ON THE RANGE OF THE Y-TRANSFORM

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The ranges of the Y-integral transform in some spaces of functions are described.

1. INTRODUCTION

The Y-transform $Y\nu f$ is defined by [8, 6]

$$f(x) = (Y\nu g)(x) = \int_0^\infty \sqrt{xy} Y\nu(xy)g(y)\,dy, \quad x \in \mathbb{R}^+ = (0, \infty),$$

if the integral converges in some sense (absolutely, improper, mean convergence), where $Y\nu(x)$ is the Bessel function of the second kind [1]. The Y-transform $Y\nu$ has been considered in $L^p_{\mu,p}$ in [3, 6, 7]. In particular, it follows that in $L_2(\mathbb{R}^+) = \mathbb{L}^2_{1/2,2}$ the Y-transform $Y\nu$ is bounded if $|\Re \nu| < 1$, and if, moreover, $0 < |\Re \nu| < 1$, then the range of the Y-transform $Y\nu$ is $L_2(\mathbb{R}^+)$. 

$$\|Y\nu g\|_{L_2(\mathbb{R}^+)} \leq C \|g\|_{L_2(\mathbb{R}^+)}, \quad |\Re \nu| < 1,$$

$$\|g\|_{L_2(\mathbb{R}^+)} \leq C \|Y\nu g\|_{L_2(\mathbb{R}^+)}, \quad 0 < |\Re \nu| < 1,$$

where $C$ is an independent constant, (but different in distinct inequalities). The H-transform $H\nu$ [8, 6] denoted by

$$g(x) = (H\nu f)(x) = \int_0^\infty \sqrt{xy} H\nu(xy)f(y)\,dy, \quad x \in \mathbb{R}^+,$$

is the inverse of Y-transform $Y\nu$ in $L_2(\mathbb{R}^+)$ if $-1 < \Re \nu < 0$. If $0 < \Re \nu < 1$ the inverse formula (4) should be replaced by formula (51) or, equivalently, (52). Here $H\nu(x)$ is the Struve function [1]. The Y- and H-transforms are of importance in many singular axially symmetric potential problems [6]. In this work we describe precisely the range of the Y-transform in some spaces of functions. The range of the Y-transform of functions with compact supports (analogous to the Paley-Wiener theorem for the Fourier transform [5]) is also considered. It is worth remarking that our Paley-Wiener

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theorem (Theorem 2) is different from the classical ones describing Fourier transform of compactly supported functions in terms of entire functions of exponential type [5]. (For the Hankel transform of compactly supported functions see [4].) The theorem stated here involves the spectral radius [12] of some differential operator obtained from the Bessel differential equation and having the kernel of the Y-transform as “eigenfunctions”, (similar ideas have been applied in [2, 11] to the Fourier transform). Nevertheless, its proof is straightforward, without referring to spectral theory. Since the H-transform \( H_\nu \) is the inverse of the Y-transform \( Y_\nu \) in all spaces we considered in this paper, corresponding theorems on the range of the H-transform can be easily derived.

2. Y-TRANSFORM OF POLYNOMIAL DECREASING FUNCTIONS

We describe the range of the Y-transform on the space of functions \( g(y) \) square integrable together with \( y^n g(y), n = 1,2,\ldots \) (polynomial decreasing functions):

**Theorem 1.** A function \( f(x) \) is the Y-transform \( Y_\nu, 0 < |\Re \nu| < 1/2, \) of a function \( g(y) \), square integrable together with \( y^n g(y), n = 1,2,\ldots \), if and only if

(i) \( f(x) \) is infinitely differentiable on \( R_+; \)

(ii) \( (d^2/dx^2 + (1/x^2)((1/4) - \nu^2))^n f(x), n = 0,1,\ldots, \) belongs to \( L_2(R_+) \);

(iii) \( (d^2/dx^2 + (1/x^2)((1/4) - \nu^2))^n f(x), n = 0,1,\ldots, \) tends to 0 as \( x \) tends both to 0 and to infinity;

(iv) \( x(d/dx)(d^2/dx^2 + (1/x^2)((1/4) - \nu^2))^n f(x), n = 0,1,\ldots, \) is bounded at 0;

(v) \( (d/dx)(d^2/dx^2 + (1/x^2)((1/4) - \nu^2))^n f(x), n = 0,1,\ldots, \) tends to 0 as \( x \) tends to infinity;

(vi) The improper integrals

\[
\int_{-\infty}^{\infty} x^{\nu-1/2} \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right)^n f(x) \, dx
\]

exist and vanish for all \( n = 1,2,\ldots, \) as well as for \( n = 0 \) if \( -1/2 < \Re \nu < 0. \)

**Proof:** (a) Let \( y^n g(y) \) belong to \( L_2(R_+) \) for all \( n = 0,1,2,\ldots, \) then \( y^n g(y) \) belongs to \( L_1(R_+) \) for all \( n = 0,1,2,\ldots. \) Let \( f(x) \) be the Y-transform \( Y_\nu, 0 < |\Re \nu| < 1/2, \) of \( g(y) \) (the Y-transform \( Y_\nu \) of \( g(y) \) with other values of \( \nu \) also appears in the proof, but it is not denoted by \( f(x) \)).

(a-i) We have [1]

\[
\left( \frac{d^n}{dx^n} Y_\nu(x) \right) = 2^{-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} Y_{\nu-n+2j}(x).
\]
Therefore,

\[ \frac{\partial^n}{\partial x^n}(\sqrt{xy}Y_\nu(xy)) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^{n+j} 2^{-k} (-1/2)^{n-k} \binom{n}{k} \binom{k}{j} x^{1/2+k-n} y^{1/2+k} Y_{\nu - k + 2j}(xy), \]

(6)

where \((a)_n = \Gamma(a+n)/\Gamma(a)\) is the Pochhammer symbol \([1]\). The Bessel function of the second kind \(Y_\nu(y)\) has the asymptotics \([1]\)

\[ Y_\nu(y) = \begin{cases} \sqrt{\frac{2}{\pi y}} \left[ \sin \left( y - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + \frac{4\nu^2 - 1}{8y} \cos \left( y - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \right] + O(y^{-5/2}), \quad y \to \infty \\ O(y^{-|\Re\nu|}), \quad y \to 0. \end{cases} \]

(7)

Consequently, \(\frac{\partial^n}{\partial x^n}[\sqrt{xy}Y_\nu(xy)]\), \(|\Re\nu| < 1\), as a function of \(y\) has the asymptotics \(O(y^{1/2-|\Re\nu|})\) in the neighbourhood of 0 and \(O(y^n)\) at infinity. Hence, \(\frac{\partial^n}{\partial x^n}[\sqrt{xy}Y_\nu(xy)]g(y), \quad |\Re\nu| < 1, \) as a function of \(y\) belongs to \(L_1(R_+)\) for all \(n = 0, 1, 2, \ldots\), and therefore, \(f(x)\) is infinitely differentiable on \(R_+\).

(a-ii) Since \(Y_\nu(x)\) satisfies the Bessel differential equation \([1]\)

\[ x^2 u'' + xu' + (x^2 - \nu^2)u = 0, \]

the function \(\sqrt{x}Y_\nu(x)\) is a solution of the equation

\[ x^2 u'' + \left( x^2 + \frac{1}{4} - \nu^2 \right) u = 0. \]

Therefore, we have

\[ \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n (\sqrt{xy}Y_\nu(xy)) = (-y^2)^n \sqrt{xy}Y_\nu(xy). \]

(10)

Consequently,

\[ \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = (-1)^n \int_0^\infty \sqrt{xy}Y_\nu(xy)y^{2n}g(y) dy, \quad |\Re\nu| < 1/2. \]

(11)

By using inequality (2) for the Y-transform (11) of \(y^{2n}g(y) \in L_2(R_+)\), we obtain that \([d^2/dx^2 + (1/x^2)((1/4) - \nu^2)]^n f(x), \quad |\Re\nu| < 1/2, \quad n = 0, 1, \ldots, \) belongs to \(L_2(R_+).\)
(a-iii) From (7) we see that the function \( \sqrt{xy} Y_\nu(xy) \), \(|\Re \nu| < 1/2\), has the asymptotics \( x^{1/2-|\Re \nu|} \) as \( x \) tends to 0, and is uniformly bounded on \( R_+ \). Because \( y^{2n} g(y) \in L_1(R_+) \), by applying the dominated convergence theorem \([12]\) we have

\[
\lim_{x \to 0} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = (-1)^n \int_0^\infty \lim_{x \to 0} [\sqrt{xy} Y_\nu(xy)] y^{2n} g(y) \, dy = 0,
\]

(12) \(|\Re \nu| < 1/2\).

Since \( \sqrt{xy} Y_\nu(xy) \), \(|\Re \nu| < 3/2\), is uniformly bounded for \( x, y \in [1, \infty) \) and \( y^n g(y) \in L_1(R_+) \), for every \( \epsilon > 0 \) and for every \( n, n = 0, 1, \ldots \), one can choose \( b \) large enough so that

\[
\int_0^\infty \sqrt{xy} Y_\nu(xy) y^n g(y) \, dy < \epsilon, \quad |\Re \nu| < 3/2,
\]

(13) uniformly with respect to \( x \in [1, \infty) \). On the other hand, from (7) one can conclude that the integral

\[
\int_{az}^{bz} \sqrt{y} Y_\nu(y) \, dy, \quad |\Re \nu| < 1/2,
\]

(14) is uniformly bounded for all non-negative \( a, b \) and \( x \). Hence,

\[
\int_a^b \sqrt{xy} Y_\nu(xy) \, dy = \frac{1}{x} \int_{az}^{bz} \sqrt{y} Y_\nu(y) \, dy, \quad |\Re \nu| < 1/2,
\]

(15) tends to 0 uniformly in \( a, b \) for \( 0 \leq a < b < \infty \) as \( x \) tends to infinity. Consequently, applying the generalised Riemann-Lebesgue lemma \([8]\) we get

\[
\lim_{x \to \infty} \int_0^b \sqrt{xy} Y_\nu(xy) y^{2n} g(y) \, dy = 0, \quad 0 < b < \infty, \quad |\Re \nu| < 1/2.
\]

(16) Because \( \epsilon \) can be taken arbitrarily small, from (13) and (16) we obtain

\[
\lim_{x \to \infty} \int_0^\infty \sqrt{xy} Y_\nu(xy) y^{2n} g(y) \, dy = 0, \quad |\Re \nu| < 1/2.
\]

(17) Hence,

\[
\lim_{x \to \infty} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = 0, \quad n = 0, 1, \ldots, \quad |\Re \nu| < 1/2.
\]

(18) (a-iv) Since \([1]\)

\[
2 \frac{d}{dx} (\sqrt{xy} \nu(x)) = \sqrt{xy} \nu_{-1}(x) - \sqrt{xy} \nu_1(x) + \frac{1}{\sqrt{x}} \nu(x),
\]

(19)
we have

\[
\begin{align*}
(20) \quad \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) &= \frac{(-1)^n}{2} \int_0^\infty \sqrt{xy} Y_{n-1}(xy) y^{2n+1} g(y) \, dy \\
&\quad + \frac{(-1)^{n+1}}{2} \int_0^\infty \sqrt{xy} Y_n(xy) y^{2n+1} g(y) \, dy \\
&\quad + \frac{(-1)^n}{2} \int_0^\infty \sqrt{xy} Y_{n+1}(xy) y^{2n+1} g(y) \, dy.
\end{align*}
\]

The function \( \sqrt{x} Y_\nu(x) \) is uniformly bounded on \([1, \infty)\), and is of the order \( O(x^{1/2-|\Re \nu|}) \) on \((0,1)\). Therefore, for \( x \in (0,1) \),

\[
\left| \int_0^\infty \sqrt{xy} Y_\mu(xy) g(y) \, dy \right| \leq \left| \int_0^{1/x} \sqrt{xy} Y_\mu(xy) g(y) \, dy \right| + \left| \int_0^\infty \sqrt{xy} Y_\mu(xy) g(y) \, dy \right|
\]

\[
\leq C x^{1/2-|\Re \mu|} \int_0^{1/x} y^{1/2-|\Re \mu|} |g(y)| \, dy + C \int_0^\infty |g(y)| \, dy
\]

\[
(21)
\]

Hence, in the neighbourhood of 0 we have

\[
\frac{1}{x} \int_0^\infty \sqrt{xy} Y_\nu(xy) y^{2n} g(y) \, dy = O(x^{-1}),
\]

\[
\int_0^\infty \sqrt{xy} Y_{n-1}(xy) y^{2n+1} g(y) \, dy = O(x^{\Re \nu - 1/2}),
\]

\[
(22)
\]

\[
\int_0^\infty \sqrt{xy} Y_{n+1}(xy) y^{2n+1} g(y) \, dy = O(x^{-\Re \nu - 1/2}), \quad |\Re \nu| < 1/2.
\]

By combining (20) and (22), we obtain

\[
\frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = O(1), \quad x \to 0;
\]

\[
(23)
\]

\[\]

(a-v) Let \( |\Re \nu| < 3/2 \). For every \( \varepsilon > 0 \) choose \( b \) so that the inequality (13) holds. Because \((xy)^{3/2} Y_\nu(xy), \ |\Re \nu| < 3/2 \), is uniformly bounded for \( x, y \in R_+, \ xy \leq 1 \), then

\[
(24)
\]

Hence,

\[
(25)
\]
Let
\[
\Phi(x, y) = \begin{cases} 
\sqrt{xy} Y_\nu(xy) dy, & y > 1/x \\
0, & y \leq 1/x.
\end{cases}
\] (26)

Then \(\Phi(x, y)\) is uniformly bounded. The integral
\[
\int_{ax}^{bx} \sqrt{y} Y_\nu(y) dy, \quad |\Re \nu| < 3/2,
\] (27)
is uniformly bounded for all non-negative \(a, b\) and \(x\) such that \(ax \geq 1\). Hence,
\[
\int_{a}^{b} \Phi(x, y) dy = \frac{1}{x} \int_{\max\{1, ax\}}^{bx} \sqrt{y} Y_\nu(y) dy, \quad |\Re \nu| < 3/2,
\] (28)
tends to 0 uniformly in \(a, b\) for \(0 \leq a < b < \infty\) as \(x\) tends to infinity. Consequently, applying again the generalised Riemann-Lebesgue lemma [8] we get
\[
\lim_{z \to \infty} \int_{0}^{b} \Phi(x, y)y^n g(y) dy = 0, \quad 0 < b < \infty,
\] (29)
This means that
\[
\lim_{z \to \infty} \int_{1/x}^{b} \sqrt{xy} Y_\nu(xy)y^n g(y) dy = 0, \quad 0 < b < \infty, \quad |\Re \nu| < 3/2.
\] (30)
Because \(\varepsilon\) can be taken arbitrarily small, from (13), (25) and (30) we obtain
\[
\lim_{z \to \infty} \int_{0}^{\infty} \sqrt{xy} Y_\nu(xy)y^{n+1} g(y) dy = 0, \quad n = 0, 1, \ldots, \quad |\Re \nu| < 3/2.
\] (31)
If \(|\Re \nu| < 1/2\), then \(|\Re \nu + 1| < 3/2\). Hence,
\[
\lim_{z \to \infty} \int_{0}^{\infty} \sqrt{xy} Y_{\nu-1}(xy)y^{2n+1} g(y) dy = 0,
\] (32)
Applying now formulas (20), (31) and (32), we have
\[
\lim_{z \to \infty} \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = 0, \quad n = 0, 1, \ldots, \quad |\Re \nu| < 1/2.
\] (33)
(a-vi) The special case \(-1/2 < \Re \nu < 0\) has been proved in [3]. We give here a proof valid for all the range of \(\nu\). Integral (11) converges uniformly with respect to \(x\)

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on every compact subset of \( R_+ \). Therefore, one can interchange the order of integration in the following formula to obtain

\[
\int_{1/N}^N x^{-\nu/2} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \, dx
= (-1)^n \int_{1/N}^N x^{-\nu/2} \int_0^\infty \sqrt{\pi} Y_\nu(xy)y^{2n}g(y) \, dy \, dx
\]

(34)

\[
= (-1)^n \int_0^\infty y^{2n-\nu-1/2}g(y) \int_{y/N}^y x^n Y_\nu(x) \, dx \, dy, \quad 0 < N < \infty.
\]

The last inner integral in (34) is uniformly bounded for all nonnegative \( N \) and \( y \), provided that \( \Re \nu < 1/2 \). For \( y^{2n-\nu-1/2}g(y) \in L_1(R_+) \) under the restriction \( \Re \nu < 0 \), and \( n \geq 1 \) otherwise, one can apply the dominated convergence theorem to obtain

\[
\lim_{N \to \infty} \int_{1/N}^N x^{-\nu/2} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \, dx
= (-1)^n \int_0^\infty y^{2n-\nu-1/2}g(y) \int_0^y x^n Y_\nu(x) \, dx \, dy, \quad n = 0,1,\ldots; -1/2 < \Re \nu < 0,
\]

\[
= (n = 1,2,\ldots; 0 \leq \Re \nu < 1/2.
\]

Applying now the formula [1]

\[
\int_0^\infty x^\mu Y_\nu(x) \, dx = \frac{2^\mu}{\pi} \sin \frac{\pi}{2} (\mu - \nu) \Gamma \left( \frac{\mu + \nu + 1}{2} \right) \Gamma \left( \frac{\mu - \nu + 1}{2} \right),
\]

(36)

\[
\Re (\mu + \nu) > -1, \Re \mu < 1/2,
\]

with \( \mu = \nu \), we see that the inner integral on the right hand side of (35) equals 0. Hence,

\[
\int_{-\infty}^\infty x^{-\nu/2} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \, dx = 0, \quad n = 0,1,\ldots; -1/2 < \Re \nu < 0,
\]

(37)

\[
= (n = 1,2,\ldots; 0 \leq \Re \nu < 1/2.
\]

(b) Suppose now that \( f \) satisfies conditions (i)-(vi) of Theorem 1. Then \( [d^2/dx^2 + (1/x^2)((1/4) - \nu^2)]^n f(x), \quad n = 0,1,\ldots, \) belongs to \( L_2(R_+) \).

(b-i) Let \(-1/2 < \Re \nu < 0\) and \( g_n(y) \) be the H-transforms \( H_\nu, \quad -1/2 < \Re \nu < 0, \) of \([d^2/dx^2 + (1/x^2)((1/4) - \nu^2)]^n f(x), \quad n = 0,1,\ldots, \) Then

\[
g_n(y) = \int_0^\infty \sqrt{\pi} Y_\nu(xy) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \, dx, \quad n = 0,1,2,\ldots,
\]

(38)
where the integrals are understood in the $L_2(R_+)$ norm. Put

$$g_n^N(y) = \int_{1/N}^N \sqrt{xy} H_\nu(xy) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \, dx, \quad n = 0, 1, 2, \ldots$$

Then $g_n^N(y)$ tends to $g_n(y)$ in $L_2$ norm as $N \to \infty$. Let $n \geq 1$. Integrating (39) by parts twice, we obtain

$$g_n^N(y) = \left\{ \sqrt{xy} H_\nu(xy) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \right\} \bigg|_{x=1/N}^{x=N}$$

$$- \left\{ \frac{\partial}{\partial x} \left( \sqrt{xy} H_\nu(xy) \right) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \right\} \bigg|_{x=1/N}^{x=N}$$

$$+ \int_{1/N}^N \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right] \left( \sqrt{xy} H_\nu(xy) \right) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \, dx.$$

Using formulas [1]

$$\frac{\partial}{\partial x} \left( \sqrt{xy} H_\nu(xy) \right) = \frac{1}{2} \nu \sqrt{xy} H_\nu(xy) + y \sqrt{xy} H_{\nu-1}(xy),$$

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right] \left( \sqrt{xy} H_\nu(xy) \right) = \frac{2^{1-\nu} y^{\nu+3/2}}{\sqrt{\pi} \Gamma(\nu+1/2)} x^{\nu-1/2} - y^2 \sqrt{xy} H_\nu(xy),$$

we have

$$g_n^N(y) = \sqrt{Ny} H_\nu(Ny) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N)$$

$$- \sqrt{Ny} H_\nu(Ny/N) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N)$$

$$+ \left( \nu - \frac{1}{2} \right) \sqrt{Ny} H_{\nu}(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N)$$

$$- y \sqrt{Ny} H_{\nu-1}(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N)$$

$$+ \left( \frac{1}{2} - \nu \right) \sqrt{Ny} H_\nu(y/N) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N)$$
Here \( P(d/dx) f(N) \) means \( P(d/dx) f(x)|_{x=N} \). As \( N \) tends to infinity, integral (49) vanishes because of property (vi). Applying the asymptotic formula for the Struve function [1]

\[
H_\nu(y) = \begin{cases} 
O(y^{-1/2}), & y \to \infty, \quad \Re \nu < 1/2, \\
O(y^{\Re \nu}), & y \to 0, \quad \forall \nu,
\end{cases}
\]

we obtain that \( \sqrt{N_y} H_\nu(N y) \), \( |\Re \nu| < 1/2 \), is uniformly bounded. The function \( (d/dx) [d^2/dx^2 + (1/x^2) ((1/4) - \nu^2)]^{n-1} f(N) \) tends to 0 as \( N \) approaches infinity (property (v)), therefore, the expression on the right hand side of (42) tends to 0 as \( N \) approaches infinity. From (iv) we see that \( (d/dx) [d^2/dx^2 + (1/x^2) ((1/4) - \nu^2)]^{n-1} f(1/N) \) has order \( O(N) \), whereas function \( \sqrt{y/N} H_\nu(y/N) \) has order \( O(N^{-3/2-\nu}) \).

Hence, expression (43) approaches 0 as \( N \) tends to infinity. Function \( \sqrt{y/N} H_\nu(y/N) \) has order \( O(N^{-1}) \), whereas the expression \( [d^2/dx^2 + (1/x^2) ((1/4) - \nu^2)]^{n-1} f(N) \) is \( o(1) \) (property (iii)), therefore, expression (44) is \( o(1) \). The function \( y/\sqrt{N} H_\nu(1/N) \) is \( O(1) \), hence, property (iii) shows that (45) is \( o(1) \). Since \( \sqrt{N_y} H_\nu(y/N) \) has the order \( O(N^{-1/2-\nu}) \), and \( [d^2/dx^2 + (1/x^2) ((1/4) - \nu^2)]^{n-1} f(1/N) \) is \( o(1) \) (property (iii)), expression (46) is also \( o(1) \). The function \( \sqrt{y/N} H_{\nu-1}(y/N) \) has the order \( O(N^{-1/2-\nu}) \), hence, property (iii) shows that (47) is \( o(1) \).

Therefore, the right hand side of (42), as well as all functions (43) - (49), except (48), vanish as \( N \) tends to infinity, whereas expression (48) converges to \( -y^2 g_{n-1}(y) \). Consequently, \( g_n(y) = -y^2 g_{n-1}(y) \), and therefore, \( g_n(y) = (-y^2)^n g_0(y) \), \( n = 0,1, \ldots \). Thus \( g(y) = g_0(y) \) such that \( y^n g(y) \in L_2(R_+) \), \( n = 0,1, \ldots \), is the H-transform \( H_\nu \) of the function \( f(x) \). But the H-transform \( H_\nu \) is the inverse of the Y-transform \( Y_\nu \) if \( -1/2 < \Re \nu < 0 \), so we obtain that and \( f \) is the Y-transform \( Y_\nu \), \( -1/2 < \Re \nu < 0 \), of a function \( g \) such that \( y^n g(y) \in L_2(R_+) \), \( n = 0,1, \ldots \).

(b-ii) Let now \( 0 < \Re \nu < 1/2 \). The inverse of the Y-transform \( Y_\nu \) in the range \( 0 < \Re \nu < 1 \) has the form [3]

\[
g(y) = y^{\nu+1/2} \int_0^\infty \sqrt{x y} \left[ H_{\nu+1}(x y) - \frac{(x y)^\nu}{2^\nu \sqrt{\pi \Gamma(\nu + 3/2)} } \right] f(x) \, dx, \quad y \in R_+,
\]
that can be expressed in an equivalent form

$$
(52) \quad g(y) = \lim_{N \to \infty} \int_{1/N}^{N} \left[ \sqrt{xy} H_\nu(xy) - \frac{(xy)^{\nu-1/2}}{2^{\nu-1} \sqrt{\pi \Gamma(\nu + 1/2)}} \right] f(x) \, dx, \quad y \in R_+,
$$

where the limit is understood in the $L_2(R_+)$ norm. Putting

$$
(53) \quad g_n^N(y) = \int_{1/N}^{N} \left[ \sqrt{xy} H_\nu(xy) - \frac{(xy)^{\nu-1/2}}{2^{\nu-1} \sqrt{\pi \Gamma(\nu + 1/2)}} \right] \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \, dx,
$$

we see that $g_n^N(y)$ tends to some functions $g_n(y)$ in the $L_2$ norm as $N \to \infty$. Let $n \geq 1$. Integrating (53) by parts twice and using formulae (40), (41) we obtain

$$
(54) \quad g_n^N(y) = \sqrt{Ny} H_\nu(Ny) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N)
$$

$$
(55) \quad - \sqrt{N} H_\nu(y/N) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N)
$$

$$
(56) \quad + \left( \nu - \frac{1}{2} \right) \sqrt{N} H_\nu(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N)
$$

$$
(57) \quad - y \sqrt{Ny} H_{\nu-1}(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N)
$$

$$
(58) \quad + \left( \frac{1}{2} - \nu \right) \sqrt{Ny} H_\nu(y/N) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N)
$$

$$
(59) \quad + y \sqrt{N} H_{\nu-1}(y/N) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N)
$$

$$
(60) \quad - y^2 \int_{1/N}^{N} \sqrt{xy} \left[ H_\nu(xy) - \frac{2^{1-\nu} (xy)^{\nu-1}}{\sqrt{\pi \Gamma(\nu + 1/2)}} \right] \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \, dx
$$

$$
(61) \quad - \frac{2^{1-\nu} (N)^{\nu-1/2}}{\sqrt{\pi \Gamma(\nu + 1/2)}} \int_{1/N}^{N} x^{\nu-1/2} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \, dx.
$$

When $N$ tends to infinity integral (61) vanishes because of property (vi) and $n \geq 1$. Reasoning the same as before, we can conclude that the right hand side of (54), as well as all functions (55)-(59), vanish as $N$ tends to infinity, whereas the expression (60)
converges to \(-y^2g_{n-1}(y)\). Consequently, \(g_n(y) = -y^2g_{n-1}(y)\), and therefore, \(g_n(y) = (-y^2)^ng_0(y), \ n = 0, 1, \ldots\). Thus \(g(y) = g_0(y)\) such that \(y^{2n}g(y) \in L_2(R_+), \ n = 0, 1, \ldots\), is the transform (52) of function \(f(x)\). But transform (52) is the inverse of the Y-transform \(Y_{v}\) if \(0 < \Re v < 1/2\), so we obtain that and \(f\) is the Y-transform \(Y_{v}\), \(0 < \Re v < 1/2\), of a function \(g\) such that \(y^ng(y) \in L_2(R_+), \ n = 0, 1, \ldots\). Theorem 1 is thus proved. 

Remark. The case \(\Re v = 0\) has been excluded from Theorem 1. It was proved in [3] that in this case the range of the Y-transform in \(L_2(R_+)\) is a proper subspace of \(L_2(R_+)\).

3. Y-T Transform of Square Integrable Functions with Compact Supports

Now we describe the Y-transform of square integrable functions with compact supports (the Paley-Wiener theorem for the Y-transform).

**Theorem 2.** A function \(f\) is the Y-transform \(Y_{v}\), \(0 < |\Re v| < 1/2\), of a square integrable function \(g\) with compact support on \([0, \infty)\) if and only if \(f\) satisfies conditions (i)-(vi) of Theorem 1 and moreover,

\[
\lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/(2n)} = \sigma_g < \infty,
\]

where

\[
\sigma_g = \sup \{ y : y \in \text{supp} \ g \},
\]

and the support of a function is the smallest closed set outside which the function vanishes almost everywhere [12].

**Proof:** (a) Let \(f(x)\) be the Y-transform of \(g(y) \in L_2(R_+)\) and \(\sigma_g < \infty\):

\[
f(x) = \int_0^{\sigma_g} \sqrt{xy} Y_{\nu}(xy)g(y) dy, \ 0 < |\Re \nu| < 1/2.
\]

One can assume that \(\sigma_g > 0\), otherwise it is trivial. Since \(\sigma_g < \infty\) we have \(y^ng(y) \in L_2(R_+)\) for all \(n = 0, 1, 2, \ldots\). Therefore, \(f\) satisfies conditions (i)-(vi) of Theorem 1. Furthermore,

\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = \int_0^{\sigma_g} \sqrt{xy} Y_{\nu}(xy)(-y^2)^ng(y) dy.
\]

Consequently, applying the inequality (2) for the Y-transform (65), we obtain

\[
\left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^2 \leq C \int_0^{\sigma_g} y^{4n} |g(y)|^2 dy \leq C\sigma_g^{4n} \int_0^{\sigma_g} |g(y)|^2 dy.
\]
Hence,

\begin{equation}
\lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/(2n)} \leq \lim_{n \to \infty} C^{1/(4n)} \sigma_g \left\{ \int_0^{\sigma_g} |g(y)|^2 dy \right\}^{1/(4n)} = \sigma_g.
\end{equation}

On the other hand, since \( \sigma_g \) is the least upper bound of the support of \( g \), for every \( \epsilon, \ 0 < \epsilon < \sigma_g \), we have

\begin{equation}
\int_{\sigma_g - \epsilon}^{\sigma_g} |g(y)|^2 dy > 0.
\end{equation}

Consequently, using now inequality (3) for the Y-transform (65), we get

\begin{equation}
\lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/(2n)} \geq \lim_{n \to \infty} C^{-1/(4n)} \left\{ \int_{\sigma_g - \epsilon}^{\sigma_g} y^n |g(y)|^2 dy \right\}^{1/(4n)} \geq (\sigma_g - \epsilon) \lim_{n \to \infty} C^{-1/(4n)} \left\{ \int_{\sigma_g - \epsilon}^{\sigma_g} |g(y)|^2 dy \right\}^{1/(4n)} = \sigma_g - \epsilon.
\end{equation}

Because \( \epsilon \) can be chosen arbitrarily small, from (69) and (67) we obtain (62).

(b) Suppose now that \( f \) satisfies the conditions of Theorem 1, and the limit in (62) exists and equals \( \sigma < \infty \). Applying Theorem 1 we see that \( f \) is the Y-transform \( Y_\nu \) of a function \( g \) with \( \sigma_g \) defined by (63) such that \( y^n g(y) \in L_2(R_+) \), \( n = 0,1,\ldots \). We prove that \( \sigma_g < \infty \) and moreover, \( \sigma = \sigma_g \). Theorem 1 implies that (11) holds. Therefore, using inequalities (2) and (3) we obtain

\begin{equation}
C^{-1} \left\| y^{2n} g(y) \right\|_2 \leq \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2 \leq C \left\| y^{2n} g(y) \right\|_2.
\end{equation}

Hence,

\begin{equation}
\lim_{n \to \infty} C^{-1/(2n)} \left\| y^{2n} g(y) \right\|_2^{1/(2n)} \leq \lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/(2n)} = \sigma
\end{equation}

\begin{equation}
\leq \lim_{n \to \infty} C^{1/(2n)} \left\| y^{2n} g(y) \right\|_2^{1/(2n)}.
\end{equation}

Consequently,

\begin{equation}
\lim_{n \to \infty} \left\| y^{2n} g(y) \right\|_2^{1/(2n)} = \sigma.
\end{equation}

Suppose that \( \sigma_g > \sigma \). Then there exists a positive \( \epsilon \) such that

\begin{equation}
\int_{\sigma_g + \epsilon}^{\infty} |g(y)|^2 dy > 0.
\end{equation}
We have
\[ \sigma = \lim_{n \to \infty} \| y^{2n} g(y) \|_{2}^{1/(2n)} \geq \lim_{n \to \infty} \left\{ \int_{\sigma + \epsilon}^{\infty} y^{4n} |g(y)|^2 \, dy \right\}^{1/(4n)} \]
\[ \geq (\sigma + \epsilon) \lim_{n \to \infty} \left\{ \int_{\sigma + \epsilon}^{\infty} |g(y)|^2 \, dy \right\}^{1/(4n)} = \sigma + \epsilon. \]
(74)
This is impossible. Hence, \( \sigma_g \leq \sigma \) and therefore, the function \( g \) has a compact support.

Suppose now that \( \sigma_g < \sigma \). Then there exists a positive \( \epsilon \) such that
\[ \int_{\sigma - \epsilon}^{\infty} |g(y)|^2 \, dy = 0. \]
(75)
We have
\[ \sigma = \lim_{n \to \infty} \| y^{2n} g(y) \|_{2}^{1/(2n)} \leq \lim_{n \to \infty} \left\{ \int_{0}^{\sigma - \epsilon} y^{4n} |g(y)|^2 \, dy \right\}^{1/(4n)} \]
\[ \leq (\sigma - \epsilon) \lim_{n \to \infty} \left\{ \int_{0}^{\sigma - \epsilon} |g(y)|^2 \, dy \right\}^{1/(4n)} = \sigma - \epsilon. \]
(76)
This is also impossible. Hence, \( \sigma_g \geq \sigma \), and consequently, \( \sigma_g = \sigma < \infty \). Theorem 2 is thus proved.

REMARK. If a function \( f \) satisfies conditions of Theorem 1, then the limit (62) always exists. It equals to infinity if \( f \) is the Y-transform \( Y_{\nu} \) of a function \( g \) with unbounded support.

4. Y-TRANSFORM OF ANALYTIC FUNCTIONS

We consider now the Y-transform \( Y_{\nu} \) of functions analytic in some angle.

THEOREM 3. The Y-transform \( Y_{\nu} \), \(-1 < \Re \nu < 1\), maps the space of all functions \( g(z) \), regular in the angle \(-\alpha < \arg z < \beta\), where \( 0 < \alpha, \beta \leq \pi \); of the order \( O\left(|z|^{-\alpha-\epsilon}\right) \) for small \( z \), and \( O\left(|z|^{-b+\epsilon}\right) \) for large \( z \), where \( a < 1/2 < b \), uniformly for any small positive \( \epsilon \) in any angle interior to the above; and satisfying conditions
\[ \int_{0}^{\infty} y^{\nu+2n+1/2} g(y) \, dy = 0, \quad n \in (-b/2 - \Re \nu/2 - 1/4, -a/2 - \Re \nu/2 - 1/4), \]
\[ \int_{0}^{\infty} y^{-\nu+2n+1/2} g(y) \, dy = 0, \quad n \in (-b/2 + \Re \nu/2 - 1/4, -a/2 + \Re \nu/2 - 1/4), \]
(77)
for all nonnegative integers \( n \), if there exists such \( n \), one-to-one onto the space of all functions \( f(z) \), regular in the angle \(-\beta < \arg z < \alpha\), of the order \( O\left(|z|^{1-b-\epsilon}\right) \) for
small $z$, and $O\left(|z|^{1-a+\varepsilon}\right)$ for large $z$, uniformly for any small positive $\varepsilon$ in any angle interior to the above; and satisfying conditions

$$\int_{0}^{\infty} x^{\nu-2n-1/2} f(x) \, dx = 0, \quad n \in (a/2 + \Re \nu - 1/4, b/2 + \Re \nu - 1/4),$$

(78)

$$\int_{0}^{\infty} x^{\nu+2n+3/2} f(x) \, dx = 0, \quad n \in (-b/2 - \Re \nu - 3/4, -a/2 - \Re \nu - 3/4),$$

for all nonnegative integers $n$, if there exists such $n$. (For example, if $\Re \nu = 0$, then $n = 0$ always belongs to the interval $(a/2 - 1/4, b/2 - 1/4)$.)

**Proof:** Let $g(z)$ satisfy the conditions of Theorem 3. Then the function $g(z)$ on $R_+$ belongs to $L_2(R_+)$ and its Mellin transform $g^*(s)$

(79)

$$g^*(s) = \int_{0}^{\infty} x^{s-1} g(x) \, dx$$

is an analytic function of $s$, regular for $a < \Re s < b$; and

(80)

$$g^*(s) = \begin{cases} O\left(e^{-(\beta+\varepsilon)\Im s}\right), & \Im s \to \infty \\ O\left(e^{(\alpha-\varepsilon)\Im s}\right), & \Im s \to -\infty \end{cases}$$

for every positive $\varepsilon$, uniformly in any strip interior to $a < \Re s < b$ (see [8]). Let $f(x)$ be the Y-transform $Y_{\nu}^{-1} < \Re \nu < 1$, of $g(y)$. Since $g(y)$ belongs to $L_2(R_+)$, the Parseval identity for the Y-transform $Y_{\nu}$ holds on the line $\Re s = 1/2$ [6]:

(81)

$$f^*(s) = 2^{s-1} \frac{\Gamma \left( \frac{1}{4} + \frac{\nu}{2} + \frac{s}{2} \right) \Gamma \left( \frac{1}{4} - \frac{\nu}{2} + \frac{s}{2} \right)}{\Gamma \left( -\frac{1}{4} - \frac{\nu}{2} + \frac{s}{2} \right) \Gamma \left( \frac{5}{4} + \frac{\nu}{2} - \frac{s}{2} \right)} g^*(1-s).$$

Because of (77) the function $g^*(1-s)$ equals 0 at the poles of function $\Gamma(1/4 + \nu/2 + s/2)$ \newline $\Gamma(1/4 - \nu/2 + s/2)$ in the strip $1-b < \Re s < 1-a$, if there exists one. Hence, from (81) one can see that $f^*(s)$ is analytic in the strip $1-b < \Re s < 1-a$. Furthermore, since the function $2^{s-1/2} \left( \Gamma(1/4 + \nu/2 + s/2) \Gamma(1/4 - \nu/2 + s/2) \right)/\left( \Gamma(-1/4 - \nu/2 + s/2) \Gamma(5/4 + \nu/2 - s/2) \right)$ is uniformly bounded in any compact domain in the strip $1-b < \Re s < 1-a$, not containing the poles of function $\Gamma(1/4 + \nu/2 + s/2) \Gamma(1/4 - \nu/2 + s/2)$, and has at most only polynomial growth as $\Im s \to \pm\infty$, from (80) we see that function $f^*(s)$ decays exponentially

(82)

$$f^*(s) = \begin{cases} O\left(e^{(\beta+\varepsilon)\Im s}\right), & \Im s \to -\infty \\ O\left(e^{-(\alpha-\varepsilon)\Im s}\right), & \Im s \to \infty \end{cases}$$

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for every positive \( \varepsilon \), uniformly in any strip interior to \( 1 - b < \Re s < 1 - a \). Hence, its inverse Mellin transform \( f(z) \) is regular for \( -\beta < \arg z < \alpha \), and of the order \( O \left( |z|^\beta - 1 - \varepsilon \right) \) for small \( z \), and \( O \left( |z|^\alpha - 1 + \varepsilon \right) \) for large \( z \), uniformly in any angle interior to the above angle for any small positive \( \varepsilon \) \[8\]. Moreover, \( f^*(s) \) has zeros at the poles of the function \( \Gamma(-1/4 - \nu/2 + s/2)\Gamma(5/4 + \nu/2 - s/2) \) in the strip \( 1 - b < \Re s < 1 - a \), if there exists one. Hence (78) holds.

Conversely, let \( f(z) \) satisfy the conditions of Theorem 3. Then \( f(z) \) on \( R_+ \) belongs to \( L_2(R_+) \) and its Mellin transform (79) \( f^*(s) \) is analytic in the strip \( 1 - b < \Re s < 1 - a \) and satisfies (82). Furthermore, because of (78) the function \( f^*(s) \) vanishes at the poles of the function \( \Gamma(-1/4 - \nu/2 + s/2)\Gamma(5/4 + \nu/2 - s/2) \) in the strip \( 1 - b < \Re s < 1 - a \), if there exists one. Therefore, if we express \( f^*(s) \) in the form (81), function \( g^*(s) \) is analytic in the strip \( a < \Re s < b \); and has the asymptotics (80) for every positive \( \varepsilon \), uniformly in any strip interior to \( a < \Re s < b \). Furthermore, \( g^*(1 - s) \) has zeros at the poles of the function \( \Gamma(1/4 + \nu/2 + s/2)\Gamma(1/4 - \nu/2 + s/2) \) in the strip \( 1 - b < \Re s < 1 - a \). Consequently, the inverse Mellin transform \( g(z) \) of \( g^*(s) \) satisfies the conditions of Theorem 3 and \( f \) is the Y-transform of \( g \).

If in Theorem 3 we take \( \alpha = \beta \) and \( 0 < a < \min \{ |\nu|, |\nu + 1|, |\nu - 1| \} \), then in the strip \( 1/2 - a < \Re s < 1/2 + a \) there are no poles or zeros of the function \( 2^{s-1/2} \left( \Gamma\left(1/4 + \nu/2 + s/2\right)\Gamma\left(1/4 - \nu/2 + s/2\right) \right)/\left( \Gamma\left(-1/4 - \nu/2 + s/2\right)\Gamma\left(5/4 + \nu/2 - s/2\right) \right) \), hence, we have

**Corollary 1.** The Y-transform \( Y_\nu, 0 < |\Re \nu| < 1, \) is a bijection in the space of all functions, regular in the angle \( |\arg z| < \alpha \), where \( 0 < \alpha \leq \pi \); of order \( O \left( |z|^\alpha - 1/2 - \varepsilon \right) \) for small \( z \), and \( O \left( |z|^{-\alpha - 1/2 + \varepsilon} \right) \) for large \( z \), uniformly for any small positive \( \varepsilon \), \( 0 < \varepsilon < a \), in any angle interior to the above, where \( 0 < a < \min \{ |\nu|, |\nu + 1|, |\nu - 1| \} \).

5. **Y-Transform in Some Other Spaces of Functions**

In [9, 10] the Y-transform is proved to be a bijection in some spaces of functions \( \mathcal{M}_c^\odot(L) \) introduced there. In this section the Y-transform in a space of functions including the spaces \( \mathcal{M}_c^{-1}(L) \) as special cases is considered.

Let \( \Phi \) be any linear subspace of either \( L_1(R) \) or \( L_2(R) \) having properties:

(i) if \( \varphi(t) \in \Phi \) then \( \varphi(-t) \in \Phi \);

(ii) functions \( \varphi(t) = 2^{it}\Gamma(1/2 + \nu/2 + it/2)\Gamma(1/2 - \nu/2 + it/2) \sin(\pi/2) \) \( (it - \nu), 0 < |\Re \nu| < 1, \) and \( \varphi^{-1}(t) \) are multipliers of \( \Phi \).

It is easy to see that \( \varphi^{-1}(-t) \) is also a multiplier of \( \Phi \). The multipliers \( \varphi(t) \) and \( \varphi^{-1}(t) \) are infinitely differentiable and uniformly bounded on \( R \), and their derivatives
grow logarithmically. Therefore, many classical spaces on $R$ are special cases of $\Phi$ (for example, any $L_1$ or $L_2$ space with $L_\infty$-weights, the Schwartz space $S(R)$, and the space of infinitely differentiable functions with compact support [12]). On $R_+$ we define $\mathcal{M}^{-1}(\Phi)$ to be the space of all functions $g$ that can be represented in the form

$$g(x) = \int_{-\infty}^{\infty} \phi(t)x^{it-1/2}dt,$$

almost everywhere, where $\phi \in \Phi$ (if $\phi \notin L_1(R)$ the integral should be understood as the inverse Mellin transform in $L_2$ [8]). The spaces $\mathcal{M}^{-1}_{\gamma,L}(L)$ [10] as well as the space of functions considered in Corollary 1 are special cases of $\mathcal{M}^{-1}(\Phi)$.

**Theorem 4.** The $Y$-transform $Y_{\nu}, 0 < |\Re \nu| < 1$, is a bijection in $\mathcal{M}^{-1}(\Phi)$.

**Proof:** From (83) we see that if $g \in \mathcal{M}^{-1}(\Phi)$ then $g$ can be expressed in the form of the inverse Mellin transform

$$g(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} g^*(s)x^{-s}ds,$$

where $g^*(1/2 + it) \in \Phi$. Using formula (36) we obtain that the Mellin transform (79) of the function $k(x) = \sqrt{x}Y_{\nu}(x)$ is $k^*(s) = \varphi(i/2 - is)$. Applying the Parseval equation for the Mellin transform

$$\int_{0}^{\infty} k(xy)g(y)dy = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} k^*(s)g^*(1-s)x^{-s}ds, \quad 0 < |\Re \nu| < 1,$$

that has been proved for $g^*(1/2 + it) \in L_2(R)$ in [8] and $g^*(1/2 + it) \in L_1(R)$ in [9], we obtain

$$\mathcal{Y}_\nu g(x) = \int_{0}^{\infty} \sqrt{xy}Y_{\nu}(xy)g(y)dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t)g^*(1/2-it)x^{-it-1/2}dt.$$

Since $\varphi(t)$ and $\varphi^{-1}(-t)$ are multipliers of $\Phi$, then $\varphi(t)g^*(1/2-it)$ belongs to $\Phi$ if and only if $g^*(1/2 + it)$ belongs to $\Phi$. Therefore, $(\mathcal{Y}_\nu g)(x) \in \mathcal{M}^{-1}(\Phi)$ if and only if $g \in \mathcal{M}^{-1}(\Phi)$. Theorem 4 is thus proved.

**References**


