EXISTENCE AND MULTIPLICITY RESULTS FOR SEMICOERCIVE
UNILATERAL PROBLEMS

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In this paper, we investigate a general class of variational inequalities. Existence
and multiplicity results are obtained by using minimax principles for lower semi-
continuous functions due to A. Szulkin.

1. INTRODUCTION

The aim of this paper is the study of problems in mechanics characterised by a
general mechanics law which may be written in the form

\[ 0 \leq \nabla W(u) + \partial \Phi(u), \quad u \in U, \]

that is, the law is the sum of a potential law and a superpotential law. \( W \) is the
potential and \( \Phi \) is a proper convex function and is the superpotential. We denote by
\( \partial \Phi \) the convex subdifferential of \( \Phi \). \( U \) is the space of all fields of possible displacements.
In this paper it will be assumed that \( W \) can be written in the following form

\[ W(u) = \langle u, Tu \rangle /2 + Cu, \]

where \( T \) is linear symmetric and \( C \) is \( C^1(U, \mathbb{R}) \).

As an example, we shall consider the following problem: let \( T > 0 \) and let
\( H^1(\Pi, \mathbb{R}) \) be the Sobolev space obtained by completing the set of \( C^\infty \) real-valued
\( T \)-periodic functions on \( \Pi := \mathbb{R}/TZ \) with the norm

\[ ||u|| = \int_0^T |u|^2 + |\dot{u}|^2 \, dt. \]

Let \( K \) be the closed convex cone defined by

\[ K := \{ u \in H^1(\Pi, \mathbb{R}) : u(x) \geq 0 \text{ on } [0, T] \}. \]

We consider the following periodic unilateral problem

\[ [P_0] \quad u \in K: \int_0^T \dot{u} (\dot{v} - \dot{u}) \, dt + \int_0^T \nabla V(t, u) (v - u) \, dt \geq 0, \quad \forall v \in K, \]

when \( V \) is \( \alpha \)-positively homogeneous with respect to \( u \) and such that

(a) \( \forall u \in \mathbb{R} \setminus \{0\}: \int_0^T V(t, u) \, dt > 0, \)

(b) \( \exists v \in \mathbb{R}^+ : V(v, u) < 0 \text{ on a non zero measure subset.} \)
In this case,
\[ W(u) := \frac{1}{2} \int_0^T |\dot{u}|^2 \, dt + \int_0^T V(t, u) \, dt, \]
where the former term correspond to the kinetic energy and the latter to the potential energy. \( \Phi(u) := I_K \) characterises the constraints on the displacement field.

We now detail the framework of our paper.

Let \((X, X^*)\) be a dual system of real Hilbert spaces, let \( T: X \to X^* \) be a symmetric bounded linear operator and let \( C \in C^1(X, \mathbb{R}) \). \( C \) is assumed to be \( \beta \)-positively homogeneous that is, \( (C'(u), u) = \beta C(u) \) and strongly continuous (that is \( C \) maps weakly converging sequences into converging sequences). Let \( \Phi: X \to (-\infty, +\infty] \) be a proper convex functional. \( \Phi \) is assumed to be \( \alpha \)-positively homogeneous and weakly lower semicontinuous.

We are looking for non trivial solutions, that is, \( x^* \notin \text{Ker}T \), of the following variational inequality:

\[ [P] \quad x^* \in X: \langle v - x^*, Tx^* + C'(x^*) \rangle + \Phi(v) - \Phi(x^*) \geq 0, \quad \forall \, v \in X. \]

In case of bilateral problems (that is, \( \Phi = 0 \)), the first result concerning this problem is due to Lassoued [2]. Recently, using a version of the Ljusternik-Schnirelman theory on \( C^1 \)-manifolds due to Szulkin [5] Ben Naoum, Troestler and Willem obtained a general abstract existence and multiplicity theory for bilateral problems [1], where homogeneous second order differential equations were considered. For a basic work on critical point theory and its applications to bilateral problems we shall refer to [3]. In this paper, we use a version of minimax principles for lower semicontinuous functions due to Szulkin, to get new results for the variational inequality \([P]\) and related unilateral problems.

2. Existence result

**Theorem 2.1.** If the following conditions hold true: \( \alpha < \beta < 2 \) and

1. \( T \) is semicoercive, that is, there exists \( c > 0 \) such that
   \[ \langle x, Tx \rangle \geq c \|Px\|^2 \quad \text{for each} \quad x \in X \]
   with \( P = I - Q \), where \( I \) denotes the identity mapping and \( Q \) denotes the orthogonal projection of \( X \) onto \( \text{Ker}(T) \).
2. \( \dim \text{Ker}T < +\infty \),
3. \( \exists z \in X: \langle C'(z), z \rangle + \Phi(z) < 0 \),
4. \( C(u) > 0, \forall \, u \in \text{Ker}T, \, u \neq 0 \),

https://doi.org/10.1017/S0004972700016592 Published online by Cambridge University Press
then problem \([P]\) has a nontrivial solution.

**PROOF:** Let \(J : X \to (\mathbb{R}, +\infty]\) be the functional defined by

\[
J(u) := \frac{1}{2} \langle u, Tu \rangle + C(u) + \Phi(u).
\]

Let \(X = \bigcup_{n \in \mathbb{N}} X_n\), where for each \(n\), \(X_n := \{x \in X : \|x\| \leq n\}\) is a weakly compact convex set in \(X\). Since \(J\) is weakly lower semicontinuous, it reaches its minimum on each \(X_n\), let us say at \(u_n\).

We have \(J(u_n) \leq J(v)\), for each \(v \in X_n\).

Let \(v \in X_n\), \(tv + (1-t)u_n \in X_n\) for each \(t \in [0, 1]\) and since \(\Phi\) is convex, we get

\[
\begin{align*}
\Phi(v) - \Phi(u_n) &+ \frac{1}{2} [(T(u_n + t(v - u_n)), u_n + t(v - u_n)) - (Tu_n, u_n)]/t \\
&+ [C(u_n + t(v - u_n)) - C(u_n)]/t \geq 0, \text{ for all } v \in X_n.
\end{align*}
\]

Computing the limit as \(t \to 0^+\) we get

\[
(2.1) \quad \langle v - u_n, Tu_n + C'(u_n) \rangle + \Phi(v) - \Phi(u_n) \geq 0, \text{ for all } v \in X_n.
\]

We show first that the sequence \(\{u_n\}\) is bounded.

(a) If \(0 < \beta < 2\). Suppose on the contrary that \(\{u_n\}\) is unbounded. Passing possibly to a subsequence, we can suppose that \(w = \lim_{n \to \infty} x_n = x^*\), where \(x_n := u_n/\|u_n\|\).

Put \(v = 0\) in (2.1); we obtain

\[
\langle u_n, Tu_n \rangle + \beta C(u_n) + \Phi(u_n) \leq 0
\]

which implies:

\[
(2.2) \quad \langle x_n, Tx_n \rangle + \beta C(x_n). \|u_n\|^{\beta - 2} + \Phi(x_n) \|u_n\|^{\alpha - 2} \leq 0.
\]

Taking the limit as \(n \to +\infty\) in (2.2), we get

\[
\langle x^*, Tx^* \rangle \leq \liminf \langle x_n, Tx_n \rangle \leq 0,
\]

and since \(\langle x^*, Tx^* \rangle \geq 0\) we obtain

\[
\langle x^*, Tx^* \rangle = 0,
\]

and thus \(Tx^* = 0\) and \(c. \liminf \|Px_n\|^2 \leq \liminf \langle x_n, Tx_n \rangle \leq 0\). Going if necessary to a subsequence we can assume that \(Px_n \to 0\), \(x_n \to x^* \in \text{Ker} T\), since \(\dim \text{Ker} T < \infty\). Moreover \(\|x^*\| = 1\).
By positivity, $\langle u_n, Tu_n \rangle \geq 0$ for each $n \in \mathbb{N}$, and thus we have from (2.1)
\begin{equation}
\langle x, Tu_n \rangle + \langle x, C'(u_n) \rangle + \Phi(x) \geq \beta C(u_n) + \Phi(u_n), \text{ for each } x \in X_n.
\end{equation}
Choosing $x = 0$, we obtain
$$\Phi(u_n) + \beta C(u_n) \leq 0.$$\hfill (*)
Dividing by $\|u_n\|^\beta$,
$$\beta C(x_n) + \Phi(x_n) \|u_n\|^\alpha - \beta \leq 0$$
and taking the limit, we obtain
$$C(x^*) \leq 0,$$
and since $x^* \in \text{Ker } T$, this is a contradiction to assumption (4).
Thus the sequence $\{u_n\}$ is bounded. Without loss of generality, we can suppose that
$$u^* = w - \lim_{n \to \infty} u_n.$$
For $y \in X$, there exists $m \in \mathbb{N}$ such that $y \in X_n$ for all $n \geq m$. Hence $J(u_n) \leq J(y)$, for all $n \geq m$ and since $J$ is weakly lower semicontinuous, we get
$$J(u^*) \leq J(y),$$
and therefore $J(u^*) = \min_X J(y)$.
We have thus
$$\langle Tu^* + C'(u^*), v - u^* \rangle + \Phi(v) - \Phi(u^*) \geq 0, \quad \forall y \in X.$$\hfill (2.3)
\begin{equation}
\langle x, Tu_n \rangle + \langle x, C'(u_n) \rangle + \Phi(x) \geq \beta C(u_n) + \Phi(u_n), \text{ for each } x \in X_n.
\end{equation}
With $v = 0$, we obtain $\beta C(u^*) + \Phi(u^*) \leq 0$, and thus by assumption (4)
$$u^* \notin \text{Ker } T \setminus \{0\}.$$\hfill (4)
Now,
$$J(u^*) \leq J(v), \quad \text{for all } v \in X,$$
and thus
$$J(u^*) \leq C(v), \quad \text{for all } v \in \text{Ker } T.$$\hfill (5)
By assumption (3), we get
$$J(u^*) \leq C(z) + \Phi(z) < 0,$$
and thus $u^* \neq 0$. \hfill $\square$
Corollary 2.1. Let \( K \) be a nonempty closed convex cone of \( X \). If the following conditions hold true: \( \alpha < \beta < 2 \) and

1. \( T \) is semicoercive.
2. \( \text{dim} \, \text{Ker} \, T < +\infty \),
3. \( \exists z \in K: \langle C'(z), z \rangle < 0 \),
4. \( C(u) > 0 \), \( \forall u \in K \cap \text{Ker} \, T, u \neq 0 \),

then there exists \( u \in K \setminus \text{Ker} \, T \) such that

\[
\langle v - x^*, T x^* + C'(x^*) \rangle \geq 0, \quad \forall v \in K.
\]

3. Multiplicity Result

We shall assume that \( C \in C^1(X, \mathbb{R}) \) is even, \( \beta \)-positively homogeneous and strongly continuous, and \( \Phi: X \to (-\infty, +\infty] \) is even, \( \alpha \)-positively homogeneous and strongly continuous.

In order to obtain a multiplicity result to prove that \([P]\) has many pairs of solutions \((-x^*, x^*)\), we verify the assumptions of Theorem 4.4 in [4] due to Szulkin.

Let us recall some definitions.

Let \( X \) be a real Banach space and \( J \) a function on \( X \) satisfying: \( J = f + g \), where \( f \in C^1(X, \mathbb{R}) \) and \( g: X \to (-\infty, +\infty] \) is convex, proper and lower semicontinuous. We say that \( J \) satisfies the Palais-Smale condition in the sense of Szulkin (PS), if \( \{u_n\} \) is a sequence such that \( J(u_n) \to c \in \mathbb{R} \), \( z_n \in f'(u_n) + \partial g(u_n) \) where \( z_n \to 0 \), then \( \{u_n\} \) possesses a convergent subsequence.

Theorem 3.1. (Szulkin [4].) Suppose that \( J \) is defined as above and satisfies (PS), \( J(0) = 0 \) and \( f, g \) are even. Assume also that

1. there exists a subspace \( X_1 \) of \( X \), of finite codimension, and numbers \( \gamma, \rho > 0 \) such that \( J|_{\rho B_\rho \cap X_1} \geq \gamma \),
2. there is a finite dimensional subspace \( X_2 \) of \( X \), \( \dim X_2 > \text{codim} \, X_1 \) such that \( J(u) \to -\infty \) as \( \|u\| \to \infty \), \( u \in X_2 \).

Then \( J \) has at least \( \dim X_2 - \text{codim} \, X_1 \) distinct pairs of nonzero critical points \((-x^*, x^*)\), that is \( 0 \in f'(x^*) + \partial g(x^*) \).

Corollary 3.1. (Szulkin [4].) Suppose that the hypotheses of Theorem 3.1 are satisfied with (2) replaced by

(2') for any positive integer \( k \) there is a \( k \)-dimensional subspace \( X_2 \) of \( X \) such that \( J(u) \to -\infty \) as \( \|u\| \to +\infty \).

Then \( J \) has infinitely many distinct pairs of nonzero critical points.

From this theorem, we obtain the following
**Theorem 3.2.** If \( \alpha > 1, \beta > \max\{2, 2^\alpha - 1\} \) and

1. \( T \) is semicoercive
2. \( \dim \text{Ker} \, T < +\infty \),
3. there exists a subspace \( X_n \) of \( X \), such that \( n := \dim X_n > \dim \text{Ker} \, T \) and \( C(y) < 0 \), for all \( y \in X_n, y \neq 0 \),
4. \( \Phi(u) > 0, \forall u \in (\text{Ker} \, T) \setminus \{0\} \); \( \Phi(u) \geq 0, \forall u \in X \).

Then there exist at least \( n - \dim \text{Ker} \, T \) distinct pairs of nontrivial solutions for problem \([P]\).

**Proof:** Let \( f(x) := \langle x, Tx \rangle / 2 + C(x), g(x) := \Phi(x) \). Let \( X_1 := (\text{Ker} \, T)^\perp \), \( X_2 := X_n \)

1. For every \( x \in X_1 \), we have

\[
\langle x, Tx \rangle / 2 + \Phi(x) + C(x) \geq c / 2 \cdot \|x\|^2 - |C(x)|
\]

and since \( \Phi \) and \( C \) are continuous and positively homogeneous, there exist \( k, k' > 0 \) such that

\[
\langle x, Tx \rangle / 2 + \Phi(x) + \beta C(x) \geq c / 2 \cdot \|x\|^2 - k' \|x\|^\beta.
\]

It is always possible to choose \( \rho \) such that \( \tau := c \rho^2 / 2 - k' \rho^\beta > 0 \) and thus

\[
J(x) \geq \tau, \quad \forall x \in \partial B_\rho \cap X_1.
\]

2. By assumption (3), there exists \( \delta > 0 \) such that

\[
C(x) \leq -\delta \|x\|^\beta, \quad \text{for all} \quad x \in X_2.
\]

We have

\[
J(x) \leq \|T\|_* \|x\|^2 - \delta \|x\|^\beta + k \|x\|^\alpha
\]

and thus

\[
\lim_{\|x\| \to +\infty, x \in X_2} J(x) = -\infty.
\]

It remains to prove that \( J \) satisfies the (PS) condition. Let \( u_n \in X \) be a sequence such that \( J(u_n) \to c \in \mathbb{R}, z_n \in f'(u_n) + \partial g(u_n) \) where \( z_n \to 0 \); that is also (see [4] for more details)

\[
\Phi(v) - \Phi(u_n) + \langle Tu_n + C' u_n, v - u_n \rangle \geq -\delta_n \cdot \|v - u_n\|,
\]

where \( \delta_n \to 0 \).
We claim that \( \{u_n\} \) is bounded. Suppose that \( \{u_n\} \) is unbounded. With \( v = 2u_n \) in (3.1) we get
\[
\langle u_n, Tu_n \rangle + \beta C(u_n) + (2^\alpha - 1)\Phi(u_n) \geq -\delta_n \|u_n\|,
\]
so that, for \( n \) large enough,
\[
\beta J(u_n) - \{\langle u_n, Tu_n \rangle + \beta C(u_n) + (2^\alpha - 1)\Phi(u_n)\} \leq \beta(c + 1) + \|u_n\|.
\]
Thus
\[
(3.2) \quad (\beta + 1 - 2^\alpha)\Phi(u_n) + (\beta/2 - 1)\langle u_n, Tu_n \rangle \leq \beta(c + 1) + \|u_n\|.
\]
By assumption (6) we have
\[
(\beta/2 - 1)\langle u_n, Tu_n \rangle \leq \beta(c + 1) + \|u_n\|.
\]
Put \( v_n := u_n/\|u_n\| \). We can suppose, by considering if necessary a subsequence, that
\[
w = \lim_{n \to \infty} v_n = v^*.
\]
We have
\[
(\beta/2 - 1)\langle v_n, Tv_n \rangle \leq \beta(c + 1)/\|u_n\|^2 + 1/\|u_n\|.
\]
Taking the limit, we get
\[
0 \leq \langle v^*, Tv^* \rangle \leq \liminf_{n \to \infty} \langle v_n, Tv_n \rangle \leq 0,
\]
and as in Theorem 2.1, going if necessary to a subsequence, we can assume that \( \|v^*\| = 1 \).

Since \( T \) is positive, from (3.2) we get also
\[
(\beta + 1 - 2^\alpha)\Phi(u_n) \leq \beta(c + 1) + \|u_n\|,
\]
and thus
\[
(\beta + 1 - 2^\alpha)\Phi(v_n) \leq \beta(c + 1)/\|u_n\|^\alpha + 1/\|u_n\|^\alpha - 1.
\]
By taking the limit, we get \( \Phi(v^*) \leq 0 \), which is a contradiction to assumption (4).

Thus \( \{u_n\} \) is bounded and by considering possibly a subsequence, we may suppose that \( u_n \) is weakly convergent. Let \( u^* = w - \lim u_n \). Put \( v = u^* \) in (3.1). We get
\[
\langle Tu_n, u^* - u_n \rangle + \langle C'u_n, u^* - u_n \rangle + \Phi(u^*) - \Phi(u_n) \geq -\delta_n \|u^* - u_n\|.
\]
Taking the limit, we get
\[
\lim_{n \to \infty} \langle Tu_n, u_n - u^* \rangle \leq 0.
\]

The orthogonal decomposition \( X \oplus \text{Ker} (T) \) allows us to write \( u_n =: \bar{u}_n + \bar{u}_n \). Thus we have
\[
\lim_{n \to \infty} c. \|\bar{u}_n - \bar{u}^*\|^2 \leq \lim_{n \to \infty} \langle T(\bar{u}_n - \bar{u}^*), \bar{u}_n - \bar{u}^* \rangle \leq 0,
\]
and \( \bar{u}_n \) is strongly convergent to \( \bar{u}^* \). Since \( \dim \text{Ker} T < +\infty \), going if necessary to a subsequence, \( \bar{u}_n \) is strongly convergent to \( Q^* \) and the conclusion follows.
4. EXAMPLES

**EXAMPLE 4.1.** Let $T > 0$ and let $X := H^1(\Omega, \mathbb{R})$. Let $K$ be the closed convex cone defined by $K := \{ u \in H^1(\Omega, \mathbb{R}) : u(x) \geq 0 \text{ in } [0, T] \}$. We consider the periodic unilateral problem

\begin{equation}
\int_0^T \dot{u}(v - u)dt + \int_0^T \nabla u V(t, u).(v - u)dt \geq 0, \quad \forall v \in K.
\end{equation}

We assume that:

(a) $\forall u \in \mathbb{R}, V(\cdot, u)$ is measurable and there exist $a, b \in L^1([0, T], \mathbb{R}^+)$ such that $\forall t \in [0, T], \forall u \in \mathbb{R}, |u| = 1, |V(t, u)| \leq a(t)$ and $|\nabla u V(t, u)| \leq b(t)$.

(b) for almost all $t \in \mathbb{R}$, $V(t, \cdot) \in C^1$.

(c) $\forall u \in \mathbb{R} \setminus \{0\} : \int_0^T V(t, u)dt > 0$.

(d) $\exists \nu \in \mathbb{R}^+ : V(\cdot, \nu) < 0$, on a non zero measure subset.

(e) $V$ is $\beta$-positively homogeneous ($\beta < 2$) with respect to $u$.

Let $T: X \to X^*$ and $C: X \to \mathbb{R}$ be defined by

\begin{equation}
\langle Tu, v \rangle := \int_0^T \dot{u}(v - u)dt, \quad C(u) := \int_0^T V(t, u)dt.
\end{equation}

We can prove that if $V$ satisfies (a)-(e), then all assumptions of Corollary 2.1 are satisfied [1], so that (1) has at least one non-constant solution.

**EXAMPLE 4.2.** We consider the problem

\begin{equation}
\int_0^T \dot{u}(v - u)dt + \int_0^T \nabla u V(t, u).(v - u)dt + \int_0^T g(t)(|v|^3 - |u|^3)dt \geq 0, \quad \forall u \in X.
\end{equation}

Let $T: X \to X^*$ and $C: X \to \mathbb{R}$ be defined as in Example 4.1 and put $\Phi(u) := \int_0^T g(t)|u|^3 dt$. We assume that $g$ is a positive ($g \neq 0$) bounded function.

We can prove that if $V$ satisfies (a)-(d) and (e) with $\beta > 7$ and even, then all assumptions of Theorem 3.2 [1] are satisfied. Therefore (1) has infinitely many distinct pairs of non-constant solutions.

**REFERENCES**


