} = W_n \oplus \widetilde{\mathfrak{g}}$. We define the category $\mathcal{J}$ consisting of $\mathfrak{g}$-modules J satisfying (J1)–(J3) and also

(J4) $$\widetilde{g}\left(fv\right) = f(\widetilde{g}v), \quad \text{for } \widetilde{g} \in \widetilde{\mathfrak{g}},\ f \in \mathcal{F}(\mathbb{T}^n),\ v \in J.$$

Theorem 5.1

(i) *Every $\mathfrak{g}$-module in category $\mathcal{J}$ is a polynomial module.*

(ii) *There exists a one-to-one correspondence between indecomposable $\mathfrak{g}$-modules in category $\mathcal{J}$ and pairs (λ, U), where $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$ and U is a finite-dimensional indecomposable module for the semidirect product $\mathfrak{g}^+ = W_n^+ \oplus \mathbb{C}[z_1, \dots, z_n] \otimes \dot{\mathfrak{g}}$. This correspondence is given by the tensor product decomposition $J = \mathcal{F}(\mathbb{T}^n) \otimes U$, and $\mathfrak{g}$ acts according to* (3.5) *and*

$$(5.1) \qquad (e^{2\pi i s x} g)(e^{2\pi i m x} v) = \sum_{\beta \geq 0} \frac{s^\beta}{\beta!} e^{2\pi i (m+s) x} \rho(z^\beta g) v, \quad g \in \dot{\mathfrak{g}}, v \in U.$$

(iii) *Irreducible $\mathfrak{g}$-modules in category $\mathcal{J}$ are in a one-to-one correspondence with pairs (λ, V), where $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$ and V is a finite-dimensional irreducible $gl_n(\mathbb{C}) \oplus \dot{\mathfrak{g}}$-module. The action of $\mathfrak{g}$ on $J = \mathcal{F}(\mathbb{T}^n) \otimes V$ is given by* (3.6) *and*

$$(5.2) \qquad (e^{2\pi i s x} g)(e^{2\pi i m x} v) = e^{2\pi i (m+s) x} (g v), \quad g \in \dot{\mathfrak{g}}, \ v \in V.$$

Proof Let us outline the proof of Theorem 5.1. In the same way as in the discussion at the beginning of Section 3, we can show that an indecomposable $\mathfrak{g}$-module in category $\mathcal{J}$ is a tensor product $J = \mathcal{F}(\mathbb{T}^n) \otimes U$ with $\dim U < \infty$, and the action of $\mathfrak{g}$ given by (3.1) and for $g \in \dot{\mathfrak{g}}$:

$$(e^{2\pi i s x} g)(e^{2\pi i m x} v) = e^{2\pi i (m+s) x} g(s) v,$$

for some operators $g(s) \in \operatorname{End} U$, $s \in \mathbb{Z}^n$. In order to prove that J is a polynomial module, we have to show that the family of operators $\{g(s)\}$ has a polynomial dependence on s.

From the commutator relation in $\mathfrak{g}$,

$$\Big[\frac{1}{2\pi i} e^{2\pi i s x} \frac{\partial}{\partial x^j}, e^{2\pi i m x} g \Big] = m_j e^{2\pi i (s+m) x} g,$$

we get that

$$(5.3) \qquad [D_j(s), g(m)] = m_j (g(s+m) - g(m)).$$

Applying the above equality to $g(1, \dots, 1)$ we see that

$$g(1 + s_1, \dots, 1 + s_n) = [D_j(s), g(1, \dots, 1)] + g(1, \dots, 1).$$

By Theorem 2.2, the operators $\{D_j(s)\}$ depend on s polynomially, hence the same is true for $\{g(s)\}$.

To prove part (ii), we expand the polynomial $g(s)$:

$$g(s) = \sum_{\beta \geq 0}^{\text{finite}} \frac{s^\beta}{\beta!} g^{(\beta)},$$

with $g^{(\beta)} \in \operatorname{End} U$ and $g^{(\beta)} = 0$ for $\beta \gg 0$.

From the expansions of the relations $[g(s), h(m)] = [g, h](s + m), g, h \in \dot{\mathfrak{g}}$, and (5.3), we see that

$$[g^{(\alpha)}, h^{(\beta)}] = [g, h]^{(\alpha+\beta)}, \tag{5.4}$$

$$[D_j^{(\alpha)}, g^{(\beta)}] = \beta_j g^{(\alpha+\beta-\epsilon_j)}. \tag{5.5}$$

This shows that every indecomposable $\mathfrak{g}$-module $J \in \mathcal{J}$ yields a finite-dimensional $\mathfrak{g}^+$-module U.

Using the same technique as in the proof of Theorem 3.3(ii), we can show that for any finite-dimensional $\mathfrak{g}^+$-module U, $\rho(z^\alpha g) = 0$ for $\alpha \gg 0$. This proves that the correspondence is bijective.

Finally, to prove part (iii) of Theorem 3.3, we note that irreducible $\mathfrak{g}$-modules J correspond to irreducible $\mathfrak{g}^+$-modules V. Let us show that V is in fact a $gl_n(\mathbb{C}) \oplus \dot{\mathfrak{g}}$-module. Let V_ν be an eigenspace for $\rho(E)$, $E = z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n}$. It is easy to see that

$$\rho\bigl(z^\alpha \frac{\partial}{\partial z_j}\bigr) V_\nu \subset V_{\nu+|\alpha|-1}, \quad \text{and} \quad \rho(z^\beta g) V_\nu \subset V_{\nu+|\beta|}.$$

Thus $\bigoplus_{k=1}^\infty V_{\nu+k}$ is a $\mathfrak{g}^+$-submodule in V. Irreducibility of V implies that $V_{\nu+k} = (0)$ for all $k \geq 1$, and hence the ideal

$$g^{++} = \operatorname{Span}\Bigl\langle z^\alpha \frac{\partial}{\partial z_j}, z^\beta g \Bigm| \alpha, \beta \in \mathbb{Z}_+^n, |\alpha| > 1, |\beta| \geq 1,\ g \in \dot{\mathfrak{g}},\ j = 1, \ldots, n \Bigr\rangle$$

vanishes in every finite-dimensional irreducible $\mathfrak{g}^+$-module V. The claim of part (iii) follows from the fact that $\mathfrak{g}^+/\mathfrak{g}^{++} \cong gl_n(\mathbb{C}) \oplus \dot{\mathfrak{g}}$. ■

Acknowledgments I thank Mikhail Kochetov for helpful discussions. I also thank Eswara Rao for keeping me updated on his research. In particular, this work was strongly influenced by Eswara Rao's paper [ER]. I am grateful to the referee for bringing the paper by Gargoubi and Ovsienko [GO] to my attention.

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https://doi.org/10.4153/CJM-2007-031-2 Published online by Cambridge University Press