NONEXPANSIVE MAPPINGS AND EXPANSIVE MAPPINGS ON THE UNIT SPHERES OF SOME $F$-SPACES

DONG-NI TAN

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Abstract

This paper gives a characterization of nonexpansive mappings from the unit sphere of $\ell^\beta(\Gamma)$ onto the unit sphere of $\ell^\beta(\Delta)$ where $0 < \beta \leq 1$. By this result, we prove that such mappings are in fact isometries and give an affirmative answer to Tingley’s problem in $\ell^\beta(\Gamma)$ spaces. We also show that the same result holds for expansive mappings between unit spheres of $\ell^\beta(\Gamma)$ spaces without the surjectivity assumption.

Keywords and phrases: nonexpansive mapping, expansive mapping, isometric extension, Tingley’s problem, $\ell^\beta(\Gamma)$ spaces.

1. Introduction

A mapping $V$ between two metric spaces $(X, d_X)$ and $(Y, d_Y)$ is called nonexpansive if it is a 1-Lipschitz map. That is,

$$d_Y(V(x), V(y)) \leq d_X(x, y) \quad \forall x, y \in X.$$  (1.1)

The mapping $V$ is called an isometry if equality holds in (1.1) for all $x, y \in X$, and it is called expansive if ‘$\leq$’ is replaced by the inverse inequality ‘$\geq$’.

By a direct compactness argument or by Freudenthal and Hurewicz’s result [9], every nonexpansive map from a compact metric space onto itself must be an isometry. This does not always hold with the assumption of compactness replaced by boundedness in infinite-dimensional metric linear spaces. For example, a map $T : B(\ell^p) \to B(\ell^p)$ defined by $T(\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) = (\xi_2, \xi_3, \ldots, \xi_n, \ldots)$ for all $\{\xi_n\}_{n \geq 1}$ in $B(\ell^p)$ where $B(\ell^p)$ denotes the unit ball of $\ell^p$ and $0 < p \leq \infty$ is such a nonexpansive but not isometric map from $B(\ell^p)$ onto itself. However, what interests us is such maps defined only on the unit sphere, which can be connected with the isometric extension problem raised by Tingley in [12] and described as follows.
Let $E$ and $F$ be normed spaces with unit spheres $S(E)$ and $S(F)$, respectively. Suppose that $V_0 : S(E) \rightarrow S(F)$ is an onto isometry. Is there a linear isometry $V : E \rightarrow F$ such that $V|_{S(E)} = V_0$?

In recent years, Ding and his students have been working on this topic and have obtained many important results (see [1–7, 10, 13, 15]). Ding [2] showed that every onto nonexpansive map between unit spheres of Hilbert spaces is an isometry and answered Tingley’s problem affirmatively for Hilbert spaces. In recent work [11], the author proved that the only nonexpansive mappings from the unit sphere of $\ell^\infty(\Gamma)$-type spaces (including $c_{00}$, $c$, $\ell^\infty$) onto the unit sphere of $\mathcal{L}^\infty(\Delta)$ are those arising from a bijection between $\Delta$ and $\Gamma$ and a sign pattern. This result yields the fact that such maps are isometries and an affirmative answer to Tingley’s problem for $\mathcal{L}^\infty(\Gamma)$-type spaces. A similar result for $\ell^p(\Gamma)$ spaces where $1 < p < \infty$ can be obtained by combining the main result in [3] with that of [8]. For the case $p = 1$, Wang [14] established that every expansive map $T$ from $S(\ell^1(\Gamma))$ onto $S(\ell^1(\Delta))$ with an additional condition $\bigcup_{y \in \Gamma} \text{supp}(e_y) = \Delta$ is an isometry and can be linearly and isometrically extended to $\ell^1(\Gamma)$. In this paper, we extend these results to $F$-spaces $\ell^\beta(\Gamma)$ where $0 < \beta \leq 1$, and in the $\ell^1(\Gamma)$ case we point out that the condition $\bigcup_{y \in \Gamma} \text{supp}(e_y) = \Delta$ in [14] can be removed.

Throughout this paper, we consider spaces over the real field. Given a nonempty index set $\Gamma$, for every $0 < \beta \leq 1$, the space

$$\ell^\beta(\Gamma) = \left\{ x = \{ \xi_y \}_{y \in \Gamma} : \sum_{y \in \Gamma} |\xi_y|^\beta < \infty \right\}$$

is known as an $F$-space with an $F$-norm $\|x\| = \sum_{y \in \Gamma} |\xi_y|^\beta$. As usual, for every $x = \{ \xi_y \}_{y \in \Gamma} \in \ell^\beta(\Gamma)$, supp $x = \{ y \in \Gamma : \xi_y \neq 0 \}$ and $S(\ell^\beta(\Gamma))$ denotes the unit sphere of $\ell^\beta(\Gamma)$.

2. Main results

**Lemma 2.1.** Let $x$, $y \in \ell^\beta(\Gamma)$. Then

$$\|x + y\| = \|x\| + \|y\|$$

if and only if supp $x \cap$ supp $y = \emptyset$ for $0 < \beta < 1$ and $x \cdot y \geq 0$ for $\beta = 1$, where $x \cdot y \geq 0$ means $x(\gamma) \cdot y(\gamma) \geq 0$ for every $\gamma \in \Gamma$.

**Proof.** The proof in the case of $\beta = 1$ is trivial. For $0 < \beta < 1$, observe that the function $f(t) = t^\beta$ is strictly concave on $(0, \infty)$. It follows that

$$|\xi + \eta|^\beta \leq |\xi|^\beta + |\eta|^\beta$$

for all $\xi, \eta \in \mathbb{R}$ and equality holds if and only if $\xi \cdot \eta = 0$. The desired result is easily obtained from this. □
**Lemma 2.2.** Let $x \in S(\ell^\beta(\Gamma))$. Then for every $\gamma \in \Gamma$, 
\[
\max\{\|x + e_\gamma\|, \|x - e_\gamma\|\} \geq 2^\beta.
\]

**Proof.** As $\|x\| = 1$, it is easy to see that 
\[
\max\{\|x + e_\gamma\|, \|x - e_\gamma\|\} = (|x(\gamma)| + 1)^\beta + 1 - |x(\gamma)|^\beta.
\]
Since the function $\varphi(t) = (1 + t)^\beta - t^\beta$ is decreasing on $[0, \infty)$, it follows that 
\[
(|x(\gamma)| + 1)^\beta + 1 - |x(\gamma)|^\beta = 1 + \varphi(|x(\gamma)|) \geq 1 + \varphi(1) = 2^\beta,
\]
which completes the proof. \hfill \Box

**Lemma 2.3.** Let $T : S(\ell^\beta(\Gamma)) \to S(\ell^\beta(\Delta))$ be a nonexpansive map. For each $\delta \in \Delta$, if $\pm e_\delta \in T(S(\ell^\beta(\Gamma)))$, then there is a unique $\gamma \in \Gamma$ and a sign $\theta_\delta$ such that 
\[
T(\pm e_\gamma) = \pm \theta_\delta e_\delta.
\]

**Proof.** The hypothesis $\pm e_\delta \in T(S(\ell^\beta(\Gamma)))$ ensures that there exist $x, y \in S(\ell^\beta(\Gamma))$ such that $T(x) = e_\delta$ and $T(y) = -e_\delta$. We first claim that $x$ and $y$ are dependent, that is, 
\[
x = -y.
\]
Assume that the claim is not true. Define a map $f : [0, 1] \to S(\ell^\beta(\Gamma))$ by 
\[
f(\lambda) = \frac{(1 - \lambda)x + \lambda y}{\|(1 - \lambda)x + \lambda y\|^{1/\beta}}.
\]
It is clear that $\{f(\lambda) : \lambda \in [0, 1]\}$ is a connected path from $x$ to $y$. Hence the map 
\[
\phi(\lambda) = \|T(f(\lambda)) + e_\delta\| - \|T(f(\lambda)) - e_\delta\|
\]
is continuous on $[0, 1]$. Since $\phi(0) = 2^\beta$ and $\phi(1) = -2^\beta$, we can find $\lambda_0 \in (0, 1)$ such that $\phi(\lambda_0) = 0$, that is, 
\[
\|T(f(\lambda_0)) + e_\delta\| = \|T(f(\lambda_0)) - e_\delta\|.\]
The definition of the norm in $\ell^\beta(\Delta)$ yields $T(f(\lambda_0))(\delta) = 0$, and thus 
\[
\|T(f(\lambda_0)) + e_\delta\| = \|T(f(\lambda_0)) - e_\delta\| = 2.
\]
This shows that 
\[
\|f(\lambda_0) - y\| = \|f(\lambda_0) - x\| = 2.
\]
By Lemma 2.1 we get that for $0 < \beta < 1$, supp $f(\lambda_0) \cap \text{supp } x \cup \text{supp } y = \emptyset$ and for $\beta = 1$, $f(\lambda_0) \cdot x \leq 0$ and $f(\lambda_0) \cdot y \leq 0$. This is impossible by the definition of $f$. Therefore the claim is proved.
Let $T$ and therefore $\text{supp } x$ be a singleton. If this does not hold, then there is a $\gamma_1 \in \Gamma$ satisfying $0 < |x(\gamma_1)| < 1$. Write $x_1 = x - 2x(\gamma_1)e_{\gamma_1}$. Then by the claim

$$
\| T(x_1) - e_\delta \| = \| T(x_1) - T(x) \| \leq \| x_1 - x \| = 2^\beta |x(\gamma_1)|^\beta < 2^\beta,
$$

$$
\| T(x_1) + e_\delta \| = \| T(x_1) - T(-x) \| \leq \| x_1 + x \| = 2^\beta (1 - |x(\gamma_1)|^\beta) < 2^\beta.
$$

This contradicts Lemma 2.2 and therefore $\text{supp } x$ is a singleton.

Let $\{ \gamma \} = \text{supp } x$ and $\theta_\delta = x(\gamma)$. Noticing that the uniqueness of $\gamma$ is easily obtained from the claim, this completes the proof.

We are now ready to present one of our main results.

**Theorem 2.4.** Let $T : S(\ell^p(\Gamma)) \to S(\ell^p(\Delta))$ be a surjective nonexpansive map. Then $T$ is an isometry and there is a family of signs $\{ \theta_\delta \}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \to \Gamma$ such that, for any element $x \in S(\ell^p(\Gamma))$,

$$
T(x)(\delta) = \theta_\delta x(\sigma(\delta)) \quad \forall \delta \in \Delta. \tag{2.1}
$$

**Proof.** It is evident that $T$ is an isometry if there is a family of signs $\{ \theta_\delta \}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \to \Gamma$ such that (2.1) holds. Thus it suffices to prove this. By Lemma 2.3 we can define $\sigma : \Delta \to \Gamma$ and $\{ \theta_\delta \}_{\delta \in \Delta}$ such that

$$
T(\pm e_{\sigma(\delta)}) = \pm \theta_\delta e_\delta \quad \forall \delta \in \Delta. \tag{2.2}
$$

It is obvious that $\sigma$ is injective. To see that $\sigma$ is surjective and that (2.1) holds, for every $y = \sum \eta_\delta e_\delta \in S(\ell^p(\Delta))$, take $x = \sum \xi_\gamma e_\gamma \in S(\ell^p(\Gamma))$ such that $T(x) = y$.

For any $\delta \in \Delta$ with $\xi_{\sigma(\delta)} \neq 0$,

$$
\| y - \text{sign}(\xi_{\sigma(\delta)}) \theta_\delta e_\delta \| = |\eta_\delta - \text{sign}(\xi_{\sigma(\delta)}) \theta_\delta|\beta + 1 - |\eta_\delta|\beta.
$$

On the other hand, clearly,

$$
\| x - \text{sign}(\xi_{\sigma(\delta)}) e_{\sigma(\delta)} \| = (1 - |\xi_{\sigma(\delta)}|)\beta + 1 - |\xi_{\sigma(\delta)}|\beta.
$$

The fact that $T$ is nonexpansive and (2.2) then give

$$
(1 - |\eta_\delta|)^\beta - |\eta_\delta|^\beta \leq |\eta_\delta - \text{sign}(\xi_{\sigma(\delta)}) \theta_\delta|\beta - |\eta_\delta|^\beta
$$

$$
\leq (1 - |\xi_{\sigma(\delta)}|)^\beta - |\xi_{\sigma(\delta)}|\beta. \tag{2.3}
$$

Noticing that $\phi(t) = (1 - t)^\beta - t^\beta$ is decreasing on $[0, 1]$, we see that

$$
|\eta_\delta| \geq |\xi_{\sigma(\delta)}|. \tag{2.4}
$$

Thus if $\text{supp } x \subset \sigma(\Delta)$, then by (2.4),

$$
1 = \sum_{\delta \in \Delta} |\xi_{\sigma(\delta)}| \leq \sum_{\delta \in \Delta} |\eta_\delta| = 1.
$$
As a result,
\[ |\eta_\delta| = |\xi_{\sigma(\delta)}| \] (2.5)
and inequality (2.3) turning out to be an equality obviously implies that
\[ \text{sign}(\eta_\delta) = \text{sign}(\xi_{\sigma(\delta)}) \theta_\delta. \] (2.6)

Since Equations (2.5) and (2.6) have already established that (2.1) holds for all \( x \in S(\mathcal{V}(\Gamma)) \) satisfying \( \text{supp} \ x \subset \sigma(\Delta) \), to finish the proof we only need to show that \( \sigma \) is surjective. Suppose to the contrary that there is a \( \gamma_0 \in \Gamma \setminus \sigma(\Delta) \). Choose \( \delta_0 \in \text{supp} \ T(e_{\gamma_0}) \) and put
\[ x_0^\pm = \frac{1}{2^{1/\beta}} e_{\gamma_0} \pm \frac{1}{2^{1/\beta}} e_{\sigma(\delta_0)} \quad \text{and} \quad \eta_\delta^\pm = T(x_0^\pm)(\delta_0). \]

It is easy to see from Lemma 2.3 that \( \text{supp} \ T(x_0^+)(\delta_1) \) cannot be a singleton. Thus we can let \( \delta_1 \in \text{supp} \ T(x_0^+) \) satisfy \( \delta_1 \neq \delta_0 \). Then write
\[ \eta_{\delta_1} = T(x_0^+)(\delta_1) \quad \text{and} \quad x_1 = \frac{1}{2^{1/\beta}} e_{\sigma(\delta_0)} - \frac{1}{2^{1/\beta}} \text{sign}(\eta_{\delta_1}) \theta_1 e_{\sigma(\delta_1)}. \]

Note from the above argument that
\[ T(x_1) = \frac{1}{2^{1/\beta}} \theta_{\delta_0} e_{\delta_0} - \frac{1}{2^{1/\beta}} \text{sign}(\eta_{\delta_1}) e_{\delta_1}. \]

It follows that
\[
1 = \|x_0^+ - x_1\| \geq \|T(x_0^+) - T(x_1)\|
\geq \left|\eta_{\delta_0}^+ - \frac{1}{2^{1/\beta}} \theta_{\delta_0}\right|^\beta + \left|\eta_{\delta_1} + \frac{1}{2^{1/\beta}} \text{sign}(\eta_{\delta_1})\right|^\beta
\geq \left|\eta_{\delta_0}^+ - \frac{1}{2^{1/\beta}} \theta_{\delta_0}\right|^\beta + \frac{1}{2}. \]

Thus \( \text{sign}(\eta_{\delta_0}^+) = \theta_{\delta_0} \). Similarly, we can also obtain \( \text{sign}(\eta_{\delta_0}^-) = -\theta_{\delta_0} \).

By (2.4), we have \( |\eta_{\delta_0}^\pm| \geq (1/2)^{1/\beta} \) and observe that
\[
2^{\beta-1} = \|x_0^+ - x_0^-\| \geq \|T(x_0^+) - T(x_0^-)\|
\geq |\eta_{\delta_0}^+ - \eta_{\delta_0}^-|^\beta = (|\eta_{\delta_0}^+| + |\eta_{\delta_0}^-|)^\beta. \] (2.7)

Consequently,
\[ \eta_{\delta_0}^+ = \frac{1}{2^{1/\beta}} \theta_{\delta_0} \quad \text{and} \quad \eta_{\delta_0}^- = -\frac{1}{2^{1/\beta}} \theta_{\delta_0}. \] (2.8)

Moreover, this and the inequality becoming an equality in (2.7) imply that
\[ T(x_0^+)(\delta) = T(x_0^-)(\delta) \quad \forall \delta \neq \delta_0. \] (2.9)
Now using the same technique as in Lemma 2.3, we define
\[
\phi(\lambda) = \|T(f(\lambda)) - T(x_0^+\| - \|T(f(\lambda)) - T(x_0^-)\|
\]
for all \( \lambda \in [0, 1] \) where \( f(\lambda) = ((1 - \lambda)x_0^+ + \lambda x_0^-) / (\|{(1 - \lambda)x_0^+ + \lambda x_0^-}\|^{1/\beta}) \).

Since \( \phi \) is continuous on \([0, 1]\) and \( \phi(0)\phi(1) < 0 \), there is a \( \lambda_0 \in (0, 1) \) such that
\[
\|T(f(\lambda_0)) - T(x_0^+)\| = \|T(f(\lambda_0)) - T(x_0^-)\|.
\]

Hence by the form of \( T(x_0^\pm) \) given by (2.8) and (2.9) we see that
\[
T(f(\lambda_0))(\delta_0) = 0.
\]

So
\[
\|f(\lambda_0) - e_{\sigma(\delta_0)}\| \geq \|T(f(\lambda_0)) - T(e_{\sigma(\delta_0)})\| = 2
\]
yields \( f(\lambda_0)(\sigma(\delta_0)) = 0 \).

It follows that \( f(\lambda_0) = e_{\gamma_0} \), that is, \( T(e_{\gamma_0})(\delta_0) = 0 \). This contradicts the choice of \( \delta_0 \). Thus the proof is complete. \( \square \)

**Remark 2.5.** In the case where \( \dim(\ell^\beta(\Gamma)) < \infty \), that is, the cardinality of \( \Gamma \) is finite, the above conclusion that \( T \) is an isometry cannot be simply obtained by a compactness argument or Freudenthal and Hurewicz’s result [9] which states that every nonexpansive map from a totally bounded metric space onto itself must be an isometry since the nonexpansive map is not assumed to be from \( S(\ell^\beta(\Gamma)) \) onto itself. The statement of Theorem 2.4 remains valid if we consider the quasi-Banach space consisting of all the points \( x = \{\xi_\gamma\}_{\gamma \in \Gamma} \in \ell^\beta(\Gamma) \) with the quasi-norm \( \|x\|_\beta = (\sum |\xi_\gamma|^\beta)^{1/\beta} \) for \( 0 < \beta < 1 \).

**Corollary 2.6.** Every surjective nonexpansive mapping \( T : S(\ell^\beta(\Gamma)) \rightarrow S(\ell^\beta(\Delta)) \) can be extended to a linear surjective isometry on \( \ell^\beta(\Gamma) \).

**Remark 2.7.** We can see from Lemma 2.3 that the surjection assumption of \( T \) in Theorem 2.4 and Corollary 2.6 in fact can reduce to \( \{\pm e_\delta\}_{\delta \in \Delta} \subseteq T(S(\ell^\beta(\Gamma))) \). On the other hand, by Theorem 2.4, we have in fact shown that every nonexpansive map \( T \) from \( S(\ell^\beta(\Gamma)) \) onto \( S(\ell^\beta(\Delta)) \) ensures that for every \( \gamma \in \Gamma \), \( \text{supp} T(e_{\gamma}) \) is a singleton. However, without the assumption of surjectivity or \( \{\pm e_\delta\}_{\delta \in \Delta} \subseteq T(S(\ell^\beta(\Gamma))) \) this is not always true. For example, let \( T : S(\ell^\beta(2)) \rightarrow S(\ell^\beta(3)) \) be defined by
\[
T(\xi_1 e_1 + \xi_2 e_2) = \xi_1(1/2^{1/\beta} e_1 + 1/2^{1/\beta} e_2) + \xi_2 e_3,
\]
where \( \{\xi_1, \xi_2\} \subseteq \mathbb{R} \) satisfies \( |\xi_1|^\beta + |\xi_2|^\beta = 1 \). Then \( T \) is an isometry, but \( e_1, e_2 \notin T(S(\ell^\beta(2))) \) and \( \text{supp} T(e_1) = \{1, 2\} \). Considering this example, we give a more general result for expansive maps on \( S(\ell^\beta(\Gamma)) \).
THEOREM 2.8. Let $T$ be an expansive map from $S(\ell^\beta(\Gamma))$ to $S(\ell^\beta(\Delta))$ such that $T(S(\ell^\beta(\Gamma))) = S(F)$, where $F$ is a linear closed subspace of $\ell^\beta(\Delta)$. Then $T$ is an isometry and can be extended to a linear isometry on $\ell^\beta(\Gamma)$.

PROOF. Since $\|T(x) - T(y)\| \geq \|x - y\|$ for all $x, y \in S(\ell^\beta(\Gamma))$, we see that $T$ is injective and its inverse $T^{-1}$ is nonexpansive. Note that $T^{-1}(T(e_\gamma)) = e_\gamma$ and $T^{-1}(T(-e_\gamma)) = -e_\gamma$ holds for all $\gamma \in \Gamma$. By the same argument as in Lemma 2.3, we deduce that

$$T(-e_\gamma) = -T(e_\gamma).$$

(2.10)

It follows that, for every $\gamma_1 \neq \gamma_2$,

$$\|T(e_{\gamma_1}) + T(e_{\gamma_2})\| \geq \|e_{\gamma_1} + e_{\gamma_2}\| = 2.$$

Hence

$$\|T(e_{\gamma_1}) + T(e_{\gamma_2})\| = \|T(e_{\gamma_1}) - T(e_{\gamma_2})\| = 2,$$

which together with Lemma 2.1 guarantees that

$$\text{supp } T(e_{\gamma_1}) \cap \text{supp } T(e_{\gamma_2}) = \emptyset.$$  
(2.11)

Thus $y = \sum \xi_\gamma T(e_\gamma)$ has norm one for every $\sum \xi_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$. Since $F$ is a linear closed subspace and $T(S(\ell^\beta(\Gamma))) = S(F)$, it follows that $y \in T(S(\ell^\beta(\Gamma)))$. Hence there is an element $x = \sum \alpha_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$ such that $T(x) = y$.

For any $\xi_\gamma \neq 0$, by (2.10) and (2.11) we get

$$\|T(x) - \text{sign}(\xi_\gamma) T(e_\gamma)\| = (1 - |\xi_\gamma|)^\beta + 1 - |\xi_\gamma|^\beta.$$

Furthermore,

$$\|x - \text{sign}(\xi_\gamma)e_\gamma\| = |\text{sign}(\xi_\gamma) - \alpha_\gamma|\beta + 1 - |\alpha_\gamma|^\beta.$$

Thus by the fact that $T$ is expansive,

$$|\text{sign}(\xi_\gamma) - \alpha_\gamma|\beta - |\alpha_\gamma|^\beta \leq (1 - |\xi_\gamma|)^\beta - |\xi_\gamma|^\beta.$$  
(2.12)

It follows that $|\alpha_\gamma| \geq |\xi_\gamma|$. This yields $1 = \sum |\alpha_\gamma|^\beta \geq \sum |\xi_\gamma|^\beta = 1$, which combined with (2.12) ensures that for every $\gamma$, $\alpha_\gamma = \xi_\gamma$ even if $\xi_\gamma = 0$. That is,

$$T\left(\sum \xi_\gamma e_\gamma\right) = \sum \xi_\gamma T(e_\gamma)  
(2.13)$$

for every $\sum \xi_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$.

Finally, by its property given by (2.13), $T$ is clearly an isometry and the desired extension $\tilde{T}$ defined by

$$\tilde{T}\left(\sum \xi_\gamma e_\gamma\right) = \sum \xi_\gamma T(e_\gamma) \quad \forall \sum \xi_\gamma e_\gamma \in \ell^\beta(\Gamma).$$

It is plain that $\tilde{T}$ is a linear isometry on $\ell^\beta(\Gamma)$ and its restriction to $S(\ell^\beta(\Gamma))$ is just $T$. The proof is complete. \qed
If $\beta = 1$, then some minor modifications of the previous example give a counterexample showing that there is an expansive map or, to be precise, an isometry between $S(\ell^1(\Gamma))$ and $S(\ell^1(\Delta))$ which cannot be linearly extended to the whole space. In fact, let $T : S(\ell^1(2)) \rightarrow S(\ell^1(3))$ be defined by

$$
T(\xi_1 e_1 + \xi_2 e_2) = \begin{cases} 
\xi_1(1/4e_1 + 3/4e_2) + \xi_2 e_3 & \text{if } \xi_1 \geq 0, \\
\xi_1(1/2e_1 + 1/2e_2) + \xi_2 e_3 & \text{otherwise,}
\end{cases}
$$

where $\{\xi_1, \xi_2\} \subset \mathbb{R}$ satisfies $|\xi_1| + |\xi_2| = 1$. It is easy to check that $T$ does not satisfy the condition of Theorem 2.8 since $-T(S(\ell^1(2))) \not\subseteq T(S(\ell^1(2)))$, and that $T$ is an isometry which cannot be linearly extended to $\ell^1(2)$ because it is not even an odd operator.

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References


DONG-NI TAN, School of Mathematical Science and LPMC, Nankai University, Tianjin 300071, PR China

e-mail: 0110127@mail.nankai.edu.cn