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On Vojta's $1+\varepsilon$ Conjecture

Xi Chen

Abstract. We give another proof of Vojta's $1+\varepsilon$ conjecture.

1 Introduction

In [V1] and [V2], P. Vojta conjectured the following.

Conjecture 1.1 ($1+\varepsilon$ Conjecture) *Let $\pi\colon X \to B$ be a flat family of projective curves over a projective curve B with connected fibers. Suppose that X has at worst quotient singularities. Then for every $\varepsilon > 0$, there exists a constant N_ε such that*

$$\omega_{X/B}\cdot C \le (1+\varepsilon)(2g(C)-2) + N_\varepsilon(X_b\cdot C) \tag{1.1}$$

for every irreducible curve $C \subset X$ that dominates B, where $\omega_{X/B}$ is the relative dualizing sheaf of X/B, X_b is a general fiber of X/B and $g(C)$ is the geometric genus of C.

Remark 1.2 From the number-theoretical point of view, one can think of X as an algebraic curve X_k over the function field $k = K(B)$ and the multi-section $C \subset X$ as an algebraic point p_C on $X_{\overline{k}} = X_k \otimes \overline{k}$. The logarithmic height $h(p_C)$ and discriminant $\Delta(p_C)$ of p_C are defined to be

$$h(p_C) = \frac{\omega_{X/B}\cdot C}{\deg(K(C)/K(B))} \quad \text{and} \quad \Delta(p_C) = \frac{2g(C)-2}{\deg(K(C)/K(B))},$$

respectively, where $\deg(K(C)/K(B)) = X_b \cdot C$, obviously. With these notations, (1.1) can be put in the form

$$h(p_C) \le (1+\varepsilon)\Delta(p_C) + N_\varepsilon. \tag{1.2}$$

Note that the definition of the height $h(p_C)$ depends on the choice of the birational model X of X_k. However, it is not hard to see that (1.2) holds regardless of the choice of the birational model (see below).

Vojta proved that (1.1) holds with $1+\varepsilon$ replaced by $2+\varepsilon$. This conjecture was settled recently by K. Yamanoi [Y]. M. McQuillan later gave an algebro-geometric proof. However, we find his proof quite hard to follow. Inspired by his idea, we will

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give another proof of this conjecture and generalize it to the log case. Compared to his proof, ours is more elementary.

It seems natural to study a (generalized) log version of the $1+\varepsilon$ conjecture. For a log variety (X, D) and a curve $C \subset X$ that meets D properly, we define $i_X(C, D)$ to be the number of the points in $\nu^{-1}(D)$, where $\nu\colon \widetilde{C} \to C \subset X$ is the normalization of C.

Theorem 1.3 *Let $\pi\colon X \to B$ be a flat family of projective curves over a projective curve B with connected fibers. Suppose that X has at worst quotient singularities and $D \subset X$ is a reduced effective divisor on X. Then for every $\varepsilon > 0$, there exists a constant N_ε such that*

$$(\omega_{X/B} + D)\cdot C \le (1+\varepsilon)(2g(C) - 2 + i_X(C, D)) + N_\varepsilon(X_b \cdot C) \tag{1.3}$$

for every irreducible curve $C \subset X$ that dominates B and $C \not\subset D$.

Conventions We work exclusively over $\mathbb{C}$ and with analytic topology wherever possible.

2 Reduction to $(\mathbb{P}^1 \times B, D)$

As a first step in our proof, we will reduce Theorem 1.3 to the case $(\mathbb{P}^1 \times B, D)$. This was also done in Yamanoi's proof [Y].

It is not hard to see that (1.3) continues to hold after applying birational transforms and/or base changes to X/B. That is, we have the following.

Lemma 2.1 *Let $\pi\colon X \to B$ and D be given as in Theorem* 1.3.

(i) *Let $f\colon X' \dashrightarrow X$ be a birational morphism and D' be the proper transform of D under f. Then* (1.3) *holds for (X, D) if and only if it holds for (X', D').*

(ii) *Let $B' \to B$ be a finite map from a smooth projective curve B' to B, $f\colon X' = X \times_B B' \to X$ be the base change of the family X, and $D' = f^{-1}(D)$. Then* (1.3) *holds for (X, D) if and only if it holds for (X', D').*

The constants N'_ε for (X', D'), though, might be different from N_ε for (X, D).

Proof For part (i), it is enough to argue for X' being the blowup of X at one point p. Let $C' \subset X'$ be the proper transform of $C \subset X$. Then

$$(\omega_{X/B} + D)\cdot C = (\omega_{X'/B} + D' + rE)\cdot C'$$

for some constant r, where E is the exceptional divisor of f. On the other hand, we have

$$E \cdot C' \le X'_b \cdot C' = X_b \cdot C = \deg(C),$$

where X'_b and X_b are the fibers of X' and X over a point $b \in B$, respectively. Consequently,

$$|(\omega_{X/B} + D)\cdot C - (\omega_{X'/B} + D')\cdot C'| \le |r| \deg C. \tag{2.1}$$

Also, it is obvious that $g(C) = g(C')$ and

$$|i_X(C,D) - i_{X'}(C',D')| \le E\cdot C' \le \deg(C). \tag{2.2}$$

Then part (i) follows from (2.1) and (2.2).

For part (ii), let d be the degree of the map $B' \to B$, $R \subset B'$ be its ramification locus and μ_r be the ramification index of a point $r \in R$. Let $C' = f^*(C)$. It is not hard to see that

$$|d(\omega_{X/B} + D)\cdot C - (\omega_{X'/B'} + D')\cdot C'| \le \sum_{r\in R}(\mu_r - 1)\deg(C) \tag{2.3}$$

$$|d(2g(C) - 2) - (2g(C') - 2)| \le \sum_{r\in R}(\mu_r - 1)\deg C \tag{2.4}$$

and

$$|d(i_X(C,D)) - i_{X'}(C',D')| \le \sum_{r\in R}(\mu_r - 1)\deg C \tag{2.5}$$

Then part (ii) follows from (2.3)–(2.5). ∎

Remark 2.2 We see from Lemma 2.1 that (1.2) holds regardless of the choice of birational models X.

Remark 2.3 If $(\omega_{X/B} + D)\cdot X_b \le 0$, (1.3) is trivially true. So we may assume that

$$(\omega_{X/B} + D)\cdot X_b > 0.$$

We may also assume that D meets every fiber properly. Using Lemma 2.1, we can apply the stable reduction to (X,D) and make X into a family of stable curves with marked points $X_b \cap D$ on each fiber. The resulting X has at worst quotient singularities, and $\omega_{X/B} + D$ is relatively ample over B.

Proposition 2.4 *If* (1.3) *fails for some* (X,D)*, then there exists* $\delta > 0$ *and a log pair* (Y,R) *such that* (1.3) *fails with* (X,D,ε) *replaced by* (Y,R,δ)*, where* R *is a reduced effective divisor on* $Y = \mathbb{P}^1 \times B$.

Proof By the above remark, we may assume that X is a family of stable curves with marked points $X_b \cap D$. In particular, $\omega_{X/B} + D$ is relatively ample over B.

Since (1.3) fails for (X,D), there exists a sequence of irreducible curves $C_1, C_2, \dots, C_n, \dots \subset X$ such that

$$\lim_{n\to\infty}\left(\frac{(\omega_{X/B} + D)\cdot C_n}{X_b\cdot C_n} - \frac{(1+\varepsilon)(2g(C_n) - 2 + i_X(C_n,D))}{X_b\cdot C_n}\right) = \infty. \tag{2.6}$$

Taking a sufficiently ample line bundle L on X, we can map $X \to \mathbb{P}^1$ with a very general pencil in $|L|$. Combining this with the projection $X \to B$, we obtain a rational map $\phi\colon X \dashrightarrow Y = B \times \mathbb{P}^1$. We can make the following happen by taking L sufficiently ample and the pencil sufficiently general:

- The indeterminancy locus I_ϕ of ϕ consists of L^2 distinct points on X, $I_\phi \cap C_n = \varnothing$ for all n and $I_\phi \cap D = \varnothing$.
- Outside of I_ϕ, ϕ is finite. Let $R_X \subset X$ be the closure of the ramification locus of $\phi\colon X\backslash I_\phi \to Y$, $R_Y = \overline{\phi(R_X)}$ be the proper transform of R_X and

$$\phi^* R_Y = 2R_X + R_\phi$$

outside of I_ϕ, where $R_\phi \subset X$ is a reduced effective divisor on X.
- ϕ is simply ramified along R_X with multiplicity 2.
- ϕ maps C_n and D birationally to $\Gamma_n = \phi(C_n)$ and $\Delta = \phi(D)$, respectively, for all n.

Since $\phi_* C_n = \Gamma_n$, we have

$$\phi^*(\omega_{Y/B} + R_Y + \Delta)\cdot C_n = (\omega_{Y/B} + R_Y + \Delta)\cdot\Gamma_n \tag{2.7}$$

On the other hand,

$$\begin{aligned}\phi^*(\omega_{Y/B} + R_Y + \Delta)\cdot C_n &= (\phi^*\omega_{Y/B} + 2R_X + R_\phi + \phi^*\Delta)\cdot C_n\\ &= (\phi^*\omega_{Y/B} + R_X + D)\cdot C_n + (R_X + R_\phi)\cdot C_n + D_\phi\cdot C_n,\end{aligned} \tag{2.8}$$

where

$$\phi^*\Delta = D + D_\phi \tag{2.9}$$

for some effective divisor $D_\phi \subset X$. By Riemann-Hurwitz,

$$\omega_{X/B} = \phi^*\omega_{Y/B} + R_X \tag{2.10}$$

holds outside of I_ϕ. Meanwhile, it is obvious that

$$(R_X + R_\phi)\cdot C_n \ge i_Y(\Gamma_n, R_Y) \tag{2.11}$$

and

$$D_\phi\cdot C_n \ge i_Y(\Gamma_n,\Delta) - i_X(C_n, D) \tag{2.12}$$

Combining (2.7) through (2.12), we obtain

$$\begin{aligned}&(\omega_{Y/B} + R_Y + \Delta)\cdot\Gamma_n - (1+\delta)\big(2g(\Gamma_n) - 2 + i_Y(\Gamma_n, R)\big)\\ &\ge (\omega_{X/B} + D)\cdot C_n - (1+\delta)\big(2g(C_n) - 2 + i_X(C_n, D)\big)\\ &\quad - \delta(R_X + R_\phi + D_\phi)C_n,\end{aligned}$$

where $R = R_Y \cup \Delta$. Since $\omega_{X/B} + D$ is relatively ample over B, there exist constants β and $\gamma > 0$ such that

$$(R_X + R_\phi + D_\phi)C \le \gamma(\omega_{X/B} + D + \beta X_b)C$$

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for all curves $C \subset Y$. Thus, it suffices to take

$$\delta = \frac{\varepsilon}{(1+\varepsilon)\gamma + 1}.$$

Then

$$\begin{aligned}
&(\omega_{X/B} + D) \cdot C_n - (1+\delta)\big(2g(C_n) - 2 + i_X(C_n, D)\big) - \delta(R_X + R_\phi + D_\phi) \cdot C_n \\
&\quad \geq (1 - \delta\gamma)(\omega_{X/B} + D) \cdot C_n - (1+\delta)\big(2g(C_n) - 2 + i_X(C_n, D)\big) \\
&\qquad - \beta\gamma\delta(X_b \cdot C_n) \\
&\quad = (1 - \delta\gamma)\Big((\omega_{X/B} + D) \cdot C_n - (1+\varepsilon)\big(2g(C_n) - 2 + i_X(C_n, D)\big)\Big) \\
&\qquad - \beta\gamma\delta(X_b \cdot C_n)
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{(\omega_{Y/B} + R) \cdot \Gamma_n}{Y_b \cdot \Gamma_n} - (1+\delta) \frac{2g(\Gamma_n) - 2 + i_Y(\Gamma_n, R)}{Y_b \cdot \Gamma_n} \right) = \infty,$$

and Proposition 2.4 follows. ∎

In the above proof, we have quite a bit of freedom to choose the map $X \dashrightarrow \mathbb{P}^1$. We can make R really "nice" by choosing L and the pencil of L sufficiently "general".

Proposition 2.5 *Let S be a finite set of points on B. In the proof of Proposition* 2.4*, for a sufficiently ample L and a general pencil $\sigma \subset |L|$ that maps $X \dashrightarrow \mathbb{P}^1$, the corresponding divisor $R = R_Y + \Delta \subset Y = \mathbb{P}^1 \times B$ has the following properties:*

- *For every fiber Y_b of Y/B,*

$$i_Y(Y_b, R) \geq Y_b \cdot R - 1 \tag{2.13}$$

  and if the equality holds, $b \in B \backslash S$ and X_b is disjoint from the base locus $\mathrm{Bs}(\sigma)$ *of σ;*
- *R is a divisor of normal crossing.*

Proof Let $\mathbb{G}(k, |L|)$ be the Grassmanian $\{\mathbb{P}^k \subset |L|\}$. For each pencil $\sigma \in \mathbb{G}(1, |L|)$, we use the notation ϕ_σ for the rational map $X \dashrightarrow Y$ induced by σ and $R_{X,\sigma}$ for the closure of its ramification locus. Let $\phi_{\sigma,b} \colon X_b \to \mathbb{P}^1$ be the restriction of ϕ_σ to X_b and let $R_{X,\sigma,b} = R_{X,\sigma} \cap X_b$ be the ramification locus of $\phi_{\sigma,b}$.

For L sufficiently ample and for each $b \in B$, we see by simple dimension counting that each of

$$\{\sigma :\ \phi_\sigma(p_1) = \phi_\sigma(p_2) = \phi_\sigma(p_3) \text{ for three distinct points } p_1, p_2, p_3 \in D \cap X_b\},$$

$$\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) = \phi_\sigma(p_3) \text{ for } p_1 \neq p_2 \in D \cap X_b \text{ and } p_3 \in R_{X,\sigma,b}\},$$

$$\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) \text{ and } X_b \cap \mathrm{Bs}(\sigma) \neq \varnothing,\ \text{for } p_1 \neq p_2 \in D \cap X_b\},$$

$$\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2),\ \text{where } p_1 \in D \cap X_b \text{ and}$$
$$\phi_{\sigma,b} \text{ ramifies at } p_2 \in R_{X,\sigma,b} \text{ with index} \geq 3\},$$

$$\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) \text{ and } X_b \cap \mathrm{Bs}(\sigma) \neq \varnothing, \text{where } p_1 \in D \cap X_b \text{ and } p_2 \in R_{X,\sigma,b}\},$$

$$\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2),\ \text{where } p_1 \neq p_2 \in R_{X,\sigma,b} \text{ and}$$
$$\phi_{\sigma,b} \text{ ramifies at } p_2 \text{ with index} \geq 3\},$$

$$\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) \text{ and } X_b \cap \mathrm{Bs}(\sigma) \neq \varnothing, \text{where } p_1 \neq p_2 \in R_{X,\sigma,b}\},$$

$$\{\sigma : \phi_{\sigma,b} \text{ ramifies at } p_1 \neq p_2 \in R_{X,\sigma,b} \text{ with indices} \geq 3\},$$

$$\{\sigma : \phi_{\sigma,b} \text{ ramifies at } p_1 \in R_{X,\sigma,b} \text{ with index} \geq 3 \text{ and } X_b \cap \mathrm{Bs}(\sigma) \neq \varnothing\},\ \text{ and}$$

$$\{\sigma : \phi_{\sigma,b} \text{ ramifies at } p_1 \in R_{X,\sigma,b} \text{ with index} \geq 4\}$$

has codimension two in $\mathbb{G}(1, |L|)$, and hence (2.13) follows. The same dimension count also shows that Y_b meets R transversely for $b \in S$ and σ general. Hence if the equality in (2.13) holds, $b \notin S$.

Already by (2.13), we see that R has at worst double points as singularities. We can further show that the singularities R_{sing} of R are all nodes.

Let $D = \sum D_i$, where D_i's are irreducible components of D, which are sections of X/B by our assumption on X. And let $\Delta_{\sigma,i} = \phi_\sigma(D_i)$ and $R_{Y,\sigma} = \phi_\sigma(R_{X,\sigma})$. To show that R has normal crossing, it is suffices to verify the following:

- $\Delta_{\sigma,i}$ and $\Delta_{\sigma,j}$ meet transversely for all $i \neq j$;
- $\Delta_{\sigma,i}$ meets $R_{Y,\sigma}$ transversely for all i;
- $R_{Y,\sigma}$ is nodal.

It is easy to see that the monodromy action on the intersections $\Delta_{\sigma,i} \cap \Delta_{\sigma,j}$ when σ varies in $\mathbb{G}(1, |L|)$ is transitive. Therefore, to show that $\Delta_{\sigma,i}$ and $\Delta_{\sigma,j}$ meet transversely, it suffices to show that they meet transversely at (at least) one point, *i.e.*,

- there exists $\sigma \in \mathbb{G}(1, |L|)$, $p_i \in D_i$ and $p_j \in D_j$ such that $\Delta_{\sigma,i}$ and $\Delta_{\sigma,j}$ meet transversely at $\phi_\sigma(p_i) = \phi_\sigma(p_j)$.

Similarly, the other two statements translate to

- there exists $\sigma \in \mathbb{G}(1, |L|)$, $p_i \in D_i$ and $q \in R_{X,\sigma}$ such that $\Delta_{\sigma,i}$ and $R_{Y,\sigma}$ meet transversely at $\phi_\sigma(p_i) = \phi_\sigma(q)$;
- there exists $\sigma \in \mathbb{G}(1, |L|)$ and $q \in R_{X,\sigma,b}$ for some b such that $\phi_{\sigma,b}$ has ramification index 3 at q and $R_{Y,\sigma}$ is smooth at $\phi_\sigma(q)$;
- there exists $\sigma \in \mathbb{G}(1, |L|)$ and $q_1 \neq q_2 \in R_{X,\sigma,b}$ for some b such that $R_{Y,\sigma}$ has a node at $\phi_\sigma(q_1) = \phi_\sigma(q_2)$.

None of these statements is hard to prove. We leave their verification to the reader. ■

Suppose that (1.3) fails for (X, D) and $\{C_n \subset X\}$ is the sequence of irreducible curves satisfying (2.6). We fix a positive $(1, 1)$ form ω on X that represents $c_1(L)$ and for every finite set of points $S \subset B$, we define

$$f_\omega(S) = \lim_{r\to 0} \varliminf_{n\to\infty} \left(\frac{1}{L \cdot C_n} \sum_{b\in S} \int_{C_n \cap \pi^{-1}(U(b,r))} \omega \right),$$

where $U(b, r) \subset B$ is the disk of radius r centered at b. Of course, we need a metric on B in order to define $U(b, r)$. But it is obvious that the choice of metric on B is irrelevant here. Although $f_\omega(S)$ depends on the choice of ω, the vanishing of $f_\omega(S)$ does not depend on ω, *i.e.*, if $f_\omega(S) = 0$ for one ω, it vanishes for all choices of ω. And it is easy to see that

$$\sum_\alpha f_\omega(S_\alpha) \leq 1 \tag{2.14}$$

for any collection $\{S_\alpha \subset B\}$ of disjoint finite sets S_α.

Let us fix a sufficient ample line bundle L on X and let $\phi_\sigma : X \dashrightarrow Y$ be the map given by a pencil $\sigma \subset |L|$ as in the proof of Proposition 2.5. This map gives rise to another log pair (Y, R) with R satisfying the conditions given in the above proposition. Let $Q_\sigma \subset B$ be the finite set of points b where the equality in (2.13) holds. This gives us a map from $\mathbb{G}(1, |L|)$ to B^N/S_N sending $\sigma \to Q_\sigma$, where $N = |Q_\sigma|$ and B^N/S_N is the space of N unordered points on B. By Proposition 2.5, $Q_\sigma \cap Q_{\sigma'} = \varnothing$ for two general pencils σ and σ'. Combining this with (2.14), we see that the set $\{\sigma : f_\omega(Q_\sigma) > r\}$ is contained in a proper subvariety of $\mathbb{G}(1, |L|)$ for every $r > 0$. Consequently, the set

$$\Big\{\sigma : f_\omega(Q_\sigma) > 0\Big\} = \bigcup_{n=1}^{\infty} \Big\{\sigma : f_\omega(Q_\sigma) > \tfrac{1}{n}\Big\}$$

is contained in a union of countably many proper subvarieties of $\mathbb{G}(1, |L|)$. In other words, $f_\omega(Q_\sigma) = 0$ for a very general pencil σ. For a very general pencil σ, C_n are disjoint from the base locus of σ. Hence $L \cdot C_n = Y_p \cdot \Gamma_n$, where $\Gamma_n = \phi_\sigma(C_n)$ and

Y_p is a fiber of $Y/\mathbb{P}^1$. In addition, we have proved that $X_b \cap \mathrm{Bs}(\sigma) = \varnothing$ for $b \in Q_\sigma$. Hence $f_\omega(Q_\sigma) = 0$ implies

$$\lim_{r\to 0} \varliminf_{n\to\infty} \left(\frac{1}{Y_p \cdot \Gamma_n} \sum_{b\in Q_\sigma} \int_{\Gamma_n \cap \pi_Y^{-1}(U(b,r))} \eta \right) = 0,$$

where η is the pullback of a positive $(1,1)$ form on $\mathbb{P}^1$ representing $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ and π_Y is the projection $Y \to B$. By taking a subsequence of $\{\Gamma_n\}$, we may as well replace $\varliminf$ by lim.

We may further apply a suitable base change to Y/B to make R_Y into a union of sections of Y/B while preserving the other properties of (Y,R). So we finally reduce the conjecture from (X,D,ε) to (Y,R,δ). Replacing (X,D,ε) by (Y,R,δ), we may assume the following holds.

(A1) $D \subset X = \mathbb{P}^1 \times B$ is a normal-crossing divisor which is a union of sections of X/B.

(A2) $\omega_{X/B} + D$ is relatively ample over B.

(A3) For every fiber X_b of X/B,

$$i_X(X_b, D) \geq X_b \cdot D - 1. \tag{2.15}$$

(A4) There is a sequence of reduced and irreducible curves $\{C_n\}$ on X that dominate B and satisfy (2.6).

(A5) Let $Q \subset B$ be the set of points b where the equality in (2.15) holds, *i.e.*, $Q = \pi(D_{\mathrm{sing}})$, where D_{sing} is the singular locus D_{sing} of D; then

$$\lim_{r\to 0} \lim_{n\to\infty} \left(\frac{1}{X_p \cdot C_n} \sum_{b\in Q} \int_{C_n \cap \pi^{-1}(U(b,r))} w \right) = 0, \tag{2.16}$$

where X_p is the fiber of X over a point $p \in \mathbb{P}^1$ and w is the pullback of a positive $(1,1)$ form on $\mathbb{P}^1$ representing $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$.

3 Proof of Theorem 1.3

3.1 Lifts to the First Jet Space

Now we can work exclusively on (X,D) with (X,D) satisfying the hypotheses (A1)–(A5) in the last section. As in Vojta's proof of $2+\varepsilon$ theorem, we start by lifting every curve $C_n \subset X$ to its first jet space.

Let $\Omega_X(\log D)$ be the sheaf of logarithmic differentials with poles along D and $T_X(-\log D) = \Omega_X(\log D)^\vee$ be its dual. Let $Y = \mathbb{P}T_X(-\log D)$ be the projectivization of $T_X(-\log D)$ with tautological line bundle L. Here we follow the traditional convention that

$$\mathbb{P}E = \mathrm{Proj}(\oplus \operatorname{Sym}^\bullet E^\vee) \quad \text{and} \quad H^0(L) \cong H^0(E^\vee).$$

We have the basic exact sequence

$$0 \to \pi^*\Omega_B \to \Omega_X(\log D) \to \Omega_{X/B}(D). \tag{3.1}$$

Note that this sequence is not right exact; $\Omega_X(\log D) \to \Omega_{X/B}(D)$ fails to be surjective along D_{sing}.

Every nonconstant map $\nu\colon C \to X$ from a smooth curve C to X can be naturally lifted to a map $\nu_Y : C \to Y$ via the map

$$\mathbb{P}T_C(-\log \nu^* D) \to \mathbb{P}T_X(-\log D).$$

Suppose that ν maps C birationally onto its image. Then the natural map $\nu^*\Omega_X(\log D) \to \Omega_C(\log \nu^* D)$ induces a map

$$\nu_Y^* L \to \Omega_C(\log \nu^* D). \tag{3.2}$$

Obviously, this map is nonzero, and $\nu_Y^* L$ is locally free; consequently, it is an injection. Therefore, we have

$$\deg \nu_Y^* L \le \deg \Omega_C(\log \nu^* D) = 2g(C) - 2 + i_X(\nu(C), D).$$

Hence (1.3) holds if

$$\deg \nu_Y^*(\pi_X^*(\omega_{X/B} + D) - (1+\varepsilon)L) \le N_\varepsilon \deg(\nu^* X_b),$$

where π_X is the projection $Y \to X$. Another way to put this is that

$$G \cdot (\nu_Y)_* C \ge 0 \tag{3.3}$$

for a sufficiently ample divisor $M \subset B$ and every $\nu\colon C \to X$ with $\nu(C)$ dominating B, where

$$G = (1+\varepsilon)L + \pi_B^* M - \pi_X^*(\omega_{X/B} + D),$$

where $\pi_B = \pi \circ \pi_X$ is the projection $Y \to B$. Or in the context of our hypothesis A4, we want to show that

$$-G \cdot \Gamma_n = O(\deg C_n) \tag{3.4}$$

and thus arrive at a contradiction, where $\Gamma_n \subset Y$ is the lift of $C_n \subset X$ via its normalization and $\deg C_n = C_n \cdot X_b$. Here by $O(\deg C_n)$, we mean a quantity $\le K \deg C_n$ for some constant K and all n.

Obviously, (3.3) holds if the divisor G is numerically effective (NEF). Unfortunately, we cannot expect this to be true in general.

The map $\Omega_X(\log D) \to \Omega_{X/B}(D)$ in (3.1) induces a rational map

$$\mathbb{P}T_{X/B}(-D) \dashrightarrow Y.$$

Let $\Delta \subset Y$ be the closure of the image of this map. As we are going to see, Δ will play a central role in our argument. Another way to characterize Δ is the following.

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Lemma 3.1 *We have*

$$\Delta = \overline{\bigcup_{b\in B} \mu_Y(X_b)}$$

and a curve $\nu\colon C \hookrightarrow X$ *is tangent to a fiber* X_b *if and only if* $\nu_Y(C)$ *intersects* Δ, *where* $\mu_Y : X_b \to Y$ *is the lifting of the embedding* $X_b \hookrightarrow X$.

Proof This is more or less trivial. ■

3.2 Some Numerical Results

Here we prove some numerical results on Δ, L, and G, which we are going to need later. First of all, it is obvious that π_X maps Δ birationally onto X; indeed, by a local analysis, we see that Δ is the blowup of X along D_{sing}, *i.e.,* the places where $\Omega_X(\log D) \to \Omega_{X/B}(D)$ fails to be surjective. In the lift of $\nu\colon C \to X$ to $\nu_Y\colon C \to Y$, if ν is a smooth embedding, we have $(\nu_Y)^*L = \omega_C + \nu^{-1}(D)$, where $\nu^{-1}(D) = \operatorname{supp}(\nu^*D)$ is the reduced pre-image of D. Namely, (3.2) is an isomorphism. Therefore, for every fiber X_b,

$$L \cdot \widetilde{X}_b = 2g(X_b) - 2 + i_X(X_b, D),$$

where $\widetilde{X}_b \subset \Delta$ is the proper transform of X_b under $\Delta \to X$. Applying this to all the fibers X_b with $X_b \cap D_{\text{sing}} \neq \varnothing$, we see that

$$L\big|_{\Delta} = \pi_X^*(\omega_{X/B} + D + \pi^*M) - E \tag{3.5}$$

for some divisor M on B, where $E = \sum_{q\in D_{\text{sing}}} E_q$ is the exceptional divisor of $\Delta \to X$. To determine M, we restrict everything to a section $X_p = \rho^{-1}(p)$ of X/B, where ρ is the projection $X \to \mathbb{P}^1$. For p general, the restriction of (3.1) to $X_p \cong B$ becomes

$$0 \to \Omega_{X_p} \to \Omega_X(\log D)\big|_{X_p} \to \mathcal{O}_{X_p}(D) \to 0. \tag{3.6}$$

Let Δ_p be the proper transform of X_p under $\Delta \to X$. Then we see from (3.6) that the restriction of L to $\Delta_p \cong B$ is

$$L\big|_{\Delta_p} = \pi_X^* D. \tag{3.7}$$

Comparing (3.5) and (3.7), we conclude that M is trivial and hence

$$L\big|_{\Delta} = \pi_X^*(\omega_{X/B} + D) - E.$$

As a consequence,

$$\begin{aligned} G\Big|_{\Delta} &= \big((1+\varepsilon)L + \pi_B^* M - \pi_X^*(\omega_{X/B} + D)\big)\Big|_{\Delta} \\ &= \varepsilon\pi_X^*(\omega_{X/B} + D) + \pi_B^* M - (1+\varepsilon)E. \end{aligned} \tag{3.8}$$

Next, we claim that

$$\Delta = L - \pi_B^*\omega_B \tag{3.9}$$

This is obviously true if (3.1) is an exact sequence of locally free sheaves, *i.e.,* $D_{\text{sing}} = \varnothing$. To see that this is true in general, we restrict everything to a smooth curve $C \subset X$ with $C \cap D_{\text{sing}} = \varnothing$. By the above reason, (3.9) holds when restricted to $\pi_X^{-1}(C)$. Such curves C obviously generate $\operatorname{Pic}(X)$ and hence (3.9) holds over Y.

By restricting (3.1) to each fiber X_b of X/B, we see that L is relatively NEF over B. Moreover, the following holds.

Lemma 3.2 *For all $m \geq k \in \mathbb{Z}$, $mL - k\Delta$ is relatively base point free over B and*

$$H^1(m(L + \pi_B^*M) - k\Delta) = 0 \tag{3.10}$$

for a sufficiently ample divisor $M \subset B$.

Proof Since $c_1(\Omega_X(\log D)) = \omega_X + D$, the restriction of $\Omega_X(\log D)$ to a fiber $X_b \cong \mathbb{P}^1$ is

$$\Omega_X(\log D)\big|_{X_b} = \mathcal{O}_{\mathbb{P}^1}(\beta) \oplus \mathcal{O}_{\mathbb{P}^1}(\gamma), \tag{3.11}$$

where $\beta + \gamma = \alpha = (\omega_{X/B} + D) \cdot X_b$. By (3.1), we must have $\beta, \gamma \geq 0$. Therefore,

$$Y_b \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(-\beta) \oplus \mathcal{O}_{\mathbb{P}^1}(-\gamma)\right), \tag{3.12}$$

and together with (3.9), we see that $mL - k\Delta$ is relatively NEF over B for $m \geq k$. Also, we see from the above argument that

$$H^1(Y_b, mL - k\Delta) = 0 \Leftrightarrow R^1(\pi_B)_*\mathcal{O}(mL - k\Delta) = 0.$$

This implies

$$\begin{aligned} H^1(m(L + \pi_B^*M) - k\Delta) &= H^1((\pi_B)_*\mathcal{O}(m(L + \pi_B^*M) - k\Delta)) \\ &= H^1((\pi_B)_*L^{m-k} \otimes \mathcal{O}_B(k\omega_B + mM)). \end{aligned}$$

By (3.12), $\operatorname{Sym}^n H^0(Y_b, L) = H^0(Y_b, L^n)$. Therefore,

$$H^1(m(L + \pi_B^*M) - k\Delta) = H^1(\operatorname{Sym}^{m-k}(\pi_B)_*L \otimes \mathcal{O}_B(k\omega_B + mM)).$$

It suffices to choose M such that all of M, $\omega_B + M$ and $(\pi_B)_*L \otimes \mathcal{O}_B(M)$ are ample and (3.10) follows. ■

Remark 3.3 It is possible to give a more precise version of (3.10) on how ample M should be in terms of ω_B and D; however, we have no need of it here. Also, in the above proof, we observe that L fails to be ample on Y_b if and only if (3.11) splits as

$$\Omega_X(\log D)\big|_{X_b} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha) \tag{3.13}$$

If (3.13) holds on a general fiber X_b, it holds everywhere, and this only happens when D consists of $\alpha + 2$ disjoint sections of X/B, in which case the conjecture is trivial. Hence we may assume that L is ample on a general fiber of Y/B. This implies that $L + \pi_B^*M$ is big for a sufficiently ample divisor $M \subset B$, in addition to being NEF as already proved. The same, of course, holds for $mL - k\Delta + \pi_B^*M$ when $m > k$.

3.3 Bergman Metric

Given a line bundle L on a compact complex manifold X and sections $s_0, s_1, \dots, s_n \in |L|$ of L, we recall that the *Bergman metric* associated with $\{s_k\}$ is the pullback of the Fubini-Study metric under the map $X \dashrightarrow \mathbb{P}^n$ given by $\{s_k\}$, *i.e.*, the pseudo-metric with associated $(1,1)$ form

$$w = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{k=0}^{n}|s_k|^2\Big).$$

Alternatively, the Fubini-Study metric can be regarded as a metric of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ and the Bergman metric is correspondingly a pseudo-metric of L with w the curvature form. In general, w is only a closed real current of type $(1,1)$ with the following properties:

- it is C^∞ outside of the base locus $\mathrm{Bs}\{s_k\}$ of $\{s_k\}$;
- it represents $c_1(L)$ if $\{s_k\}$ is base point free;
- we always have

$$\nu^* w \text{ is } C^\infty, \quad \nu^* w \geq 0, \quad \text{and} \quad \deg(\nu^* L) \geq \int_C \nu^* w \tag{3.14}$$

  for any morphism $\nu\colon C \to X$ from a smooth and irreducible projective curve C to X with $\nu(C) \not\subset \mathrm{Bs}\{s_k\}$.

The indeterminancy of the rational map $\phi\colon X \dashrightarrow \mathbb{P}^n$ given by $\{s_k\}$ can be resolved by a sequence of blowups along smooth centers over $\mathrm{Bs}\{s_k\}$. That is, there exists a birational map $\pi\colon Y \to X$ such that $f = \phi\circ\pi$ is regular. Let $\widetilde{s}_k$ be the proper transform of s_k under π. Then $\{\widetilde{s}_k\}$ span a base point free linear system of $\widetilde{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let $\widetilde{w}$ be the Bergman metric associated with $\{\widetilde{s}_k\}$. Then $\widetilde{w} = \pi^* w$ outside of exceptional locus of π. Indeed, the current w is defined in the way of

$$\langle w, \gamma\rangle = \int_Y \widetilde{w}\wedge\pi^*\gamma$$

Then (3.14) follows easily.

3.4 Construction of the First Chern Classes

Let $\pi_X^*(\omega_{X/B} + D) = \alpha Y_p + \pi_B^* N$ for some divisor $N \subset B$, where Y_p is a fiber of $Y/\mathbb{P}^1$. We replace M by $M+N$ and write G in the form

$$G = (1+\varepsilon)L + \pi_B^* M - \alpha Y_p.$$

Our purpose remains, of course, to show (3.4).

We write the left-hand side of (3.4) in the integral form:

$$G\cdot\Gamma_n = \int_{\Gamma_n} c_1(G) = \int_{\Gamma_n\setminus U} c_1(G) + \int_{\Gamma_n\cap U} c_1(G), \tag{3.15}$$

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where U is an (analytic) open neighborhood of Δ. Here we have to work with the forms that represent the first chern classes instead of cohomology classes themselves, *i.e.*, $c_1(G)$ refers to a $(1,1)$ form representing the first chern class of G; otherwise, the integrals in (3.15) do not make sense. The construction of appropriate $c_1(G)$ is one of the main parts of our proof. Basically, by a proper choice of $c_1(G)$ with

$$c_1(G) = c_1((1+\varepsilon)L + \pi_B^*M) - c_1(\alpha Y_p)$$

we will show that both

$$-\int_{\Gamma_n \setminus U} c_1(G) \quad \text{and} \quad -\int_{\Gamma_n \cap U} c_1(G)$$

are of order $O(\deg C_n)$. The forms representing $c_1((1+\varepsilon)L + \pi_B^*M)$ and $c_1(\alpha Y_p)$ are constructed via the Bergman metric mentioned above.

Let us first fix a sufficiently large integer m with $m\varepsilon \in \mathbb{Z}$; obviously, we may assume $\varepsilon \in \mathbb{Q}$. Since $H^0(m\alpha Y_p) = H^0(\mathcal{O}_{\mathbb{P}^1}(m\alpha))$, a general pencil of $m\alpha Y_p$ is base point free. To construct a form w representing $c_1(m\alpha Y_p)$, it is enough to choose a base point free pencil of $m\alpha Y_p$ with basis $\{s_0, s_1\}$ and let

$$w = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(|s_0|^2 + |s_1|^2\right)$$

be the Bergman metric associated with $\{s_0, s_1\}$. Obviously, w is C^∞ and represents $c_1(m\alpha Y_p)$. Next we will construct a Bergman metric on the line bundle $\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M)$.

Let $S_i = \{s_i = 0\}$ for $i = 0,1$ and let $\{\sigma_{0j} : j \in J\}$ be a basis of the linear system of $m(1+\varepsilon)L + m\pi_B^*M$ consisting of sections σ with

$$\sigma\Big|_{S_0} \in H^0(S_0, m(1+\varepsilon)L + m\pi_B^*M - 2\Delta)$$

Or equivalently, σ_{0j} are the sections tangent to S_0 along $S_0 \cap \Delta$.

Lemma 3.4 *For each j, there exists a section σ_{1j} of $m(1+\varepsilon)L + m\pi_B^*M$ such that $s_0\sigma_{1j} - s_1\sigma_{0j}$ vanishes to the order of 2 along Δ,* i.e.,

$$s_0\sigma_{1j} - s_1\sigma_{0j} \in H^0(m(1+\varepsilon)L + m\pi_B^*M + m\alpha Y_p - 2\Delta). \tag{3.16}$$

In addition, $\{\sigma_{1j}\}$ can be chosen to be a basis of the linear system consisting of sections σ with

$$\sigma\Big|_{S_1} \in H^0(S_1, m(1+\varepsilon)L + m\pi_B^*M - 2\Delta).$$

Proof Let F_0 be the subscheme of Y given by $F_0 = S_0 \cap 2\Delta$. Then we have the Koszul complex for the ideal sheaf I_{F_0} of $F_0 \subset Y$:

$$0 \to \mathcal{O}(-S_0 - 2\Delta) \to \mathcal{O}(-S_0) \oplus \mathcal{O}(-2\Delta) \to I_{F_0} \to 0$$

Obviously,

$$\Sigma_0 = H^0(\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^* M) \otimes I_{F_0}) \tag{3.17}$$

is exactly the linear system $\mathrm{Span}\{\sigma_{0j}\}$ generated by $\{\sigma_{0j}\}$. By Lemma 3.2,

$$H^1(m(1+\varepsilon)L + m\pi_B^* M + m\alpha Y_p - S_0 - 2\Delta) = H^1(m(1+\varepsilon)L + m\pi_B^* M - 2\Delta) = 0.$$

Therefore, $AF + BG$ holds for

$$s_1\sigma_{0j} \in H^0(\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^* M + m\alpha Y_p) \otimes I_{F_0}).$$

That is, $s_1\sigma_{0j} = s_0\sigma_{1j} + s_\Delta^2 l_j$ for some σ_{1j}, where $\Delta = \{s_\Delta = 0\}$ and l_j is a section of

$$\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^* M + m\alpha Y_p - 2\Delta).$$

And (3.16) follows. Obviously, σ_{1j} are members of the linear sytem,

$$\Sigma_1 = H^0(\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^* M) \otimes I_{F_1}), \tag{3.18}$$

where I_{F_1} is the ideal sheaf of the subscheme $F_1 = S_1 \cap 2\Delta \subset Y$. It is obvious that

$$H^0(m(1+\varepsilon)L + m\pi_B^* M - 2\Delta) \subset \Sigma_0 \cap \Sigma_1.$$

It is not hard to see that $\{\sigma_{1j}\}$ spans the quotient

$$\Sigma_1 / H^0(m(1+\varepsilon)L + m\pi_B^* M - 2\Delta) = \mathrm{Span}\{\sigma_{1j}\}. \tag{3.19}$$

Without loss of generality, we may assume that $\{\sigma_{0j} : j \in J\}$ contains a subset $\{\sigma_{0j} : j \in J_\Delta\}$, which is a basis of $H^0(m(1+\varepsilon)L + m\pi_B^* M - 2\Delta)$, where $J_\Delta \subset J$. Then it is enough to choose $\sigma_{1j} = \sigma_{0j}$ for $j \in J_\Delta$. Combining this with (3.19), we see that $\{\sigma_{1j}\}$ is a basis of Σ_1. ■

Let σ_{1j} be the sections given in the above lemma. Together with $\{\sigma_{0j}\}$ we have the Bergman metric associated with $\{\sigma_{ij} : 0 \le i \le 1, j \in J\}$

$$\gamma = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{ij}|\sigma_{ij}|^2\Big).$$

And we let $\eta = \gamma - w$.

Proposition 3.5 *Let Σ_i be the linear system generated by $\{\sigma_{ij} : j \in J\}$ as in* (3.17) *and* (3.18)*. For each i, the base locus of Σ_i is contained in $(Y_Q \cup S_i) \cap \Delta$, where $Q = \pi(D_{\mathrm{sing}}) \subset B$ is the finite set defined in* (A5) *and $Y_Q = \pi_B^{-1}(Q)$.*

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Proof Since $H^0(m(1+\varepsilon)L + m\pi_B^*M - 2\Delta) \subset \Sigma_i$, the base locus $\mathrm{Bs}(\Sigma_i)$ of Σ_i is contained in Δ by Lemma 3.2. So it suffices to show that $\mathrm{Bs}(\Sigma_i) \subset Y_Q \cup S_i$.

Let $F_i = S_i \cap 2\Delta$ be the subscheme of Y defined in the proof of Lemma 3.4. We have the exact sequence

$$\begin{aligned}0 &\to \mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M - 2\Delta) \to \mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M) \otimes I_{F_i}\\ &\to \underbrace{\mathcal{O}_\Delta(m(1+\varepsilon)L + m\pi_B^*M - S_i)}_{\mathcal{O}_\Delta(mG)} \otimes \mathcal{O}_Y/I_\Delta^2 \to 0,\end{aligned}$$

where $I_\Delta = \mathcal{O}_Y(-\Delta)$ is the ideal sheaf of $\Delta \subset Y$. Again by Lemma 3.2,

$$H^1(\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M - 2\Delta)) = 0,$$

and hence we have the surjection

$$\begin{aligned}\Sigma_i &\twoheadrightarrow H^0(\mathcal{O}_\Delta(m(1+\varepsilon)L + m\pi_B^*M - S_i) \otimes \mathcal{O}_Y/I_\Delta^2)\\ &\cong H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2).\end{aligned} \tag{3.20}$$

Composing the above map with

$$\varphi : H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2) \to H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta) = H^0(\mathcal{O}_\Delta(mG)),$$

we have a natural map $f\colon \Sigma_i \to H^0(\mathcal{O}_\Delta(mG))$. To show that $\mathrm{Bs}(\Sigma_i) \subset Y_Q \cup S_i$, it is enough to show that $\mathrm{Bs}(f(\Sigma_i)) \subset Y_Q$, which is equivalent to $\mathrm{Bs}(\mathrm{Im}(\varphi)) \subset Y_Q$ by (3.20). For $M \subset B$ sufficiently ample, we have the diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2) & \twoheadrightarrow & H^0(\Delta_b, \mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2) \\ \downarrow \varphi & & \downarrow \varphi_b \\ H^0(\mathcal{O}_\Delta(mG)) & \twoheadrightarrow & H^0(\Delta_b, \mathcal{O}_\Delta(mG)) \end{array}$$

with rows being surjections when we restrict φ to each fiber Δ_b of Δ/B. Therefore, it suffices to show that

$$\mathrm{Bs}(\mathrm{Im}(\varphi_b)) = \varnothing$$

for all $b \notin Q$. This is more or less obvious, since we have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\Delta/I_\Delta^2 & \longrightarrow & \mathcal{O}_Y/I_\Delta^2 & \longrightarrow & \mathcal{O}_Y/I_\Delta & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathcal{O}_\Delta(-\Delta) & & & & \mathcal{O}_\Delta & & \end{array}$$

When we tensor the sequence by $\mathcal{O}_\Delta(mG)$ and restrict it to $\Delta_b \cong \mathbb{P}^1$ with $b \notin Q$, we have

$$h^1(\Delta_b, \mathcal{O}_\Delta(mG - \Delta)) = h^1(\mathcal{O}_{\mathbb{P}^1}((m\varepsilon - 1)\alpha)) = 0$$

by (3.8) and (3.9). Consequently, φ_b is surjective and

$$\mathrm{Bs}(\mathrm{Im}(\varphi_b)) = \mathrm{Bs}(H^0(\Delta_b, \mathcal{O}_\Delta(mG))) = \mathrm{Bs}(H^0(\mathcal{O}_{\mathbb{P}^1}(m\varepsilon\alpha))) = \varnothing. \qquad \blacksquare$$

Remark 3.6 It is not hard to see that the above proposition continues to hold with tangency 2 replaced by any $\mu \leq m\varepsilon$. Moreover, being a little more careful, we can actually show that

$$\mathrm{Bs}(\Sigma_i) = \widetilde{X}_Q \cup (S_i \cap \Delta),$$

where $\widetilde{X}_Q \subset \Delta$ is the proper transform of $X_Q = \pi^{-1}(Q)$ under the map $\Delta \to X$. However, we have no need for these generalizations here.

By the above proposition, we see that the base locus of $\{\sigma_{ij} : i, j\}$ is supported on $Y_Q \cap \Delta$. Consequently, γ is a closed $(1,1)$ current that is C^∞ on $Y \backslash (Y_Q \cap \Delta)$. By (3.14),

$$-mG \cdot \Gamma_n \leq -\int_{\Gamma_n} \eta = -\int_{\Gamma_n \backslash U} \eta - \int_{\Gamma_n \cap U} \eta \leq \int_{\Gamma_n \backslash U} w - \int_{\Gamma_n \cap U} \eta \tag{3.21}$$

The fact that the first integral has order $O(\deg C_n)$ is a consequence of the following lemma.

Lemma 3.7 *Let $U \subset Y$ be an open neighborhood of Δ, w be a smooth $(1,1)$ form on X and κ be a positive smooth $(1,1)$ form on B. Then there exists a constant $A_U > 0$ such that at every point $(p, v) \in Y \backslash U$*

$$\left| \langle w, v \wedge \overline{v} \rangle \right| \leq A_U \langle \pi^* \kappa, v \wedge \overline{v} \rangle \tag{3.22}$$

where $p \in X$ and $v \in T_{X,p}(-\log D)$.

Proof By Lemma 3.1, $\langle \pi^*\kappa, v \wedge \overline{v} \rangle$ does not vanish for $(p, v) \notin \Delta$ and hence the function

$$f(p, v) = \frac{\langle w, v \wedge \overline{v} \rangle}{\langle \pi^* \kappa, v \wedge \overline{v} \rangle}$$

is continuous on $Y \backslash \Delta$. Then (3.22) follows from the compactness of $Y \backslash U$. $\blacksquare$

Note that w is the pullback of a form on X; indeed, it is the pullback of a form on $\mathbb{P}^1$. So Lemma 3.7 applies, and we conclude that $w \leq A_U \pi_B^* \kappa$ on $\Gamma_n \backslash U$ for some constant A_U depending only on U, where we choose κ to be a positive $(1,1)$ form on B representing $c_1(\mathcal{O}_B(b))$ for a point $b \in B$. Therefore,

$$\int_{\Gamma_n \backslash U} w \leq A_U \int_{\Gamma_n} \pi_B^* \kappa = A_U \deg(C_n) \tag{3.23}$$

Next, we claim that $\eta > 0$ everywhere on $\Delta \backslash Y_Q$.

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Lemma 3.8 *The current $\eta > 0$ at every point $p \in \Delta\backslash Y_Q$.*

By (2.16), there exists an open neighborhood V of Y_Q such that

$$\int_{\Gamma_n \cap V} w \le \varepsilon(m\alpha Y_p \cdot \Gamma_n)$$

By the above lemma and the compactness of $\Delta\backslash V$, we see that $\eta > 0$ in $U\backslash V$ for some open neighborhood U of Δ. The second integral in (3.21) becomes

$$(3.24) \qquad -\int_{\Gamma_n \cap U} \eta \le -\int_{\Gamma_n \cap (U\backslash V)} \eta + \int_{\Gamma_n \cap V} w$$
$$\le \varepsilon(m\alpha Y_p \cdot \Gamma_n) = m\varepsilon(\omega_{X/B} + D) \cdot C_n + O(\deg C_n)$$

Combining (3.23) and (3.24), we have

$$-G \cdot \Gamma_n = \varepsilon(\omega_{X/B} + D) \cdot C_n + O(\deg C_n) \quad \Longrightarrow$$
$$-\big((1+\varepsilon)L + \pi_B^* M - (1-\varepsilon)\pi_X^*(\omega_{X/B} + D)\big) \cdot \Gamma_n = O(\deg C_n).$$

Replace ε by $\varepsilon/(2+\varepsilon)$ and we are done. It remains to verify Lemma 3.8.

Proof of Lemma 3.8 At least one of $s_0(p)$ and $s_1(p)$ does not vanish. Let us assume that $s_0(p) \ne 0$ WLOG. Let $r_j = \sigma_{0j}/s_0$; r_j is holomorphic at p, of course. Let $\delta_j = \sigma_{1j} - s_1 r_j$. By our construction of σ_{1j}, we see that δ_j vanishes to the order 2 along Δ. We may write

$$(3.25) \qquad \gamma = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_j (|s_0 r_j|^2 + |s_1 r_j + \delta_j|^2)\Big)$$
$$= \underbrace{\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|s_0|^2 + |s_1|^2)}_{w} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_j |r_j|^2\Big)$$
$$-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(1 + \sum_j \frac{s_1 r_j \overline{\delta}_j + \overline{s}_1 \overline{r}_j \delta_j + |\delta_j|^2}{(|s_0|^2 + |s_1|^2)\sum_j |r_j|^2}\Big).$$

Basically, we want to show that the last term in (3.25) vanishes along Δ. Then

$$\eta\Big|_\Delta = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_j |r_j|^2\Big)$$

locally at p, which is positive.

Since η is C^∞ at p, it is enough to show that $\eta > 0$ at p when η is restricted to every curve passing through p, *i.e.,* to show that $f^*\eta > 0$ at q for every nonconstant morphism $f\colon C \to Y$ from a smooth and irreducible projective C to Y with $f(q) = p$. Indeed, it is enough to show the following:

For every tangent vector $\xi \in T_{Y,p}$, there exists a morphism $f\colon C \to Y$ from a smooth irreducible curve C to Y with $f(q) = p$, $\xi \in f_* T_{C,q}$ and $f^*\eta > 0$ at q.

Therefore, we can also exclude the curves contained in a fixed proper subvariety of Y. So we may assume that $f(C) \not\subset \Delta \cup W$, where $W \subsetneq Y$ is the subvariety such that

$$L \cdot \Gamma = 0 \text{ for a curve } \Gamma \Leftrightarrow \Gamma \subset W.$$

Such W exists because L is big and NEF (see Remark 3.3). Let $\hat{\mathcal{O}}_{C,q} \cong \mathbb{C}[[t]]$ be the formal local ring of C at q and μ be its valuation, *i.e.*, $\mu(t^n) = n$. Let $\mu(f^* s_\Delta) = \lambda$, where $\Delta = \{s_\Delta = 0\}$. Then $\mu(f^*\delta_j) \geq 2\lambda$. And since $\{\sigma_{0j}\}$ and hence $\{r_j\}$ are base point free at p, we have

$$f^*\left(\frac{s_1 r_j \overline{\delta}_j + \bar{s}_1 \bar{r}_j \delta_j + |\delta_j|^2}{(|s_0|^2 + |s_1|^2)\sum_j |r_j|^2}\right) = O(t^{2\lambda} + \bar{t}^{2\lambda} + |t|^{4\lambda}).$$

Therefore, we obtain

$$\frac{\sqrt{-1}}{2\pi} f^* \partial\overline{\partial} \log\left(1 + \sum_j \frac{s_1 r_j \overline{\delta}_j + \bar{s}_1 \bar{r}_j \delta_j + |\delta_j|^2}{(|s_0|^2 + |s_1|^2)\sum_j |r_j|^2}\right)\Bigg|_{t=0} = 0$$

by the Taylor expansion of the left-hand side. Consequently,

$$f^*\eta\Big|_q = \frac{\sqrt{-1}}{2\pi} f^* \partial\overline{\partial} \log\Big(\sum_j |r_j|^2\Big)\Bigg|_q = \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log\Big(\sum_j |f^*\sigma_{0j}|^2\Big)\Bigg|_q .$$

Since $H^0(m(1+\varepsilon)L + m\pi_B^* M - 2\Delta) \subset \Sigma_0$ and $f(C) \not\subset \Delta \cup W$, the linear system $f^*\Sigma_0$ is big on C. Therefore, $f^*\eta > 0$ at q and $\eta > 0$ at p. ■

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References

[V1] P. Vojta, *Diophantine inequalities and Arakelov theory.* In: S. Lang, *Introduction to Arakelov theory.*, Springer-Verlag, New York, 1988, 155–178.

[V2] P. Vojta, *On algebraic points on curves.* Compositio Math. **78**(1991), no. 1, 29–36.

[Y] K. Yamanoi, *The second main theorem for small functions and related problems.* Acta. Math. **192**(2004), no. 2, 225–294. doi:10.1007/BF02392741

University of Alberta, Edmonton, AB, T6G 2G1
e-mail: xichen@math.ualberta.ca