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ADJOINT INTERIOR-POINT BOUNDARY CONDITIONS FOR LINEAR DIFFERENTIAL OPERATORS

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Introduction. The investigations reported in this paper were prompted by a remark by A. M. Krall in [2] that certain functionals which appear in the boundary conditions of the system adjoint to a given linear differential boundary value problem seem artificial in that setting.

First, we show that those functionals arise naturally when the problem of finding adjoint boundary conditions is given the appropriate setting. Then, after using a standard procedure for eliminating the functionals from the adjoint boundary conditions so that the possibility of self adjointness of the system is clear, we give a set of self-adjointness criteria, simpler and easier to check than those in [2].

1. **The problem.** We review the notation in [2]. Let the compact interval $[a, b]$ be partitioned by $a = a_0 < a_1 < \cdots < a_{m-1} < a_m = b$, and let H denote the Hilbert space $L_n^2[a, b] = \{Y = (y_1, y_2, \ldots, y_n)^T$: y_i is in $L^2[a, b]$ for $1 \le i \le n\}$ with inner product $(X, Y) = \int_a^b Y^*X = \sum_{i=1}^n x_i\bar{y}_i$. Define the boundary operators M_i by

$$M_iY = \sum_{j=0}^{m} [A_{ij}Y(a_j+) + B_{ij}Y(a_j-)] \quad \text{for} \quad 1 \le i \le k,$$

where A_{ij} and B_{ij} are $1 \times n$ constant matrices, $A_{im} = B_{i0} = 0$ for $1 \le i \le k$, and $Y(a_j\pm)$ indicate the appropriate limits. We assume that the partitioned matrices $[A_{i0}\, A_{i1} \cdots A_{im} B_{i0} \cdots B_{im}]$, $1 \le i \le k$, are linearly independent so that $k \le 2mn$. Let A_0 and A_1 be continuous $n \times n$ matrices, and assume that A_1' is also continuous on $[a, b]$.

We denote by D the set of all Y in H satisfying
(1) Y is absolutely continuous on (a_{j-1}, a_j) for $1 \le j \le m$;
(2) $M_iY = 0$ for $1 \le i \le k$; and
(3) $A_1Y' + A_0Y$ is in H.
The operator L is defined by $LY = A_1Y' + A_0Y$ for all Y in D.

2. **The adjoint.** As Krall pointed out, D is dense in H, so that L has an adjoint L^+, and, in fact [2, Theorem 1], if Z is in D^+, the domain of L^+, then Z is absolutely continuous on each interval (a_{j-1}, a_j) and $L^+Z = -(A_1^*Z)' + A_0^*Z$

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on each (a_{j-1}, a_j) for $1 \le j \le m$. In the proof of Theorem 2 of [2], Krall establishes that if Y is in D and Z is in D^+, then

$$\sum_{j=1}^{m} Z^* A_1 Y \Big|_{a_{j-1^+}}^{a_j} = 0. \tag{4}$$

It follows from his proof that if Z is absolutely continuous on each (a_{j-1}, a_j) and (4) holds for every Y in D, then Z is in D^+.

At this point we introduce some notation which will allow us to present an alternative proof of Theorem 2 of [2] in which the functionals $\phi_i(Z)$, $1 \le i \le k$, arise naturally. We begin by setting $X = A_1^* Z$. Then (4) becomes

$$\sum_{j=1}^{m} X^*(a_{j^-}) Y(a_{j^-}) - \sum_{j=1}^{m} X^*(a_{j-1^+}) Y(a_{j-1^+}) = 0. \tag{5}$$

Using the notations

$$\beta[Y] = \begin{bmatrix} Y(a_{0^+}) \\ Y(a_{1^-}) \\ Y(a_{1^+}) \\ \cdot \\ \cdot \\ \cdot \\ Y(a_{m^-}) \end{bmatrix}$$

and $Q = \operatorname{diag}(-I_n, I_n, \ldots, -I_n, I_n)$, a $2mn \times 2mn$ matrix in which I_n is the $n \times n$ identity matrix, we can write equation (5) as

$$\beta^*[X] Q \beta[Y] = 0. \tag{6}$$

Letting M denote the $k \times 2mn$ matrix whose i-th row is $[A_{i0} B_{i1} A_{i1} \cdots B_{im-1} A_{im-1} B_{im}]$, we note that M has rank k, since the boundary operators M_i are linearly independent, and that the boundary condition (2) may be written as $\mu[Y] = M\beta[Y] = 0$.

It is easily seen that $\{\beta[Y] : Y \text{ is in } D\}$ is the same as $\mathcal{K}(M) = \{v \in V_{2mn} : Mv = 0\}$, since for each v in V_{2mn}, there is a vector Y in H satisfying (1) and (3) for which $\beta[Y] = v$.

THEOREM 1 [cf. 2, Theorem 2]. *If Z is in D^+, then there exists a vector $\Phi = (\phi_1, \phi_2, \ldots, \phi_k)^T$ satisfying*

$$\begin{cases} A_1^*(a_j) Z(a_{j^-}) = \sum_{i=1}^{k} B_{ij}^* \bar{\phi}_i \\ -A_1^*(a_{j-1}) Z(a_{j-1^+}) = \sum_{i=1}^{k} A_{ij-1}^* \bar{\phi}_i, \end{cases} \tag{7}$$

for $1 \leq j \leq m$. Furthermore, if Z is absolutely continuous on each (a_{j-1}, a_j) and there exists a vector Φ for which (7) holds, then Z is in D^+.

Proof. Let $Z \in D^+$ and $X = A_1^* Z$. Then, for every Y in D, equation (6) holds; i.e., $\beta^*[X]Q$ is orthogonal to every vector in $\mathcal{K}(M)$ and hence, must be in the row space of M. Thus, a vector Φ exists satisfying $\beta^*[X]Q = \Phi^T M$, which is equivalent to the conclusion. Reversing the argument yields the converse.

Cole [1] and Reid [5] call special cases of conditions (7) the parametric form of the adjoint boundary conditions, and they both indicate how to eliminate the parameter. We imitate their approach here, thus providing some motivation for the formulas (3) of Locker [4, p. 563].

Let S be the row space of M and $S^\perp$ its orthogonal complement. Let N be any $2mn - k \times 2mn$ matrix whose rows form a basis of $S^\perp$. Then, clearly, $MN^* = 0$. We define μ^+ on H by $\mu^+[Z] = NQ\beta[A_1^* Z]$.

THEOREM 2. *Let $Z \in H$ be absolutely continuous on (a_{j-1}, a_j) for $1 \leq j \leq m$. Then $\mu^+[Z] = 0$ if and only if Z is in D^+.*

Proof. For Z in D^+, a vector Φ exists satisfying $\beta^*[X]Q = \Phi^T M$, so that $\mu^{+*}[Z] = \beta^*[X]Q^* N^* = \Phi^T MN^* = 0$. Reversing the argument and applying Theorem 1 yields the converse.

We note in passing, that the i-th row of $\mu^+[Z]$ may be written as

$$M_i^+ Z = \sum_{j=0}^{m} [-A_{ij}^+ A_1^*(a_j) Z(a_{j^+}) + B_{ij}^+ A_1^*(a_j) Z(a_{j^-})],$$

where A_{ij}^+ and B_{ij}^+ are $1 \times n$ vectors and $A_{im}^+ = B_{i0}^+ = 0$.

3. Self-adjointness. We now characterize self-adjointness of L without resorting to reducing the original problem as Krall does in Section 2 of [2]. To do this, we introduce the $2mn \times 2mn$ matrix $C = \operatorname{diag}(A_1^*(a_0), A_1^*(a_1), A_1^*(a_1), \ldots, A_1^*(a_m))$, so that we may write $\beta[X] = \beta[A_1^* Z] = C\beta[Z]$. We have immediately the following.

THEOREM 3. *The operator L is self-adjoint in H if and only if $A_1 = -A_1^*$, $A_0 = A_0^* - A_1^{*\prime}$, and NQC and M have the same kernel.*

REFERENCES

1. R. H. Cole, *Theory of Ordinary Differential Equations*, Appleton-Century-Crofts, New York, 1968.

2. A. M. Krall, *Boundary value problems with interior point boundary conditions*, Pacific J. Math., vol. **29** (1969) pp. 161–166.

3. A. M. Krall, *Differential-boundary operators*, Trans. Amer. Math. Soc., vol. **154** (1971) pp. 429–458.

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4. J. Locker, *Self-adjointness for multi-point differential operators*, Pacific J. Math., vol. **45** (1973) pp. 561–570.

5. W. T. Reid, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1971.

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