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Isospectrality for Orbifold Lens Spaces

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Abstract. We answer Mark Kac's famous question, "Can one hear the shape of a drum?" in the positive for orbifolds that are 3-dimensional and 4-dimensional lens spaces; we thus complete the answer to this question for orbifold lens spaces in all dimensions. We also show that the coefficients of the asymptotic expansion of the trace of the heat kernel are not sufficient to determine the above results.

1 Introduction

Given a closed Riemannian manifold (M, g), the eigenvalue spectrum of the associated Laplace Beltrami operator is referred to as the *spectrum of* (M, g). The inverse spectral problem asks the extent to which the spectrum encodes the geometry of (M, g). While various geometric invariants such as dimension, volume, and total scalar curvature are spectrally determined, numerous examples of isospectral Riemannian manifolds, *i.e.*, manifolds with the same spectrum, show that the spectrum does not fully encode the geometry. Not surprisingly, the earliest examples of isospectral manifolds were manifolds of constant curvature including flat tori [M], hyperbolic manifolds [V], and spherical space forms [IY, Gi, I1, I2]. Lens spaces are spherical space forms that are quotients of round spheres by cyclic groups of orthogonal transformations that act freely on the sphere. Lens spaces have provided a rich source of isospectral manifolds with interesting properties. In addition to the work of Ikeda and Yamamoto cited above, see the results of Gornet and McGowan [GoM].

In this paper we generalize this theme to the category of Riemannian orbifolds. A smooth *orbifold* is a topological space that is locally modelled on an orbit space of $\mathbf{R}^n$ under the action of a finite group of diffeomorphisms. *Riemannian* orbifolds are spaces that are locally modelled on quotients of Riemannian manifolds by finite groups of isometries. Orbifolds have wide applicability, for example, in the study of 3-manifolds and in string theory [DHVW, ALR].

The tools of spectral geometry can be transferred to the setting of Riemannian orbifolds by using their well-behaved local structure (see [Chi, S1, S2]). As in the manifold setting, the spectrum of the Laplace operator of a compact Riemannian orbifold is a sequence $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \nearrow \infty$, where each eigenvalue is repeated according to its finite multiplicity. We say that two orbifolds are *isospectral* if their Laplace spectra agree.

The literature on inverse spectral problems on orbifolds is less developed than that for manifolds. Examples of isospectral orbifolds include pairs with boundary [BCDS, BW]; isospectral flat 2-orbifolds [DR]; arbitrarily large finite families of isospectral

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orbifolds [BSW]; isospectral orbifolds with different maximal isotropy orders [RSW]; and isospectral deformation of metrics on an orbifold quotient of a nilmanifold [PS].

In the study of inverse isospectral problem spherical space forms provide a rich and important set of orbifolds with interesting results. For the 2-dimensional case, it is known [DGGW] that the spectrum determines the spherical orbifolds of constant curvature $R > 0$. Lauret [L] found examples in dimensions 5 through 8 of orbifold lens spaces (spherical orbifold spaces with cyclic fundamental groups) that are isospectral but not isometric. For dimension 9 and higher, the author proved the existence of isospectral orbifold lens spaces that are non-isometric [Ba]. The problem was unsolved for 3 and 4-dimensional orbifold lens spaces. For 3-dimensional manifold lens spaces, Ikeda and Yamamoto (see [IY, Y, I1]) proved that the spectrum determines the lens space. In [I2], Ikeda further proved that for general 3-dimensional manifold spherical space forms, the spectrum determines the space form. In the manifold case, it is also known that even dimensional spherical space forms are only the canonical sphere and the real projective space. For orbifold spherical space forms, this is not the case, as there are many even dimensional orbifold spherical space forms. In this article, we will prove the following results.

***Theorem* 3.1** *Any two three-dimensional isospectral orbifold lens spaces are isometric.*

***Theorem* 4.3** *Any two four-dimensional isospectral orbifold lens spaces are isometric.*

***Theorem* 5.6** *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two (orbifold) spherical space forms. Suppose G is cyclic and G' is not cyclic. Then $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ cannot be isospectral.*

The above results will complete the classification of the inverse spectral problem on orbifold lens spaces in all dimensions.

In addition to the above theorems, we also prove that one of the traditonal methods of obtaining geomeric invariants from the spectrum, *i.e.*, from the coefficients of the trace of the heat kernel, is not sufficient to prove the above results. We will show that we can have two non-isospectral orbifold lens spaces with identical coefficients of the trace of heat kernel.

2 Orbifold Lens Spaces

In this section, we will generalize the idea of manifold lens spaces to orbifold lens spaces. Note that lens spaces are special cases of *spherical space forms*, which are *connected complete Riemannian manifolds of positive constant curvature* 1. An m-dimensional spherical space form can be written as $\mathbb{S}^m/G$ where G is a finite subgroup of the orthogonal group $O(m+1)$. In fact, the definition of spherical space forms can be generalized to allow G to have fixed points making $\mathbb{S}^m/G$ an orbifold. Manifold lens spaces are spherical space forms where the m-dimensional sphere $\mathbb{S}^m$ of constant curvature 1 is acted upon by a cyclic group of fixed point free isometries on $\mathbb{S}^m$. We will generalize this notion to orbifolds by allowing the cyclic group of isometries to have fixed points. For details of spectral geometry on orbifolds, see Stanhope [S1] and E. Dryden, C. Gordon, S. Greenwald, and D. Webb [DGGW].

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2.1 Orbifold Lens Spaces and their Generating Functions

We now reproduce the background work developed by Ikeda in [I1, I2] for manifold spherical space forms. For the most part, we will keep the notation used by Ikeda. We will note that, with slight modifications, the results are valid for orbifold spherical space forms. This is the background work we will need to develop our results for orbifold lens spaces.

We will first consider general $m = 2n-1$ dimensional lens spaces. Let q be a positive integer. Set

$$q_0 = \begin{cases} \frac{q-1}{2} & \text{if } q \text{ is odd,} \\ \frac{q}{2} & \text{if } q \text{ is even.} \end{cases}$$

Throughout this section we assume that $q_0 \geq 4$.

For $n \leq q_0$, let $p_1, \ldots, p_n$ be n integers. Note, if $\gcd(p_1, \ldots, p_n, q) \neq 1$, we can divide all the $p_i's$ and q by this gcd to get a case where the gcd is 1. So, without loss of generality, we can assume that $\gcd(p_1, \ldots, p_n, q) = 1$. We denote by g the orthogonal matrix given by

$$g = \begin{pmatrix} M(p_1/q) & & 0 \\ & \ddots & \\ 0 & & M(p_n/q) \end{pmatrix},$$

where

$$M(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Then g generates a cyclic subgroup $G = \{g^l\}_{l=1}^{q}$ of order q of the special orthogonal group $SO(2n)$, since $\det g = 1$. Note that g has eigenvalues $\gamma^{p_1}, \gamma^{-p_1}, \gamma^{p_2}, \gamma^{-p_2}, \ldots, \gamma^{p_n}, \gamma^{-p_n}$, where γ is a primitive q-th root of unity. We define the lens space $L(q : p_1, \ldots, p_n)$ as follows:

$$L(q : p_1, \ldots, p_n) = S^{2n-1}/G.$$

Note that if $\gcd(p_i, q) = 1$ for all i, then $L(q : p_1, \ldots, p_n)$ is a smooth manifold; Ikeda and Yamamoto have answered Kac's question in the affirmative for 3-dimensional manifold lens spaces [IY, Y]. To get an orbifold in this setting with non-trivial singularities, we must have $\gcd(p_i, q) > 1$ for some i. In such a case $L(q : p_1, \ldots, p_n)$ is a good smooth orbifold with S^{2n-1} as its covering manifold. Let π be the covering projection of $\mathbb{S}^{2n-1}$ onto $\mathbb{S}^{2n-1}/G$,

$$\pi : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^{2n-1}/G.$$

Since the round metric of constant curvature one on $\mathbb{S}^{2n-1}$ is G-invariant, it induces a Riemannian metric on $\mathbb{S}^{2n-1}/G$. Henceforth, the term "lens space" will refer to this generalized definition.

Ikeda proved the following result for manifold spherical space forms ([I2, Lemma 1.2]). We note that the proof does not require the groups to be fixed-point free, and it reproduces the result for orbifold spherical space forms.

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Lemma 2.1 *Let $\mathbb{S}^m/G$ and $\mathbb{S}^m/G'$ be spherical space forms for any integer $m \geq 2$. Then $\mathbb{S}^m/G$ is isometric to $\mathbb{S}^m/G'$ if and only if G is conjugate to G' in $O(m+1)$.*

Note that if we have a lens space $\mathbb{S}^{2n-1}/G = L(q : p_1, \ldots, p_n)$, with $G = \langle g \rangle$, permuting the p_is does not change the underlying group G; similarly, if we multiply all the p_is by some number $\pm l$, where $\gcd(l, q) = 1$, that simply means we have mapped the generator g to the generator g^l, and so we still have the same group G. Also note that if two lens spaces $\mathbb{S}^{2n-1}/G = L(q : p_1, \ldots, p_n)$ and $\mathbb{S}^{2n-1}/G' = L(q : s_1, \ldots, s_n)$ are isometric, then by the above lemma, G and G' must be conjugate. So, the lift of the isometry on $\mathbb{S}^{2n-1}$ maps a generator, g of G to a generator g'^l of G'. This means that the eigenvalues of g and g'^l are the same, which means that each p_i is equivalent to some ls_j or $-ls_j \pmod q$. These facts give us the following corollary for Lemma 2.1.

Corollary 2.2 *Let $L = L(q : p_1, \ldots, p_n)$ and $L' = L(q : s_1, \ldots, s_n)$ be lens spaces. Then L is isometric to L' if and only if there is a number l coprime with q and numbers $e_i \in \{-1, 1\}$ such that $(p_1, \ldots, p_n)$ is a permutation of $(e_1 l s_1, \ldots, e_n l s_n) \pmod q$.*

Assume we have a spherical space form $\mathbb{S}^m/G$ for any integer $m \geq 2$. For any $f \in C^\infty(\mathbb{S}^m/G)$, we define the Lapacian on the spherical space form as $\widetilde{\Delta}(\pi^* f) = \pi^*(\Delta f)$. We now construct the spectral generating function associated with the Laplacian on $\mathbb{S}^{2n-1}/G$ analogous to the construction in the manifold case (see [IY, I1, I2]). Let $\widetilde{\Delta}$, Δ, and Δ_0 denote the Laplacians of $\mathbb{S}^{2n-1}$, $\mathbb{S}^{2n-1}/G$, and $\mathbb{R}^{2n}$, respectively.

Definition 2.3 For any non-negative real number λ, we define the *eigenspaces* $\widetilde{E}_\lambda$ and E_λ as follows:

$$\widetilde{E}_\lambda = \left\{ f \in C^\infty(\mathbb{S}^{2n-1}) \,\middle|\, \widetilde{\Delta} f = \lambda f \right\},$$
$$E_\lambda = \left\{ f \in C^\infty(\mathbb{S}^{2n-1}/G) \,\middle|\, \Delta f = \lambda f \right\}.$$

The next lemma follows from the definitions of Δ and smooth function.

Lemma 2.4 *Let G be a finite subgroup of $O(2n)$.*

(i) *Forany $f \in C^\infty(\mathbb{S}^{2n-1}/G)$, we have $\widetilde{\Delta}(\pi^* f) = \pi^*(\Delta f)$.*
(ii) *For any G-invariant function F on $\mathbb{S}^{2n-1}$, there exists a unique function $f \in C^\infty(\mathbb{S}^{2n-1}/G)$ such that $F = \pi^* f$.*

Corollary 2.5 *Let $(\widetilde{E}_\lambda)_G$ be the space of all G-invariant functions of $\widetilde{E}_\lambda$. Then $\dim(E_\lambda) = \dim(\widetilde{E_\lambda})_G$.*

Let Δ_0 be the Laplacian on $\mathbf{R}^{2n}$ with respect to the flat Kähler metric. Set $r^2 = \sum_{i=1}^{2n} x_i^2$, where $(x_1, x_2, \ldots, x_{2n})$ is the standard coordinate system on $\mathbf{R}^{2n}$. For $k \geq 0$, let P^k denote the space of complex valued homogeneous polynomials of degree k on $\mathbb{R}^{2n}$. Let H^k be the subspace of P^k consisting of harmonic polynomials on $\mathbb{R}^{2n}$,

$$H^k = \left\{ f \in P^k \,\middle|\, \Delta_0 f = 0 \right\}.$$

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Each orthogonal transformation of $\mathbb{R}^{2n}$ canonically induces a linear isomorphism of P^k.

Proposition 2.6 *The space H^k is $O(2n)$-invariant, and P^k has the direct sum decomposition $P^k = H^k \oplus r^2 P^{k-2}$.*

The injection map $i\colon \mathbb{S}^{2n-1} \to \mathbb{R}^{2n}$ induces a linear map $i^* : C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{S}^{2n-1})$. We denote $i^*(H^k)$ by $\mathcal{H}^k$.

Proposition 2.7 *$\mathcal{H}^k$ is an eigenspace of $\widetilde{\Delta}$ on $\mathbb{S}^{2n-1}$ with eigenvalue $k(k+2n-2)$, and $\sum_{k=0}^{\infty} \mathcal{H}^k$ is dense in $C^\infty(\mathbb{S}^{2n-1})$ in the uniform convergence topology. Moreover, $\mathcal{H}^k$ is isomorphic to H^k. That is, $i^*\colon H^k \xrightarrow{\simeq} \mathcal{H}^k$.*

For proofs of these propositions, see [BGM].

Now Corollary 2.5 and Proposition 2.7 imply that if we denote by $\mathcal{H}_G^k$ the space of all G-invariant functions in $\mathcal{H}^k$, then

$$\dim E_{k(k+2n-2)} = \dim \mathcal{H}_G^k.$$

Moreover, for any integer k such that $\dim \mathcal{H}_G^k \neq 0$, $\overline{\lambda}_k = k(k+2n-2)$ is an eigenvalue of Δ on $\mathbb{S}^{2n-1}/G$ with multiplicity equal to $\dim \mathcal{H}_G^k$, and no other eigenvalues appear in the spectrum of Δ.

Definition 2.8 Let O be a closed compact Riemannian orbifold with the Laplace spectrum, $0 \leq \overline{\lambda}_1 < \overline{\lambda}_2 < \overline{\lambda}_3 \cdots \nearrow \infty$. For each $\overline{\lambda}_k$, let the eigenspace be

$$E_{\overline{\lambda}_k} = \left\{ f \in C^\infty(O) \,\middle|\, \Delta f = \overline{\lambda}_k f \right\}.$$

We define the spectrum generating function associated with the spectrum of the Laplacian on O as

$$F_O(z) = \sum_{k=0}^{\infty} \left(\dim E_{\overline{\lambda}_k}\right) z^k.$$

In terms of spherical space forms, the definition becomes the following.

Definition 2.9 The *generating function* $F_G(z)$ associated with the spectrum of the Laplacian on $\mathbb{S}^m/G$ is the generating function associated with the infinite sequence $\{\dim \mathcal{H}_G^k\}_{k=0}^{\infty}$, *i.e.*,

$$F_G(z) = \sum_{k=0}^{\infty} \left(\dim \mathcal{H}_G^k\right) z^k.$$

By Corollary 2.5, Proposition 2.7, and subsequent discussion, we know that the generating function determines the spectrum of $\mathbb{S}^m/G$. This fact gives us the following proposition.

Proposition 2.10 *Let $\mathbb{S}^m/G$ and $\mathbb{S}^m/G'$ be two spherical space forms. Let $F_G(z)$ and $F_{G'}(z)$ be their respective spectrum generating functions. Then $\mathbb{S}^m/G$ is isospectral to $\mathbb{S}^m/G'$ if and only if $F_G(z) = F_{G'}(z)$.*

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Our first goal is to find an alternative expression for $F_G(z)$ that will allow us to compare $F_G(z)$ and $F_{G'}(z)$.

If G is a finite subgroup of $O(2n)$ with orientation preserving action on S^{2n-1}, then G is a subgroup of $SO(2n)$. In the sequel, we will consider orientation-preserving group actions.

The following theorem, proved for manifold spherical space forms in [I2, Theorem 2.2], holds true for the orbifold spherical space forms as well, since the proof does not require the group action to be free.

Theorem 2.11 *Let G be a finite subgroup of $SO(2n)$, and let $\mathbb{S}^{2n-1}/G$ be a spherical space form with spectrum generating function $F_G(z)$. Then, on the domain $\{z \in \mathbb{C} \mid |z| < 1\}$, $F_G(z)$ converges to the function*

$$F_G(z) = \frac{1}{|G|} \sum_{g \in G} \frac{1 - z^2}{\det(I_{2n} - gz)},$$

where $|G|$ denotes the order of G and I_{2n} is the $2n \times 2n$ identity matrix.

We denote the generating function for a lens space $L = L(q : p_1, \ldots, p_n)$ by $F_q(z : p_1, \ldots, p_n)$.

Corollary 2.12 *Let $L(q : p_1, \ldots, p_n)$ be a lens space and let $F_q(z : p_1, \ldots, p_n)$ be the generating function associated with the spectrum of $L(q : p_1, \ldots, p_n)$. Then, on the domain $\{z \in \mathbb{C} \mid |z| < 1\}$,*

$$F_q(z : p_1, \ldots, p_n) = \frac{1}{q} \sum_{l=1}^{q} \frac{1 - z^2}{\prod_{i=1}^{n} (z - \gamma^{p_i l})(z - \gamma^{-p_i l})},$$

where γ is a primitive q-th root of unity.

Proof Let χ_k denote the character of the natural representation of $SO(2n)$ on H^k. Then, in the notation of the Theorem 2.11, we get

$$\dim \mathcal{H}_G^k = \frac{1}{|G|} \sum_{g \in G} \chi_k(g) = \frac{1}{q} \sum_{l=1}^{q} \chi_k(g^l).$$

So

$$\begin{aligned} F_q(z : p_1, \ldots, p_n) &= \frac{(1 - z^2)}{|G|} \sum_{g \in G} \frac{1}{\prod_{i=1}^{n} (1 - \gamma^{p_i} z)(1 - \gamma^{-p_i} z)} \\ &= \frac{(1 - z^2)}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n} (z - \gamma^{p_i l})(z - \gamma^{-p_i l})}, \end{aligned}$$

since multiplying through by $1 = (-\gamma^{-p_i l})(-\gamma^{p_i l})$ gives $(1 - \gamma^{p_i l} z)(1 - \gamma^{-p_i l} z) = (z - \gamma^{-p_i l})(z - \gamma^{p_i l})$. ∎

Remark By Theorem 2.11 and unique analytic continuation, we can consider the generating function to be a meromorphic function on the whole complex plane $\mathbb{C}$ with poles on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

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From this remark, we have the following corollary.

Corollary 2.13 *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two spherical space forms. Let $E(g)$ denote the set of eigenvalues of g, with multiplicity counted. If there is a one to one mapping ϕ of G onto G' such that the set $E(g)$ equals the set $E(\phi(g))$ for all $g \in G$, then $\mathbb{S}^{2n-1}/G$ is isospectral to $\mathbb{S}^{2n-1}/G'$.*

Proof The proof follows from the fact that

$$\prod_{\gamma \in E(g)} (1-\gamma z) = \prod_{\gamma \in E(g)} (z-\gamma) = \det(I_{2n} - gz). \qquad \blacksquare$$

Corollary 2.14 *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two isospectral spherical space forms. Then $|G| = |G'|$.*

3 3-Dimensional Orbifold Lens Spaces

For 3-dimensional manifold lens spaces, it is known that if two lens spaces are isospectral, then they are also isometric ([IY] and [Y]). We will generalize this result to the orbifold case.

Using the notation adopted in the previous section, we write the two isospectral lens spaces as $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. Now there are only five possibilities.

Case 1: Both L_1 and L_2 are manifolds. In this case, $\gcd(p_i, q) = 1 = \gcd(s_i, q)$ for $i = 1, 2$.

Case 2: One of the two lens spaces, say L_1, is a manifold, while the other, L_2 is an orbifold with non-trivial isotropy groups. This means that $\gcd(p_1, q) = \gcd(p_2, q) = 1$, while at least one of s_1 or s_2 is not coprime to q.

Case 3: Both L_1 and L_2 are orbifolds with non-trivial isotropy groups so that exactly one of p_1 or p_2 is coprime to q and exactly one of s_1 or s_2 is coprime to q.

Case 4: Both L_1 and L_2 are orbifolds with non-trivial isotropy groups, but in one case, say for L_1, exactly one of p_1 or p_2 is coprime to q, while for the other lens space, L_2, neither s_1 nor s_2 is coprime to q.

Case 5: None of p_1, p_2, s_1, and s_2 is coprime to q.

With these five cases in mind, we will prove our main theorem.

Theorem 3.1 *Given two 3-dimensional lens spaces $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. If L_1 is isospectral to L_2, then the two lens spaces are isometric.*

Proof We will consider each case separately:

Case 1: In this case L_1 and L_2 are both manifolds. Ikeda and Yamamoto proved this case (see [IY, Y]).

Case 2: We know that whenever two isospectral good orbifolds share a common Riemannian cover, their respective singular sets are either both trivial or both non-trivial [GR]. Therefore, for orbifold lens spaces we cannot have a situation where two lens

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spaces are isospectral, but one has a trivial singular set while the other has a non-trivial singular set. So this case is not possible.

Case 3: By multiplying the entries of L_1 and L_2 by appropriate numbers coprime to q, we can rewrite $L_1 = L(q:1,x)$ and $L_2 = L(q:1,y)$, where x and y are not coprime to q. Let $F_1(z)$ (resp. $F_2(z)$) be the generating function associated with the spectrum of L_1 (resp. L_2). Let γ be a primitive q-th root of unity.

Now,

$$\begin{aligned}(3.1)\qquad &\lim_{z\to\gamma}(z-\gamma)F_1(z)\\ &\quad=\lim_{z\to\gamma}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma)(1-z^2)}{(1-\gamma^l z)(1-\gamma^{-l}z)(1-\gamma^{xl}z)(1-\gamma^{-xl}z)}\\ &\quad=\lim_{z\to\gamma}\frac{-\gamma}{q}\sum_{l=1}^{q}\frac{(1-\gamma^{-1}z)(1-z^2)}{(1-\gamma^l z)(1-\gamma^{-l}z)(1-\gamma^{xl}z)(1-\gamma^{-xl}z)}.\end{aligned}$$

Each term of the sum vanishes unless $(1-\gamma^{-1}z)$ cancels one of the four terms in the denominator. This occurs if one of the following congruences has a solution:

(1) $l+1\equiv 0 \pmod q$,
(2) $-l+1\equiv 0 \pmod q$,
(3) $xl+1\equiv 0 \pmod q$,
(4) $-xl+1\equiv 0 \pmod q$.

Congruences (3) and (4) have no solution as x is not coprime to q. The solution to (1) is $l=q-1$, and the solution to (2) is $l=1$. Substituting into (3.1), we get

$$\lim_{z\to\gamma}(z-\gamma)F_1(z)=\frac{-2\gamma}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})}.$$

By the same argument, we get

$$\lim_{z\to\gamma}(z-\gamma)F_2(z)=\frac{-2\gamma}{q(1-\gamma^{-y+1})(1-\gamma^{y+1})}.$$

Since

$$\lim_{z\to\gamma}(z-\gamma)F_1(z)=\lim_{z\to\gamma}(z-\gamma)F_2(z),$$

we get

$$\begin{aligned}&\frac{-2\gamma}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})}=\frac{-2\gamma}{q(1-\gamma^{-y+1})(1-\gamma^{y+1})},\\ &\quad\Longrightarrow\ \frac{1}{[1-(\gamma^{-x+1}+\gamma^{x+1})+\gamma^2]}=\frac{1}{[1-(\gamma^{-y+1}+\gamma^{y+1})+\gamma^2]},\\ &\quad\Longrightarrow\ \gamma^{-x+1}+\gamma^{x+1}=\gamma^{-y+1}+\gamma^{y+1}.\end{aligned}$$

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Since $\gamma \neq 0$, we get

$$\begin{aligned}
&\gamma^{-x} + \gamma^{x} = \gamma^{-y} + \gamma^{y}, \\
&\Longrightarrow \frac{1}{\gamma^{x}} + \gamma^{x} = \frac{1}{\gamma^{y}} + \gamma^{y}, \\
&\Longrightarrow \frac{1+\gamma^{2x}}{\gamma^{x}} = \frac{1+\gamma^{2y}}{\gamma^{y}}, \\
&\Longrightarrow \gamma^{y} + \gamma^{2x+y} = \gamma^{x} + \gamma^{x+2y}, \\
&\Longrightarrow \gamma^{y} - \gamma^{x+2y} = \gamma^{x} - \gamma^{2x+y}, \\
&\Longrightarrow \gamma^{y}(1-\gamma^{x+y}) = \gamma^{x}(1-\gamma^{x+y}), \\
&\Longrightarrow (\gamma^{y} - \gamma^{x})(1-\gamma^{x+y}) = 0, \\
&\Longrightarrow \gamma^{y} - \gamma^{x} = 0 \text{ or } 1-\gamma^{x+y} = 0, \\
&\Longrightarrow x \equiv y \pmod{q} \text{ or } x \equiv -y \pmod{q}.
\end{aligned}$$

Thus, by Corollary 2.2, we get that L_1 and L_2 are isometric.

Case 4: By the same argument as in Case 3, we get

$$\lim_{z\to\gamma}(z-\gamma)F_1(z) = \frac{-2\gamma}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})}.$$

However,

$$\lim_{z\to\gamma}(z-\gamma)F_2(z) = 0,$$

since the congruences (1)–(4) in Case 3 become

(1′) $s_1 l + 1 \equiv 0 \pmod{q}$,
(2′) $-s_1 l + 1 \equiv 0 \pmod{q}$,
(3′) $s_2 l + 1 \equiv 0 \pmod{q}$,
(4′) $-s_2 l + 1 \equiv 0 \pmod{q}$,

and these congruences have no solutions, because s_1 and s_2 are not coprime to q.

Thus, in this case, L_1 cannot be isospectral to L_2.

Case 5: This is the hardest of all the cases. First, we can simplify the forms of the two lens spaces as follows.

Let $\gcd(p_1, q) = x > 1$, $\gcd(p_2, q) = y > 1$, $\gcd(s_1, q) = u > 1$, and $\gcd(s_2, q) = v > 1$. Also, without loss of generality, we can assume that $y > x$ and $v > u$, because if $x = y$ (resp. $u = v$), then $|G| = q/x$ (resp. $|G| = q/u$), which contradicts our assumption that $|G| = q$.

We rewrite $L_1 = L(q : ax, by)$ and $L_2 = L(q : cu, dv)$. Since $\gcd(ax, q) = \gcd(x, q) = x$ and $\gcd(cu, q) = \gcd(u, q) = u$, we can multiply the entries of L_1 and L_2 by appropriate numbers coprime to q and rewrite $L_1 = L(q : x, py)$ and $L_2 = L(q : u, sv)$ (see [GP]). We will also assume that $\gcd(x, py) = 1 = \gcd(u, sv)$, because if say $\gcd(x, py) = e > 0$, then we could divide x, py, and q by e and get a lens space with fundamental group of order q/e instead of q, which is a contradiction.

In this case, we again want to consider a limit of the spectral generating functions for L_1 and L_2.

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Proposition 3.2 *Suppose $L = L(q : x, py)$ is an orbifold lens space with spectrum generating function $F_q(z)$. Then $\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z) \neq 0$, where $\gamma = e^{2\pi i/q}$ is a primitive q-th root of unity.*

Proof We denote $q_{/x} = \frac{q}{x}$ and $q_{/y} = \frac{q}{y}$. Then

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z) \tag{3.2}$$
$$= \lim_{z\to\gamma^x}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma^x)(1-z^2)}{(1-\gamma^{xl}z)(1-\gamma^{-xl}z)(1-\gamma^{pyl}z)(1-\gamma^{-pyl}z)}$$
$$= \lim_{z\to\gamma^x}\frac{-\gamma^x}{q}\sum_{l=1}^{q}\frac{(1-\gamma^{-x}z)(1-z^2)}{(1-\gamma^{xl}z)(1-\gamma^{-xl}z)(1-\gamma^{pyl}z)(1-\gamma^{-pyl}z)}.$$

As before, the terms in the above sum are non-zero if and only if one of the following congruences has a solution:

$(1'')$ $xl + x \equiv 0 \pmod q$,
$(2'')$ $-xl + x \equiv 0 \pmod q$,
$(3'')$ $pyl + x \equiv 0 \pmod q$,
$(4'')$ $-pyl + x \equiv 0 \pmod q$,

$(3'')$ implies that $pyl + x \equiv 0 \pmod y$, so, if $(3'')$ has a solution, it would violate the fact that $\gcd(x, y) = 1$. Therefore, $(3'')$ has no solution. Similarly, $(4'')$ has no solution.

The solution to $(1'')$ is $l = tq_{/x} - 1$, and the solution to $(2'')$ is $l = tq_{/x} + 1$ for $t \in \{1, \ldots, x\}$. Note that for $l = tq_{/x} \pm 1$,

$$\lim_{z\to\gamma^x}\frac{(1-\gamma^{-x}z)(1-z^2)}{(1-\gamma^{xl}z)(1-\gamma^{-xl}z)} = 1.$$

We can therefore write (3.2) as

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z) = \frac{-2\gamma^x}{q}\sum_{t=1}^{x}\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})}.$$

Writing $\alpha_t = py(tq_{/x} - 1)$, we get

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z)$$
$$= \frac{-2\gamma^x}{q}\sum_{t=1}^{x}\frac{1}{(1-\gamma^{(\alpha_t+x)})(1-\gamma^{-(\alpha_t-x)})}$$
$$= \frac{-2\gamma^x}{q}\sum_{t=1}^{x}\frac{1}{\gamma^{(\alpha_t+x)}(\gamma^{-(\alpha_t-x)}-\gamma^{-(\alpha_t+x)})}$$
$$\times\left[\frac{1}{1-\gamma^{-(\alpha_t+x)}}-\frac{1}{1-\gamma^{-(\alpha_t-x)}}\right]$$
$$= \frac{-2\gamma^x}{q(\gamma^{2x}-1)}\sum_{t=1}^{x}\left[\frac{1}{1-\gamma^{-(\alpha_t+x)}}-\frac{1}{1-\gamma^{-(\alpha_t-x)}}\right]$$
$$= \frac{-2}{i2q\sin\frac{2\pi x}{q}}\sum_{t=1}^{x}\left[\frac{1}{1-e^{-i2\pi(\alpha_t+x)/q}}-\frac{1}{1-e^{-i2\pi(\alpha_t-x)/q}}\right].$$

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By writing $a_t = \alpha_t + x$ and $b_t = \alpha_t - x$, we can rewrite the above as

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z) = \frac{-2}{i2q\sin\frac{2\pi x}{q}}\sum_{t=1}^{x}\Big[\frac{1}{1-e^{-i2\pi a_t/q}} - \frac{1}{1-e^{-i2\pi b_t/q}}\Big]$$
$$= \frac{1}{2q\sin\frac{2\pi x}{q}}\sum_{t=1}^{x}\Big[\frac{2i}{1-e^{-i2\pi a_t/q}} - \frac{2i}{1-e^{-i2\pi b_t/q}}\Big]$$

Now, using the identity $\cot\theta + i = \frac{2i}{1-e^{-2i\theta}}$, we get

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z) = \frac{1}{2q\sin\frac{2\pi x}{q}}\sum_{t=1}^{x}\Big[\cot\frac{\pi a_t}{q} - \cot\frac{\pi b_t}{q}\Big]. \tag{3.3}$$

The above limit can only be 0 if

$$\sum_{t=1}^{x}\Big[\cot\frac{\pi a_t}{q} - \cot\frac{\pi b_t}{q}\Big]$$
$$= \sum_{t=1}^{x}\Big[\cot\frac{\pi}{q}[tpyq_{/x} - (py-x)] - \cot\frac{\pi}{q}[tpyq_{/x} - (py+x)]\Big] = 0.$$

Suppose A_t (resp. B_t) is the remainder when a_t (resp. b_t) is divided by q. Then $\frac{A_t}{q}$ is between 0 and π.

Consider the following two sets of remainders of positive integers (mod q) when a_t and b_t are divided by q:

$$A = \{A_t : A_t = a_t \pmod q, t = 1, 2, \ldots, x\},$$
$$B = \{B_t : B_t = b_t \pmod q, t = 1, 2, \ldots, x\}.$$

Suppose $\min\{A\} = A_j$ and $\min\{B\} = B_k$. Now we have the following possibilities:

(i) $A_j > B_k$. Then it is easy to check that $A_{j+t} > B_{k+t}$ for $t = 0, 1, \ldots, x-1$, since $a_{j+t} - b_{k+t} = a_j - b_k$. So we can re-write (3.3) as

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_q(z) = \frac{1}{2q\sin\frac{2\pi x}{q}}\sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}A_{j+t} - \cot\frac{\pi}{q}B_{k+t}\Big]. \tag{3.4}$$

We know that if $0 < B < A < \pi$, then $\cot A - \cot B < 0$. Since in the above equation $0 < B_{k+t} < A_{j+t} < \pi$ for all t, each pair gives us a negative value, and therefore (3.4) is negative.

(ii) $A_j < B_k$. Then using a similar argument as above, equation (3.4) will be positive.

(iii) $A_j = B_k$. This means $a_j - b_k \equiv (j-k)pyq_{/x} + 2x \equiv 0 \pmod q$. But this means that $y|2x$, which is not possible, since we are assuming that $\gcd(x, y) = 1$ and $x < y$.

This proves the proposition. ∎

We will also need the following results to prove the theorem for Case 5.

Proposition 3.3 *Suppose $L_1 = L(q : x, py)$ and $L_2 = L(q : u, sv)$ are two isospectral lens orbifolds where $\gcd(x, q) = x$, $\gcd(py, q) = y$, $\gcd(u, q) = u$, and $\gcd(sv, q) = v$. Then either $u = x$ and $v = y$, or $u = y$ and $v = x$.*

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Note 1 If $u = x$ and $v = y$, then $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$; if $u = y$ and $v = x$, then $L_1 = L(q : x, py)$ and $L_2 = L(q : y, sx) = L(q : s^{-1}y, x) = L(q : x, s^{-1}y)$. In either case, this implies that we can write $L_1 = L(q : x, py)$ and $L_2 = L(q : x, s'y)$, where $s' = s$ or $s' = s^{-1}$.

We now prove the above claim.

Proof We denote $q_{/x} = \frac{q}{x}$ and $q_{/y} = \frac{q}{y}$. Then, writing F_i to denote the spectrum generating function of L_i, we have

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_1(z) = \lim_{z\to\gamma^x}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma^x)(1-z^2)}{(1-\gamma^{xl}z)(1-\gamma^{-xl}z)(1-\gamma^{pyl}z)(1-\gamma^{-pyl}z)}.$$

Recall that the only non-zero terms in this limit will be the ones where $xl + x \equiv 0 \pmod q$ or $-xl + x \equiv 0 \pmod q$, which gives $l = tq_{/x} - 1$ or $l = tq_{/x} + 1$ for $t \in \{1, \dots, x\}$. Also note that for such a t, we have

$$\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})} = \frac{1}{(1-\gamma^{py[(x-t)q_{/x}+1]+x})(1-\gamma^{-py[(x-t)q_{/x}+1]+x})}.$$

These two facts, along with Proposition 3.2, give

$$0 \neq \frac{-2\gamma^x}{q}\sum_{t=1}^{x}\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})} = \lim_{z\to\gamma^x}(z-\gamma^x)F_1(z).$$

Since

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_1(z) = \lim_{z\to\gamma^x}(z-\gamma^x)F_2(z),$$

we get

$$0 \neq \frac{-2\gamma^x}{q}\sum_{t=1}^{x}\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})} = \lim_{z\to\gamma^x}(z-\gamma^x)F_2(z)$$
$$= \lim_{z\to\gamma^x}\frac{-\gamma^x}{q}\sum_{l=1}^{q}\frac{(1-\gamma^{-x}z)(1-z^2)}{(1-\gamma^{ul}z)(1-\gamma^{-ul}z)(1-\gamma^{svl}z)(1-\gamma^{-svl}z)}.$$

So there must be an l such that one of the following holds:

$$ul + x \equiv 0 \pmod q,$$
$$-ul + x \equiv 0 \pmod q,$$
$$svl + x \equiv 0 \pmod q,$$
$$-svl + x \equiv 0 \pmod q.$$

Recall that $u|q$. Then $ul + x \equiv 0 \pmod q$ or $-ul + x \equiv 0 \pmod q$ imply that $u|x$. Similarly, since $v|q$, we can show that if $svl + x \equiv 0 \pmod q$ or $-svl + x \equiv 0 \pmod q$, then $v|x$. So either $u|x$ or $v|x$.

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Now by multiplying the elements of L_1 by an appropriate number we can rewrite $L_1 = L(q : y, p'x)$. Then applying the same argument as above where we swap the roles of x and y, we get either $u|y$ or $v|y$.

Suppose $u|x$. Then since $\gcd(x, y) = 1$, we cannot have $u|y$. Similarly, if $v|x$, then we cannot have $v|y$. Therefore, either $u|x$ and $v|y$, or $v|x$ and $u|y$, since if u or v divide both, then it contradicts $\gcd(q, x, py) = 1$.

We can swap the roles of L_1 and L_2 and repeat the above arguments again to get either $x|u$ and $y|v$, or $y|u$ and $x|v$.

If $u|x$ and $v|y$, and at the same time $x|v$ and $y|u$, then $x|y$, which contradicts the fact that $\gcd(q, x, y) = 1$. So the only possibilities are the following:

(i) $u|x$,$v|y$, $x|u$, and $y|v$. This means $x = u$ and $y = v$.
(ii) $v|x$, $u|y$, $x|v$, and $y|u$. This means $x = v$ and $y = u$.

This completes the proof of the proposition. ∎

Corollary 3.4 *Without loss of generality, we can write the two lens spaces as* $L_1 = L(q : x, py)$ *and* $L_2 = L(q : x, sy)$ *with* $\gcd(q, py) = y = \gcd(q, sy)$ *and* $y > x$.

Note 2 If one of x or y is even (in case when q is even), we will also assume that x is always odd. This will not violate our assumption that $x < y$, because if we have a lens space $L(q : x', p'y')$ with even x' and odd y' and $x' > y'$, then we can replace x' and y' with $y = q - x'$ and $x = q - y'$, respectively; we can then multiply by an appropriate number to re-write the lens space as $L(q : x, py)$.

Proposition 3.5 *Suppose* $L_1 = L(q : x, py)$ *is a lens space as in Corollary 3.4. Then*

$$\sum_{t=1}^{x} \cot \frac{\pi}{q}\big[\, tpyq_{/x} - (py - x)\big] = \sum_{t=1}^{x} \cot \frac{\pi}{q}\big[\, tq_{/x} - (py - x)\big], \tag{3.5}$$

$$\sum_{t=1}^{x} \cot \frac{\pi}{q}\big[\, tpyq_{/x} - (py + x)\big] = \sum_{t=1}^{x} \cot \frac{\pi}{q}\big[\, tq_{/x} - (py + x)\big]. \tag{3.6}$$

Proof We will only prove the proposition for (3.5), as the proof for (3.6) is similar.

We denote $z = py - x$. Since x and py are coprime, we have $\gcd(x, z) = \gcd(py, z) = 1$.

We claim that $t_1 z q_{/x} \equiv q_{/x} \pmod{q}$ for some $t_1 \in \{1, 2, \ldots, x - 1\}$.

Since $\gcd(z, x) = 1$, there exist numbers α and β such that $\alpha z + \beta x = 1$, which means $\alpha z \equiv 1 \pmod{x}$. Without loss of generality, we can also assume that $\alpha < x$, for if $\alpha > x$, then we can write $\alpha = \gamma x + \alpha'$ (with $\alpha' < x$); this would give us $(\gamma x + \alpha')z \equiv 1 \pmod{x}$, which gives $\alpha' z \equiv 1 \pmod{x}$ with $\alpha' < x$. Multiplying both sides of the congruence by $q_{/x}$ proves our claim; *i.e.*, we have a $t_1 (\equiv \alpha \pmod{x}) \in \{1, 2, \ldots, x-1\}$ such that $t_1 z q_{/x} \equiv q_{/x} \pmod{q}$.

Now, multiplying the congruence $\alpha z \equiv 1 \pmod{x}$ by $j \in \{2, 3, \ldots, x - 1\}$, we get $j\alpha z \equiv j \pmod{x}$. As before, we can find $t_j (\equiv j\alpha \pmod{x}) \in \{1, 2, \ldots, x - 1\}$ such that $t_j z q_{/x} \equiv j q_{/x} \pmod{q}$.

Now suppose that for $i > j$, we get $t_i = t_j$. This implies that $i \equiv j \pmod{x}$, or $i - j \equiv 0 \pmod{x}$; but since i and j are smaller than x, this is not possible. So we now

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have a 1–1 correspondance between the sets $\{1, 2, \ldots, x-1\}$ and $\{t_1, t_2, \ldots, t_{x-1}\}$, *i.e.* the set $\{t_1, t_2, \ldots, t_{x-1}\}$ is just a permutation of the set $\{1, 2, \ldots, x-1\}$.

Now, for each $t_j \in \{1, 2, \ldots, x-1\}$, we get

$$\begin{aligned}
&t_j z q_{/x} \equiv j q_{/x} \pmod{q},\\
&\Longrightarrow t_j(py - x) q_{/x} \equiv j q_{/x} \pmod{q},\\
&\Longrightarrow t_j p y q_{/x} \equiv j q_{/x} \pmod{q},\\
&\Longrightarrow t_j p y q_{/x} - (py - x) \equiv j q_{/x} - (py - x) \pmod{q}.
\end{aligned}$$

This proves (3.5). ∎

To prove Case 5, we will use the fact that between 0 and π, the cotangent function is strictly decreasing. Using this property along with the fact that the cotangent function is periodic with period π, we will line up the minimum values (and hence all the values due to periodicity) of $tq_{/x} + \alpha y + x$, $tq_{/x} + \alpha y - x$, $tq_{/x} + \beta y - x$, and $tq_{/x} + \beta y + x$, respectively.

Proposition 3.6 *Suppose $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$ are isospectral lens spaces as in Corollary 3.4, where $y \neq q_{/x}$, with spectrum generating functions $F_1(z)$ and $F_2(z)$ respectively. Suppose $\alpha y \equiv (q-p)y \pmod{q}$ and $\beta y \equiv (q-s)y \pmod{q}$. Consider the following four sets of positive integers $\pmod{q}$:*

$$\begin{aligned}
A &= \{A_t : A_t \equiv [tq_{/x} + \alpha y + x] \pmod{q},\ t = 0, 1, \ldots, x-1\},\\
B &= \{B_t : B_t \equiv [tq_{/x} + \alpha y - x] \pmod{q},\ t = 0, 1, \ldots, x-1\},\\
C &= \{C_t : C_t \equiv [tq_{/x} + \beta y - x] \pmod{q},\ t = 0, 1, \ldots, x-1\},\\
D &= \{D_t : D_t \equiv [tq_{/x} + \beta y + x] \pmod{q},\ t = 0, 1, \ldots, x-1\}.
\end{aligned}$$

Then the minimum values of A and B occur for the same value $t' \in \{0, 1, \ldots, x-1\}$, and the minimum values of C and D occur for the same value $t'' \in \{0, 1, \ldots, x-1\}$.

Proof We note that the only time the minimum values for A and B (resp. C and D) will occur at different values of t is when for some t, $tq_{/x} + \alpha y + x > 0$ and $tq_{/x} + \alpha y - x < 0$. But this would mean that $y(tq_{/xy} + \alpha) < x$, which cannot be true, because we are assuming $y > x$. Therefore, for every t, both $tq_{/x} + \alpha y + x$ and $tq_{/x} + \alpha y - x$ are positive (with the only exception happening when $y = q_{/x}$, which we will look at a little later). This implies that the minimum values of $tq_{/x} + \alpha y + x$ and $tq_{/x} + \alpha y - x$ occur for the same value of $t = t'$, and in such a case, the difference between the minimum values would be $2x$. The same will be the case for the minimum values of $tq_{/x} + \beta y + x$ and $tq_{/x} + \beta y - x$, which occur for some value of $t = t''$. ∎

Corollary 3.7 *Suppose $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$ are isospectral lens spaces as above. Also suppose $\min\{A\} = A_{t'}$, $\min\{B\} = B_{t'}$, $\min\{C\} = C_{t''}$, and*

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$\min\{D\} = D_{t''}$. *Then for all* $t \in \{0, 1, \ldots, \frac{(x-3)}{2}\}$,

$$\begin{aligned}
&\Big[\cot\frac{\pi}{q}(A_{t'} + tq_{/x}) - \cot\frac{\pi}{q}(B_{t'} + tq_{/x})\Big] \\
&\quad - \Big[\cot\frac{\pi}{q}(C_{t''} + tq_{/x}) - \cot\frac{\pi}{q}(D_{t''} + tq_{/x})\Big] \\
&\quad + \Big[\cot\frac{\pi}{q}\big(A_{t'} + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(B_{t'} + (x-t-1)q_{/x}\big)\Big] \\
&\quad - \Big[\cot\frac{\pi}{q}\big(C_{t''} + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(D_{t''} + (x-t-1)q_{/x}\big)\Big] = 0,
\end{aligned}$$

and for $t = \frac{(x-1)}{2}$,

$$\begin{aligned}
&\Big[\cot\frac{\pi}{q}\Big(A_{t'} + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(B_{t'} + \frac{(x-1)}{2}q_{/x}\Big)\Big] \\
&\quad - \Big[\cot\frac{\pi}{q}\Big(C_{t''} + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(D_{t''} + \frac{(x-1)}{2}q_{/x}\Big)\Big] = 0.
\end{aligned}$$

Proof Using a similar argument to that in Proposition 3.2 and the fact that $F_1(z) = F_2(z)$, we will get

$$\begin{aligned}
&\sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}\big[tpyq_{/x} - (py - x)\big] - \cot\frac{\pi}{q}\big[tpyq_{/x} - (py + x)\big]\Big] = \\
&\qquad\qquad \sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}\big[tsyq_{/x} - (sy - x)\big] - \cot\frac{\pi}{q}\big[tsyq_{/x} - (sy + x)\big]\Big].
\end{aligned}$$

Using (3.5) and (3.6), the above equation can be written as

$$\begin{aligned}
&\sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}\big[tq_{/x} - (py - x)\big] - \cot\frac{\pi}{q}\big[tq_{/x} - (py + x)\big]\Big] = \\
&\qquad\qquad \sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}\big[tq_{/x} - (sy - x)\big] - \cot\frac{\pi}{q}\big[tq_{/x} - (sy + x)\big]\Big].
\end{aligned}$$

Finally, by writing $\alpha y \equiv (q-p)y \pmod q$ and $\beta y \equiv (q-s)y \pmod q$, we can rewrite the above equality as

$$\begin{aligned}
(3.7)\quad &\sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}[tq_{/x} + \alpha y + x] - \cot\frac{\pi}{q}[tq_{/x} + \alpha y - x]\Big] = \\
&\qquad\qquad \sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}[tq_{/x} + \beta y + x] - \cot\frac{\pi}{q}[tq_{/x} + \beta y - x]\Big].
\end{aligned}$$

Since the minimum values for A and B (resp. C and D) occur at the same value of t, $A_t > B_t$ (resp. $C_t > D_t$) for all values of $t \in \{0, 1, \ldots, x-1\}$.

Also note that $\pi A_{t'}/q$, $\pi B_{t'}/q$, $\pi C_{t''}/q$, and $\pi D_{t''}/q$ lie between 0 and $\pi/x (= \pi q_{/x}/q)$, and each subsequent $\pi A_{t'+t}/q$, $\pi B_{t'+t}/q$, $\pi C_{t''+t}/q$, and $\pi D_{t''+t}/q$ for $t \in \{1, 2, \ldots, x-1\}$ is simply a translation of $\pi A_{t'+t-1}/q$, $\pi B_{t'+t-1}/q$, $\pi C_{t''+t-1}/q$,

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and $\pi D_{t''+t-1}/q$, respectively, to the right by π/x. This means that we can re-write equation (3.7) as

$$\sum_{t=0}^{x-1}\Big[\cot\frac{\pi}{q}(A_{t'}+tq_{/x})-\cot\frac{\pi}{q}(B_{t'}+tq_{/x})\Big] - \Big[\cot\frac{\pi}{q}(C_{t''}+tq_{/x})-\cot\frac{\pi}{q}(D_{t''}+tq_{/x})\Big]=0 \tag{3.8}$$

Further note that the distance between $A_{t'+t}$ and $B_{t'+t}$ = $2x$ = distance between $C_{t''+t}$ and $D_{t''+t}$ for all values of t.

Now we have the following possibilities:

Case I: Suppose $A_{t'} > C_{t''}$, $A_{t'} > B_{t'}$, and $C_{t''} > D_{t''}$. We have two possibilities in this case: $A_{t'} > C_{t''} > B_{t'} > D_{t''}$, or $A_{t'} > B_{t'} > C_{t''} > D_{t''}$.

We will prove the result for the case when $A_{t'} > B_{t'} > C_{t''} > D_{t''}$, and will note that the case when $A_{t'} > C_{t''} > B_{t'} > D_{t''}$ can be proved similarly with a slight modification.

For $t < (x-1)/2$, we have

$$\cot\frac{\pi}{q}(A_{t'}+tq_{/x})-\cot\frac{\pi}{q}(B_{t'}+tq_{/x})<0,$$
$$\cot\frac{\pi}{q}(C_{t''}+tq_{/x})-\cot\frac{\pi}{q}(D_{t''}+tq_{/x})<0,$$

And since the distance between $A_{t'}+tq_{/x}$ and $B_{t'}+tq_{/x}$ is the same as the distance between $C_{t''}+tq_{/x}$ and $D_{t''}+tq_{/x}$, and the slope of the cotangent function is negative and continuously increasing between 0 and $\frac{\pi}{2}$, we also have

$$\Big|\cot\frac{\pi}{q}(C_{t''}+tq_{/x})-\cot\frac{\pi}{q}(D_{t''}+tq_{/x})\Big| > \Big|\cot\frac{\pi}{q}(A_{t'}+tq_{/x})-\cot\frac{\pi}{q}(B_{t'}+tq_{/x})\Big|.$$

These three conditions imply

$$\Big[\cot\frac{\pi}{q}(A_{t'}+tq_{/x})-\cot\frac{\pi}{q}(B_{t'}+tq_{/x})\Big] - \Big[\cot\frac{\pi}{q}(C_{t''}+tq_{/x})-\cot\frac{\pi}{q}(D_{t''}+tq_{/x})\Big]>0. \tag{3.9}$$

Similarly, for $t < (x-1)/2$, we have

$$\cot\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big)<0,$$
$$\cot\frac{\pi}{q}\big(C_{t''}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(D_{t''}+(x-t-1)q_{/x}\big)<0, \text{ and}$$
$$\Big|\cot\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big)\Big| > \Big|\cot\frac{\pi}{q}\big(C_{t''}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(D_{t''}+(x-t-1)q_{/x}\big)\Big|.$$

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These three conditions imply that

$$\begin{aligned}(3.10)\quad &\Big[\cot\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big)\Big]\\ &\quad-\Big[\cot\frac{\pi}{q}\big(C_{t''}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(D_{t''}+(x-t-1)q_{/x}\big)\Big]<0.\end{aligned}$$

Adding the two left sides of (3.9) and (3.10), we get the following expression for $t < (x-1)/2$:

$$\begin{aligned}(3.11)\quad &\Big[\cot\frac{\pi}{q}(A_{t'}+tq_{/x})-\cot\frac{\pi}{q}(B_{t'}+tq_{/x})\Big]\\ &\quad-\Big[\cot\frac{\pi}{q}(C_{t''}+tq_{/x})-\cot\frac{\pi}{q}(D_{t''}+tq_{/x})\Big]\\ &\quad+\Big[\cot\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big)\Big]\\ &\quad-\Big[\cot\frac{\pi}{q}\big(C_{t''}+(x-t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(D_{t''}+(x-t-1)q_{/x}\big)\Big].\end{aligned}$$

We know that the distance between $A_{t'}+tq_{/x}$ and $B_{t'}+tq_{/x}$ (resp. $C_{t''}+tq_{/x}$ and $D_{t''}+tq_{/x}$) is $2x$. Suppose the distance between $B_{t'}+tq_{/x}$ and $C_{t''}+tq_{/x}$ is r. We can thus write $C_{t''}+tq_{/x}=D_{t''}+tq_{/x}+2x$, $B_{t'}+tq_{/x}=D_{t''}+tq_{/x}+2x+r$, and $A_{t'}+tq_{/x}=D_{t''}+tq_{/x}+4x+r$. Now, letting $d=D_{t''}+tq_{/x}$, we can view the expression (3.11) as a function of d as follows:

$$\begin{aligned}f(d)=&\Big[\cot\frac{\pi}{q}(d+4x+r)-\cot\frac{\pi}{q}(d+2x+r)\Big]-\Big[\cot\frac{\pi}{q}(d+2x)-\cot\frac{\pi}{q}(d)\Big]\\ &+\Big[\cot\frac{\pi}{q}\big(d+4x+r+(x-2t-1)q_{/x}\big)\\ &\qquad-\cot\frac{\pi}{q}\big(d+2x+r+(x-2t-1)q_{/x}\big)\Big]\\ &-\Big[\cot\frac{\pi}{q}\big(d+2x+(x-2t-1)q_{/x}\big)-\cot\frac{\pi}{q}\big(d+(x-2t-1)q_{/x}\big)\Big].\end{aligned}$$

Now, using the property

$$\cot Y\pm\cot X=\frac{\cot X\cot Y\mp 1}{\cot(X\pm Y)},$$

we can rewrite the above function as

$$\begin{aligned}f(d)=&\frac{1}{\cot\frac{\pi}{q}(2x)}\Big[-\cot\frac{\pi}{q}(d+4x+r)\cot\frac{\pi}{q}(d+2x+r)\\ &+\cot\frac{\pi}{q}(d+2x)\cot\frac{\pi}{q}(d)\\ &-\cot\frac{\pi}{q}\big(d+4x+r+(x-2t-1)q_{/x}\big)\\ &\times\cot\frac{\pi}{q}\big(d+2x+r+(x-2t-1)q_{/x}\big)\\ &+\cot\frac{\pi}{q}\big(d+2x+(x-2t-1)q_{/x}\big)\cot\frac{\pi}{q}\big(d+(x-2t-1)q_{/x}\big)\Big].\end{aligned}$$

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We claim that the function $f(d)$ is decreasing. To see this, we look at the first derivative of this function as follows:

$$\begin{aligned}
f'(d) = &\frac{1}{\cot\frac{\pi}{q}(2x)}\Big[\cot\frac{\pi}{q}(d+4x+r)\csc^2\frac{\pi}{q}(d+2x+r)\\
&+\csc^2\frac{\pi}{q}(d+4x+r)\cot\frac{\pi}{q}(d+2x+r)\\
&-\cot\frac{\pi}{q}(d+2x)\csc^2\frac{\pi}{q}(d)-\csc^2\frac{\pi}{q}(d+2x)\cot\frac{\pi}{q}(d)\\
&+\cot\frac{\pi}{q}\big(d+4x+r+(x-2t-1)q_{/x}\big)\csc^2\frac{\pi}{q}(d+2x+r+(x-2t-1)q_{/x})\\
&+\csc^2\frac{\pi}{q}\big(d+4x+r+(x-2t-1)q_{/x}\big)\cot\frac{\pi}{q}\big(d+2x+r+(x-2t-1)q_{/x}\big)\\
&-\cot\frac{\pi}{q}\big(d+2x+(x-2t-1)q_{/x}\big)\csc^2\frac{\pi}{q}\big(d+(x-2t-1)q_{/x}\big)\\
&-\csc^2\frac{\pi}{q}\big(d+2x+(x-2t-1)q_{/x}\big)\cot\frac{\pi}{q}(d+(x-2t-1)q_{/x})\Big].
\end{aligned}$$

It will be easier to see that between 0 and π, $f'(d) < 0$ if we rewrite $f'(d)$ back in terms of $D_{t''}$, $C_{t''}$, $B_{t'}$, and $A_{t'}$:

$$\begin{aligned}
f'(d) = &\frac{1}{\cot\frac{\pi}{q}(2x)}\Big[\cot\frac{\pi}{q}(A_{t'}+tq_{/x})\csc^2\frac{\pi}{q}(B_{t'}+tq_{/x})\\
&+\csc^2\frac{\pi}{q}(A_{t'}+tq_{/x})\cot\frac{\pi}{q}(B_{t'}+tq_{/x})\\
&-\cot\frac{\pi}{q}(C_{t''}+tq_{/x})\csc^2\frac{\pi}{q}(D_{t''}+tq_{/x})\\
&-\csc^2\frac{\pi}{q}(C_{t''}+tq_{/x})\cot\frac{\pi}{q}(D_{t''}+tq_{/x})\\
&+\cot\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big)\csc^2\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big)\\
&+\csc^2\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big)\cot\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big)\\
&-\cot\frac{\pi}{q}\big(C_{t''}+(x-t-1)q_{/x}\big)\csc^2\frac{\pi}{q}\big(D_{t''}+(x-t-1)q_{/x}\big)\\
&-\csc^2\frac{\pi}{q}\big(C_{t''}+(x-t-1)q_{/x}\big)\cot\frac{\pi}{q}\big(D_{t''}+(x-t-1)q_{/x}\big)\Big].
\end{aligned}$$

Recall that we are assuming $A_{t'} > B_{t'} > C_{t''} > D_{t''}$. This implies the following two facts:

(a) $\frac{\pi}{2} > A_{t'}+tq_{/x} > B_{t'}+tq_{/x} > C_{t''}+tq_{/x} > D_{t''}+tq_{/x} > 0$, and
(b) $\pi > A_{t'}+(x-t-1)q_{/x} > B_{t'}+(x-t-1)q_{/x} > C_{t''}+(x-t-1)q_{/x} > D_{t''}+(x-t-1)q_{/x}$.

We know that $\csc^2$ is positive, decreasing between 0 and $\frac{\pi}{2}$, and increasing between $\frac{\pi}{2}$ and π. We also know that cot is positive and decreasing between 0 and $\frac{\pi}{2}$, and cot is negative and decreasing between $\frac{\pi}{2}$ and π. Combining these facts with our assumptions noted above, we can now easily conclude that between 0 and $\frac{\pi}{2}$, $f'(d) < 0$,

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and that $f(d)$ is a decreasing function. This implies that if the distance between 0 and $D_{t''}$ is greater than (resp. less than) the distance between $A_{t'} + (x-1)q_{/x}$ and π, then the expression (3.11) is greater than (resp. less than) 0.

We note that the case where $A_{t'} > C_{t''} > B_{t'} > D_{t''}$ is similar with the only difference being that in this case, $B_{t'} + tq_{/x} = D_{t''} + tq_{/x} + 2x - r$, and $A_{t'} + tq_{/x} = D_{t''} + tq_{/x} + 4x - r$. The remaining arguments will be the same.

Since x is odd, then we can rewrite (3.8) as

$$
\begin{aligned}
(3.12)\qquad \sum_{t=0}^{(x-3)/2} \Big\{ & \Big[\cot\frac{\pi}{q}(A_{t'} + tq_{/x}) - \cot\frac{\pi}{q}(B_{t'} + tq_{/x}) \Big] \\
& - \Big[\cot\frac{\pi}{q}(C_{t''} + tq_{/x}) - \cot\frac{\pi}{q}(D_{t''} + tq_{/x}) \Big] \\
& + \Big[\cot\frac{\pi}{q}(A_{t'} + (x-t-1)q_{/x}) - \cot\frac{\pi}{q}\big(B_{t'} + (x-t-1)q_{/x} \big) \Big] \\
& - \Big[\cot\frac{\pi}{q}\big(C_{t''} + (x-t-1)q_{/x} \big) - \cot\frac{\pi}{q}\big(D_{t''} + (x-t-1)q_{/x} \big) \Big] \Big\} \\
& + \Big[\cot\frac{\pi}{q}\Big(A_{t'} + \frac{(x-1)}{2}q_{/x} \Big) - \cot\frac{\pi}{q}\Big(B_{t'} + \frac{(x-1)}{2}q_{/x} \Big) \Big] \\
& - \Big[\cot\frac{\pi}{q}\Big(C_{t''} + \frac{(x-1)}{2}q_{/x} \Big) - \cot\frac{\pi}{q}\Big(D_{t''} + \frac{(x-1)}{2}q_{/x} \Big) \Big] = 0.
\end{aligned}
$$

If (3.11) is greater than (resp. less than) for some t, then (3.11) is greater than (resp. less than) for all t, since if the distance between 0 and $D_{t''}$ is greater than (resp. less than) the distance between $A_{t'} + (x-1)q_{/x}$ and π, then the distance between 0 and $D_{t''} + tq_{/x}$ will be greater than (resp. less than) the distance between $A_{t'} + (x-t-1)q_{/x}$ and π. Further, it can be easily seen that if (3.11) is greater than (resp. less than) 0 for all t, then

$$
\begin{aligned}
& \Big[\cot\frac{\pi}{q}\Big(A_{t'} + \frac{(x-1)}{2}q_{/x} \Big) - \cot\frac{\pi}{q}\Big(B_{t'} + \frac{(x-1)}{2}q_{/x} \Big) \Big] \\
& \qquad - \Big[\cot\frac{\pi}{q}\Big(C_{t''} + \frac{(x-1)}{2}q_{/x} \Big) - \cot\frac{\pi}{q}\Big(D_{t''} + \frac{(x-1)}{2}q_{/x} \Big) \Big] > 0
\end{aligned}
$$

(resp. < 0). This would mean that (3.12) will not be satisfied. Therefore, we conclude that for all t, (3.11) equals 0. This also means that from (3.12), we have

$$
\begin{aligned}
& \Big[\cot\frac{\pi}{q}\Big(A_{t'} + \frac{(x-1)}{2}q_{/x} \Big) - \cot\frac{\pi}{q}\Big(B_{t'} + \frac{(x-1)}{2}q_{/x} \Big) \Big] \\
& \qquad - \Big[\cot\frac{\pi}{q}\Big(C_{t''} + \frac{(x-1)}{2}q_{/x} \Big) - \cot\frac{\pi}{q}\Big(D_{t''} + \frac{(x-1)}{2}q_{/x} \Big) \Big] = 0.
\end{aligned}
$$

Case II: Suppose $A_{t'} > C_{t''}$, $A_{t'} > B_{t'}$ and $C_{t''} < D_{t''}$. We have two possibilities in this case: $A_{t'} > D_{t''} > B_{t'} > C_{t''}$ or $A_{t'} > B_{t'} > D_{t''} > C_{t''}$.

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Since the cotangent function is strictly decreasing, we have, for $t < (x-1)/2$,

$$\cot\frac{\pi}{q}(A_{t'} + tq_{/x}) - \cot\frac{\pi}{q}(B_{t'} + tq_{/x}) < 0,$$
$$\cot\frac{\pi}{q}(D_{t''} + tq_{/x}) - \cot\frac{\pi}{q}(C_{t''} + tq_{/x}) < 0.$$

This means that

$$\Big[\cot\frac{\pi}{q}(A_{t'} + tq_{/x}) - \cot\frac{\pi}{q}(B_{t'} + tq_{/x})\Big] - \Big[\cot\frac{\pi}{q}(C_{t''} + tq_{/x}) - \cot\frac{\pi}{q}(D_{t''} + tq_{/x})\Big] < 0.$$

Similarly, for $t < (x-1)/2$, we have

$$\cot\frac{\pi}{q}\big(A_{t'} + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(B_{t'} + (x-t-1)q_{/x}\big) < 0,$$
$$\cot\frac{\pi}{q}\big(D_{t''} + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(C_{t''} + (x-t-1)q_{/x}\big) < 0.$$

This means that

$$\Big[\cot\frac{\pi}{q}\big(A_{t'} + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(B_{t'} + (x-t-1)q_{/x}\big)\Big] - \Big[\cot\frac{\pi}{q}\big(C_{t''} + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(D_{t''} + (x-t-1)q_{/x}\big)\Big] < 0.$$

Rewriting (3.8)) as (3.12) as in Case I, we can see that the entire expression will be negative, unless (3.11) equals 0 for every value of t. Reasoning as in Case I, we also get

$$\Big[\cot\frac{\pi}{q}\Big(A_{t'} + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(B_{t'} + \frac{(x-1)}{2}q_{/x}\Big)\Big] - \Big[\cot\frac{\pi}{q}\Big(C_{t''} + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(D_{t''} + \frac{(x-1)}{2}q_{/x}\Big)\Big] = 0.$$

Case III: $B_{t'} > A_{t'} > D_{t''} > C_{t''}$, or $B_{t'} > D_{t''} > A_{t'} > C_{t''}$. In this case, we can take the minus sign out of expression (3.11) and swap the roles of $B_{t'}$ and $A_{t'}$ (resp. $D_{t''}$ and $C_{t''}$) to see that the function $g(c)$ (defined very similarly to the function $f(d)$ in Case I) is increasing. This would mean again that (3.12) would not be satisfied unless (3.11) equals 0 for all t, and subsequently

$$\Big[\cot\frac{\pi}{q}\Big(A_{t'} + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(B_{t'} + \frac{(x-1)}{2}q_{/x}\Big)\Big] - \Big[\cot\frac{\pi}{q}\Big(C_{t''} + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(D_{t''} + \frac{(x-1)}{2}q_{/x}\Big)\Big] = 0.$$

Case IV: $B_{t'} > C_{t''} > A_{t'} > D_{t''}$, or $B_{t'} > A_{t'} > C_{t''} > D_{t''}$. In this case we notice that (3.12) will be positive unless (3.11) = 0 for every value of t. Reasoning, as in Case II,

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we also get

$$\Big[\cot\frac{\pi}{q}\Big(A_{t'}+\frac{(x-1)}{2}q_{/x}\Big)-\cot\frac{\pi}{q}\Big(B_{t'}+\frac{(x-1)}{2}q_{/x}\Big)\Big]$$
$$-\Big[\cot\frac{\pi}{q}\Big(C_{t''}+\frac{(x-1)}{2}q_{/x}\Big)-\cot\frac{\pi}{q}\Big(D_{t''}+\frac{(x-1)}{2}q_{/x}\Big)\Big]=0.$$

Note 3 We will get four more possible cases, where the roles of $A_{t'}$ and $C_{t''}$ (resp. $B_{t'}$ and $D_{t''}$) are swapped with each other. With slight modifications, these four cases can be proved in the same manner as the above four cases.

This proves the result. ∎

Corollary 3.8 *One of the following holds for all* $t\in\{0,1,\dots,\frac{(x-1)}{2}\}$*:*

(i) $\cot\frac{\pi}{q}A_{t'+t}=\cot\frac{\pi}{q}C_{t''+t}$ *and* $\cot\frac{\pi}{q}B_{t'+t}=\cot\frac{\pi}{q}D_{t''+t}$,

(ii) $\cot\frac{\pi}{q}A_{t'+t}=-\cot\frac{\pi}{q}D_{t''-(t+1)}$ *and* $\cot\frac{\pi}{q}B_{t'+t}=-\cot\frac{\pi}{q}C_{t''-(t+1)}$.

Proof Recall that between 0 and $\frac{\pi}{x}$ the slope of the cotangent function is strictly and continuously increasing, and since the distance between A'_t and $B'_t=2x=$ distance between C''_t and D''_t, the only way for

$$\Big[\cot\frac{\pi}{q}(A_{t'}+tq_{/x})-\cot\frac{\pi}{q}(B_{t'}+tq_{/x})\Big]-\Big[\cot\frac{\pi}{q}(C_{t''}+tq_{/x})-\cot\frac{\pi}{q}(D_{t''}+tq_{/x})\Big]$$
$$+\Big[\cot\frac{\pi}{q}(A_{t'}+(x-t-1)q_{/x})-\cot\frac{\pi}{q}(B_{t'}+(x-t-1)q_{/x})\Big]$$
$$-\Big[\cot\frac{\pi}{q}(C_{t''}+(x-t-1)q_{/x})-\cot\frac{\pi}{q}(D_{t''}+(x-t-1)q_{/x})\Big]=0$$

is if one of the following conditions (a) or (b) is met:

(a)
$$\cot\frac{\pi}{q}(A_{t'}+tq_{/x})=\cot\frac{\pi}{q}(C_{t''}+tq_{/x}),$$
$$\cot\frac{\pi}{q}(B_{t'}+tq_{/x})=\cot\frac{\pi}{q}(D_{t''}+tq_{/x}).$$

Note that these two statements are equivalent to

(b)
$$\cot\frac{\pi}{q}\Big(A_{t'}+(x-t-1)q_{/x}\Big)=\cot\frac{\pi}{q}\Big(C_{t''}+(x-t-1)q_{/x}\Big),$$
$$\cot\frac{\pi}{q}\Big(B_{t'}+(x-t-1)q_{/x}\Big)=\cot\frac{\pi}{q}\Big(D_{t''}+(x-t-1)q_{/x}\Big).$$
$$\cot\frac{\pi}{q}(A_{t'}+tq_{/x})=-\cot\frac{\pi}{q}\Big(D_{t''}+(x-t-1)q_{/x}\Big),$$
$$\cot\frac{\pi}{q}(B_{t'}+tq_{/x})=-\cot\frac{\pi}{q}\Big(C_{t''}+(x-t-1)q_{/x}\Big).$$

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Note that these two statements are equivalent to

$$\cot\frac{\pi}{q}\big(A_{t'}+(x-t-1)q_{/x}\big) = -\cot\frac{\pi}{q}(D_{t''}+tq_{/x}),$$
$$\cot\frac{\pi}{q}\big(B_{t'}+(x-t-1)q_{/x}\big) = -\cot\frac{\pi}{q}(C_{t''}+tq_{/x}).$$

Also note that for $t=\frac{(x-1)}{2}$, since

$$\Big[\cot\frac{\pi}{q}\Big(A_{t'}+\frac{(x-1)}{2}q_{/x}\Big)-\cot\frac{\pi}{q}\Big(B_{t'}+\frac{(x-1)}{2}q_{/x}\Big)\Big] - \Big[\cot\frac{\pi}{q}\Big(C_{t''}+\frac{(x-1)}{2}q_{/x}\Big)-\cot\frac{\pi}{q}\Big(D_{t''}+\frac{(x-1)}{2}q_{/x}\Big)\Big]=0,$$

we again get

$$\cot\frac{\pi}{q}(A_{t'}+tq_{/x}) = \cot\frac{\pi}{q}(C_{t''}+tq_{/x}), \text{ and}$$
$$\cot\frac{\pi}{q}(B_{t'}+tq_{/x}) = \cot\frac{\pi}{q}(D_{t''}+tq_{/x}).$$

This means one of the following two conditions must be true for all values of $t\in\{0,1,\dots,\frac{(x-1)}{2}\}$:

(I) $\cot\frac{\pi}{q}A_{t'+t}=\cot\frac{\pi}{q}C_{t''+t}$ and $\cot\frac{\pi}{q}B_{t'+t}=\cot\frac{\pi}{q}D_{t''+t}$ when $A'_t>B'_t$ and $C''_t>D''_t$ or when $A'_t<B'_t$ and $C''_t<D''_t$.

(II) $\cot\frac{\pi}{q}A_{t'+t}=-\cot\frac{\pi}{q}D_{t''-(t+1)}$ and $\cot\frac{\pi}{q}B_{t'+t}=-\cot\frac{\pi}{q}C_{t''-(t+1)}$ when $A'_t>B'_t$ and $C''_t<D''_t$ or when $A'_t<B'_t$ and $C''_t>D''_t$. ■

We are now ready to prove that isospectrality in Case 5 implies that the lens spaces are isometric.

Proposition 3.9 *Under Corollary 3.8(i), L_1 and L_2 are isometric.*

Proof Condition (i) implies that $A_{t'+t}\equiv C_{t''+t} \pmod q$ and $B_{t'+t}\equiv D_{t''+t} \pmod q$, *i.e.*, $\exists t_1,t_2\in\{0,1,\dots,\frac{(x-1)}{2}\}$ with

$$\begin{aligned} pyt_1q_{/x}-py+x &\equiv A_{t'} \pmod q,\\ pyt_1q_{/x}-py-x &\equiv B_{t'} \pmod q,\\ syt_2q_{/x}-sy+x &\equiv C_{t''} \pmod q,\\ syt_2q_{/x}-sy-x &\equiv D_{t''} \pmod q \end{aligned}$$

such that

$$\begin{aligned} py(t_1+t)q_{/x}-py+x &\equiv sy(t_2+t)q_{/x}-sy+x \pmod q,\\ &\qquad \forall\, t\in\{0,1,\dots,x-1\},\\ py(t_1+t)q_{/x}-py-x &\equiv sy(t_2+t)q_{/x}-sy-x \pmod q,\\ &\qquad \forall\, t\in\{0,1,\dots,x-1\} \end{aligned}$$

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These congruences imply that for all $t \in \{0, 1, \ldots, x-1\}$

$$py\big[(t_1+t)q_{/x}-1\big] \equiv sy\big[(t_2+t)q_{/x}-1\big] \pmod{q}.$$

Now, if $t = x - t_1$, then by writing $t_3 = x - t_1 + t_2$, the above congruence can be written as

$$py(q-1) \equiv sy(t_3 q_{/x}-1) \pmod{q}.$$

We know that $\gcd(q-1, q) = 1$. We claim that $\gcd(t_3 q_{/x}-1, q) = 1$. To see this, suppose $\gcd(t_3 q_{/x}-1, q) = d > 1$. But this means

$$py(q-1) \equiv sy(t_3 q_{/x}-1) \pmod{d} \equiv 0 \pmod{d}).$$

Now d does not divide $q_{/x}$, since $d|(t_3 q_{/x}-1)$, which means $d|x$, since $d|q$. Now, since $\gcd(x, py) = 1$, this would imply that $(q-1) \equiv 0 \mod d$, which is a contradiction. Therefore, $\gcd(t_3 q_{/x}-1, q) = 1$. Now we see that the corresponding lens spaces are isometric, because

$$L(q; x, py) \sim L(q; -x, -py) \sim L\big(q; -x, (t_3 q_{/x}-1)sy\big) \sim L(q; x, sy). \qquad \blacksquare$$

Proposition 3.10 *Under Corollary 3.8(ii), L_1 and L_2 are isometric.*

Proof Condition (ii) implies that $A_{t'+t} \equiv -D_{t''+t} \pmod{q}$ and $B_{t'+t} \equiv -C_{t''+t} \pmod{q}$, *i.e.*, $\exists t_1, t_2 \in \{0, 1, \ldots, \frac{(x-1)}{2}\}$ with

$$\begin{aligned}
pyt_1 q_{/x} - py + x &\equiv A_{t'} \pmod{q},\\
pyt_1 q_{/x} - py - x &\equiv B_{t'} \pmod{q},\\
syt_2 q_{/x} - sy + x &\equiv C_{t''} \pmod{q},\\
syt_2 q_{/x} - sy - x &\equiv D_{t''} \pmod{q}
\end{aligned}$$

such that

$$\begin{aligned}
py(t_1+t)q_{/x} - py + x &\equiv -sy(t_2 - t - 1)q_{/x} + sy + x \pmod{q},\\
&\qquad \forall\, t \in \{0, 1, \ldots, x-1\},\\
py(t_1+t)q_{/x} - py - x &\equiv -sy(t_2 - t - 1)q_{/x} + sy - x \pmod{q},\\
&\qquad \forall\, t \in \{0, 1, \ldots, x-1\}.
\end{aligned}$$

These congruences imply that for $t \in \{0, 1, \ldots, x-1\}$,

$$py[(t_1+t)q_{/x}-1] \equiv -sy[(t_2 - t - 1)q_{/x}+1] \pmod{q}.$$

As before, if $t = x - t_1$, then, by writing $t_3 = x + 1 - t_1 - t_2$, the above congruence can be written as

$$py(q-1) \equiv -sy(t_3 q_{/x}+1) \pmod{q}.$$

With a similar argument as in Proposition 3.9, we get that $\gcd(t_3 q_{/x}+1, q) = 1$, and, as before, the corresponding lens spaces are isometric because

$$L(q; x, py) \sim L(q; -x, -py) \sim L\big(q; -x, -(t_3 q_{/x}+1)sy\big) \sim L(q; x, sy). \qquad \blacksquare$$

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We note that in the above setting, when $y = q_{/x}$, the minimum positive value for $tq_{/x} + x$ is x, which occurs when $t = 0$, and the minimum positive value for $tq_{/x} - x$ is $q_{/x} - x$, which occurs when $t = 1$. If $q_{/x} > 2x$ (resp. $q_{/x} < 2x$), then the minimum value of $tq_{/x} - x$ (*i.e.*, $q_{/x} - x$) is greater than (resp. less than) the minimum value of $tq_{/x} + x$ (*i.e.*, x). Consequently, $A_t < B_{t+1}$ and $C_t < D_{t+1}$ for all $t \in \{0, 1, \ldots, x-1\}$ (resp. $A_t > B_{t+1}$ and $C_t > D_{t+1}$ for all $t \in \{0, 1, \ldots, x-1\}$). This means that for each t, $B_{t+1} - A_t = q_{/x} - 2x = D_{t+1} - C_t$ (resp. $A_t - B_{t+1} = 2x - q_{/x} = C_t - D_{t+1}$). Now by substituting 0 (resp. 1) in place of t' and t'' in the subscripts of A (resp. B) and C (resp. D), respectively, we can prove the previous results with similar arguments for the case when $y = q_{/x}$.

Proposition 3.11 *Suppose lens spaces $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$ are isospectral where $\gcd(q, py) = y = \gcd(q, sy)$ and $y = q_{/x}$; then for all $t \in \{0, 1, \ldots, \frac{(x-3)}{2}\}$,*

$$\begin{aligned}
&\Big[\cot\frac{\pi}{q}(A_0 + tq_{/x}) - \cot\frac{\pi}{q}(B_1 + tq_{/x})\Big] \\
&\quad - \Big[\cot\frac{\pi}{q}(C_0 + tq_{/x}) - \cot\frac{\pi}{q}(D_1 + tq_{/x})\Big] \\
&\quad + \Big[\cot\frac{\pi}{q}\big(A_0 + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(B_1 + (x-t-1)q_{/x}\big)\Big] \\
&\quad - \Big[\cot\frac{\pi}{q}\big(C_0 + (x-t-1)q_{/x}\big) - \cot\frac{\pi}{q}\big(D_1 + (x-t-1)q_{/x}\big)\Big] = 0,
\end{aligned}$$

and for $t = \frac{(x-1)}{2}$,

$$\begin{aligned}
&\Big[\cot\frac{\pi}{q}\Big(A_0 + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(B_1 + \frac{(x-1)}{2}q_{/x}\Big)\Big] \\
&\qquad - \Big[\cot\frac{\pi}{q}\Big(C_0 + \frac{(x-1)}{2}q_{/x}\Big) - \cot\frac{\pi}{q}\Big(D_1 + \frac{(x-1)}{2}q_{/x}\Big)\Big] = 0.
\end{aligned}$$

Corollary 3.12 *One of the following holds for all $t \in \{0, 1, \ldots, \frac{(x-1)}{2}\}$:*

(i) $\cot\frac{\pi}{q}A_t = \cot\frac{\pi}{q}C_t$ *and* $\cot\frac{\pi}{q}B_{t+1} = \cot\frac{\pi}{q}D_{t+1}$,
(ii) $\cot\frac{\pi}{q}A_t = -\cot\frac{\pi}{q}D_{-t}$ *and* $\cot\frac{\pi}{q}B_{t+1} = -\cot\frac{\pi}{q}C_{-t-1}$.

Proof The proof is similar to the proof of Corollary 3.8. ∎

Proposition 3.13 *Under Conditions* (i) *and* (ii) *of Corollary* 3.12, *lens spaces L_1 and L_2 are isometric.*

Proof The proof is similar to the proofs of Propositions 3.9 and 3.10. ∎

This completes the proof of our theorem for Case 5. ∎

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4 4-Dimensional Orbifold Lens Spaces

It is known that in the manifold case, even-dimensional spherical space forms are only the sphere and the real projective spaces [I2]. It is also known that the sphere $\mathbb{S}^n$ is not isospectral to the real projective space $P^n(\mathbb{R})$ [BGM].

In the orbifold case, there are many even dimensional spherical space forms with fixed points. We will focus on the 4-dimensional orbifold lens spaces. In [L], Lauret classified cyclic subgroups of $SO(2n+1)$ up to conjugation. According to this classification, any cyclic subgroup G of $SO(2n+1)$ is represented by $G = \langle\gamma\rangle$, where $\gamma = \mathrm{diag}(M(\frac{2\pi p_1}{q}), \ldots, M(\frac{2\pi p_n}{q}), 1)$ and $M(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$.

In order to prove our theorem for 4-dimensional orbifold lens spaces, we need a couple of results from [Ba]. We define

$$\widetilde{g}_{W+} = \begin{pmatrix} M(p_1/q) & & 0 \\ & \ddots & \\ & & M(p_n/q) \\ 0 & & & I_W \end{pmatrix}$$

and

$$\widetilde{g}'_{W+} = \begin{pmatrix} M(s_1/q) & & 0 \\ & \ddots & \\ & & M(s_n/q) \\ 0 & & & I_W \end{pmatrix},$$

where I_W is the $W \times W$ identity matrix for some integer W. We can define $\widetilde{G}_{W+} = \langle\widetilde{g}_{W+}\rangle$ and $\widetilde{G}'_{W+} = \langle\widetilde{g}'_{W+}\rangle$. Then $\widetilde{G}_{W+}$ and $\widetilde{G}'_{W+}$ are cyclic groups of order q. We define lens spaces $\widetilde{L}_{W+} = S^{2n+W-1}/\widetilde{G}_{W+}$ and $\widetilde{L}'_{W+} = S^{2n+W-1}/\widetilde{G}'_{W+}$. Further suppose that the corresponding $2n-1$-dimensional orbifold lens spaces are given by $L = L(q : p_1, p_2, \ldots, p_n)$ and $L' = L(q : s_1, s_2, \ldots, s_n)$. Then by [Ba, Lemma 3.2.2], we get the following lemma.

Lemma 4.1 *Let L, L', $\widetilde{L}_{W+}$, and $\widetilde{L}'_{W+}$ be as defined above. Then L is isometric to L' if and only if $\widetilde{L}_{W+}$ is isometric to $\widetilde{L}'_{W+}$.*

And by [Ba, Theorem 3.2.3] we get the following theorem.

Theorem 4.2 *Let $F_q^{W+}(z : p_1, \ldots, p_n, 0)$ be the generating function associated with the spectrum of $\widetilde{L}_{W+}$. Then on the domain $\{z \in \mathbf{C} \mid |z| < 1\}$,*

$$F_q^{W+}(z : p_1, \ldots, p_n, 0) = \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})}.$$

Now suppose $n = 2$. Let

$$\widetilde{g}_1 = \begin{pmatrix} M(p_1/q) & & 0 \\ & M(p_2/q) & \\ 0 & & 1 \end{pmatrix}$$

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and

$$\widetilde{g}_2 = \begin{pmatrix} M(s_1/q) & & 0 \\ & M(s_2/q) & \\ 0 & & 1 \end{pmatrix}.$$

Suppose there are 4-dimensional orbifold lens spaces $O_1 = \mathbb{S}^4/\widetilde{G}_1$ (denoted by $L(q : p_1, p_2, 0)$) and $O_2 = \mathbb{S}^4/\widetilde{G}_2$ (denoted by $L(q : s_1, s_2, 0)$), where $\widetilde{G}_1 = \langle \widetilde{g}_1 \rangle$ and $\widetilde{G}_2 = \langle \widetilde{g}_2 \rangle$. Further suppose the corresponding 3-dimensional orbifold lens spaces are given by $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$.

We now prove the following theorem for 4-dimensional orbifold lens spaces.

Theorem 4.3 *We have O_1, O_2, $\widetilde{G}_1$, and $\widetilde{G}_2$ as above. If O_1 and O_2 are isospectral, then they are isometric.*

Proof From Theorem 4.2 we know that on the domain $\left\{ z \in \mathbf{C} \middle| \, |z| < 1 \right\}$, the spectrum generating functions of O_1 and O_2, respectively, are,

$$F_q(z : p_1, p_2, 0) = \frac{1}{q} \sum_{l=1}^{q} \frac{(1+z)}{\prod_{i=1}^{2}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})},$$

$$F_q(z : s_1, s_2, 0) = \frac{1}{q} \sum_{l=1}^{q} \frac{(1+z)}{\prod_{i=1}^{2}(z - \gamma^{s_i l})(z - \gamma^{-s_i l})}.$$

Notice that $F_q(z : p_1, p_2) = (1-z)F_q(z : p_1, p_2, 0)$ and $F_q(z : s_1, s_2) = (1-z)F_q(z : s_1, s_2, 0)$, where $F_q(z : p_1, p_2)$ and $F_q(z : s_1, s_2)$ are respectively the spectrum generating functions for the 3-dimensional orbifold lens spaces $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. This means that if O_1 and O_2 are isospectral, then L_1 and L_2 are also isospectral.

Now, from Theorem 3.1, we know that L_1 and L_2 are isometric. By Lemma 4.1 we know that L_1 is isometric to L_2 if and only if O_1 is isometric to O_2. This proves the theorem. ∎

5 Lens Spaces and Other Spherical Space Forms

One question still remains: Is an orbifold lens space ever isospectral to an orbifold spherical space form that has non-cyclic fundamental group?

Our next result proves that an orbifold lens space cannot be isospectral to a general spherical space form with non-cyclic fundamental group. We will use some results from [I2] noting that in some cases his assumption that the acting group is fixed-point free is not used in certain proofs, and therefore, the results hold true for orbifolds. The notation is also borrowed from [I2].

Definition 5.1 Let G be finite group and let G_k be the subset of G consisting of all elements of order k in G. Let $\sigma(G)$ denote the set consisting of orders of elements in G. Then we have

$$G = \bigcup_{k \in \sigma(G)} G_k \text{ (disjoint union).}$$

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The following lemma is proved in [I2, Lemma 2.5] for fixed-point free subgroups of $SO(2n)$, but we note that the proof does not require this condition, and we reproduce the proof from [I2].

Lemma 5.2 *Let G be a finite subgroup of $SO(2n)$ $(n \geq 2)$. Then the subset G_k is divided into the disjoint union of subsets $C_k^1, \ldots, C_k^{i_k}$ such that each C_k^t $(t = 1, 2, \ldots, i_k)$ consists of all generic elements of some cyclic subgroup of order k in G.*

Proof For any $g \in G_k$, we denote by A_g the cyclic subgroup of G generated by g. Now, for $g, g' \in G_k$ the cyclic group $A_g \cap A_{g'}$ is of order k if and only if $A_g = A_{g'}$. Now the lemma follows from this observation immediately. ∎

We now state another lemma (see [I2, Lemma 2.6] for proof) that will be used to prove our result.

Lemma 5.3 *Let g be an element in $SO(2n)$ $(n \geq 2)$ and of order q $(q \geq 3)$. Set $\gamma = e^{2\pi\sqrt{-1}/q}$. Assume g has eigenvalues $\gamma, \gamma^{-1}, \gamma^{p_1}, \gamma^{-p_1}, \ldots, \gamma^{p_k}, \gamma^{-p_k}$, with multiplicities $l, l, i_1, i_1, \ldots, i_k, i_k$, respectively, where $p_1, \ldots, p_k$ are integers prime to q with $p_i \not\equiv \pm p_j (mod q)$ (for $1 \leq i < j \leq k$), $p_i \not\equiv \pm 1 (mod q)$ (for $i = 1, \ldots, k$) and $l + i_1 + \cdots + i_k = n$. Then the Laurent expansion of the meromorphic function $\frac{1-z^2}{\det(1_{2n} - gz)}$ at $z = \gamma$ is*

$$\frac{1}{(z-\gamma)^l} \frac{(\sqrt{-1})^{n+l}\gamma^l}{2^{n-l}(1-\gamma^2)^{n-1}} \prod_{j=1}^{k} \left\{ \cot\frac{\pi}{q}(p_j+1) - \cot\frac{\pi}{q}(p_j-1) \right\}^{i_j} + \textit{lower order terms.}$$

The following proposition is proved by Ikeda for a group G that acts freely. However, we note that the proposition is true even if G does not act freely since the proof does not use the property that G acts freely.

Proposition 5.4 *Let G be a finite subgroup of $SO(2n)$ $(n \geq 2)$, and let $k \in \sigma(G)$. We define a positive integer k_0 by*

$$\begin{aligned} k_0 &= 2n-1 \textit{ if } k = 1 \textit{ or } 2, \\ &= \max_{g \in G_k} \{\textit{multiplicities of eigenvalues of } g\} \textit{ if } k \geq 3. \end{aligned}$$

Then the generating function $F_G(z)$ has a pole of order k_0 at any primitive k-th root of 1.

Proof At $z = 1$, we notice that for $g = I_{2n} \in G_1$, we get

$$\lim_{z \to 1} (1-z)^{2n-1} F_G(z) = \frac{2}{|G|},$$

as g has eigenvalue 1 with multiplicity $2n$. So $F_G(z)$ has a pole of order $2n-1$ at $z = 1$.

At $z = -1$ we notice that for $g = -I_{2n} \in G_2$, we get

$$\lim_{z \to -1} (1+z)^{2n-1} F_G(z) = \frac{2}{|G|},$$

as g has eigenvalue -1 with multiplicity $2n$. Also, for any other $g' \in G_2$, the eigenvalue -1 has multiplicity at most $2n$. So $F_G(z)$ has a pole of order $2n-1$ at $z = -1$ as well.

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We now assume $k \geq 3$. Now let $G_k, C_k^1, \ldots, C_k^{i_k}$ be as in Lemma 5.2. Then we have

$$|G|F_G(z) = \sum_{g \in G_k} \frac{1-z^2}{\det(I_{2n} - gz)} + \sum_{g \in G - G_k} \frac{1-z^2}{\det(I_{2n} - gz)} \tag{5.1}$$
$$= \sum_{j=1}^{i_k} \sum_{g \in C_k^j} \frac{1-z^2}{\det(I_{2n} - gz)} + \sum_{g \in G - G_k} \frac{1-z^2}{\det(I_{2n} - gz)}.$$

Set $\gamma = e^{2\pi\sqrt{-1}/k}$. For any primitive k-th root γ^t of 1, where t is an integer prime to k, let

$$\frac{a_{k_0}(t)}{(z-\gamma^t)^{k_0}} + \frac{a_{k_0-1}(t)}{(z-\gamma^t)^{k_0-1}} + \cdots + \frac{a_1(t)}{(z-\gamma^t)}$$

be the principal part of the Laurent expansion of $F_G(z)$ at $z = \gamma^t$. Then each coefficient $a_i(t)$ is an element in the k-th cyclotomic field $\mathbb{Q}(\gamma)$ over the rational number field $\mathbb{Q}$. The automorphisms σ_t of $\mathbb{Q}(\gamma)$ defined by $\gamma \to \gamma^t$ transforms $a_i(1)$ to $a_i(t)$ by equation (5.1). Hence, it is sufficient to show that the generating function $F_G(z)$ has a pole of order k_0 at $z - \gamma$, that is, to show that $a_{k_0}(1) \neq 0$.

Recall that if $0 < b < a < \pi$, then $\cot a - \cot b < 0$. Now the proposition follows immediately from Lemma 5.3 and equation (5.1). ∎

From Proposition 5.4, we get the following corollary.

Corollary 5.5 *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two isospectral orbifold spherical space forms. Then $\sigma(G) = \sigma(G')$.*

We now prove our result.

Theorem 5.6 *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two (orbifold) spherical space forms. Suppose G is cyclic and G' is not cyclic. Then $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ cannot be isospectral.*

Proof By Corollary 2.14, we already know that if $|G| \neq |G'|$, then $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ cannot be isospectral. So let us assume that $|G| = |G'| = q$.

Suppose $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ are isospectral. If G is cyclic, then it has an element of order q. Now, by Corollary 5.5, G' must also have an element of order q, but since $|G'| = q$, that implies that G' is cyclic, which is not true by assumption, and we arrive at a contradiction. This proves the theorem. ∎

The above results will complete the classification of the inverse spectral problem on orbifold lens spaces in all dimensions, and also will imply that orbifold lens spaces cannot be isospectral to any other spherical space forms.

6 Heat Kernel For Orbifold Lens Spaces

In the mathematical study of heat conduction and diffusion, a heat kernel is the fundamental solution to the heat equation on a specified domain with appropriate boundary conditions. It is also one of the main tools in the study of the spectrum of the Laplace operator, and is thus of some auxiliary importance throughout mathematical physics. The heat kernel represents the evolution of temperature in a region whose boundary

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is held fixed at a particular temperature (typically zero), such that an initial unit of heat energy is placed at a point at time t = 0.

In this section we will show that the coefficients of the asymptotic expansion of the heat trace of the heat kernel are not sufficient to obtain the results in the previous sections. More specifically, if two orbifold lens spaces have the same asymptotic expansion of the heat trace, that does not imply that the two orbifolds are isospectral.

Definition 6.1 Let M be a Riemannian manifold. *A heat kernel*, or alternatively, a fundamental solution to the heat equation, is a function

$$K\colon (0,\infty)\times M\times M \longrightarrow M$$

that satisfies

(i) $K(t,x,y)$ is C^1 in t and C^2 in x and y;
(ii) $\partial K/\partial t + \Delta_2(K) = 0$, where Δ_2 is the Laplacian with respect to the second variable (*i.e.*, the first space variable);
(iii) $\lim_{t\to 0^+}\int_M K(t,x,y)f(y)dy = f(x)$ for any compactly supported function f on M.

The heat kernel exists and is unique for compact Riemannian manifolds. Its importance stems from the fact that the solution to the heat equation

$$\frac{\partial u}{\partial t} + \Delta(u) = 0,$$
$$u\colon [0,\infty)\times M \longrightarrow \mathbb{R},$$

(where Δ is the Laplacian with respect to the second variable) with initial condition $u(0,x) = f(x)$ is given by

$$u(t,x) = \int_M K(t,x,y)f(y)dy.$$

If $\{\lambda_i\}$ is the spectrum of M and $\{\zeta_i\}$ are the associated eigenfunctions (normalized so that they form an orthonormal basis of $L^2(M)$), then we can write

$$K(t,x,y) = \sum_i e^{-\lambda_i t}\zeta_i(x)\zeta_i(y).$$

From this, it is clear that the heat trace,

$$Z(t) = \sum_i e^{-\lambda_i t},$$

is a spectral invariant. The heat trace has an asymptotic expansion as $t \to 0+$,

$$Z(t) = (4\pi t)^{\dim(M)/2}\sum_{j=1}^{\infty} a_j t^j,$$

where the a_j are integrals over M of universal homogeneous polynomials in the curvature and its covariant derivatives ([G], see [Gi2] or [CPR] for details). The first few

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of these are

$$a_0 = \mathrm{vol}(M),$$
$$a_1 = \frac{1}{6}\int_M \tau,$$
$$a_2 = \frac{1}{360}\int_M (5\tau^2 - 2|\rho|^2 - 10|R|^2),$$

where $\tau = \sum_{a,b=1}^{\dim(M)} R_{abab}$ is the scalar curvature, $\rho = \sum_{c=1}^{\dim(M)} R_{acbc}$ is the Ricci tensor, and R is the curvature tensor. The dimension, the volume, and the total scalar curvature are thus completely determined by the spectrum. If M is a surface, then the Gauss–Bonnet Theorem implies that the Euler characteristic of M is also a spectral invariant.

6.1 Heat Trace Results for Orbifolds

In the case of a *good* Riemannian orbifold, Donnelly [D] proved the existence of the heat kernel and also proved the following results.

Theorem 6.2 *Let $f: M \to M$ be an isometry of a manifold M, with fixed point set Ω.*
(i) *There is an asymptotic expansion as $t \downarrow 0$*

$$\sum_{\lambda} Tr(f_\lambda{}^\sharp) e^{t\lambda} \approx \sum_{N\in\Omega} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \int_N b_k(f,a)\, d\,\mathrm{vol}_N(a),$$

where N is a subset of Ω (and a submanifold of M), λ is an eigenvalue of Δ, $f_\lambda{}^\sharp$ is a linear map from λ-eigenspace to itself induced by f, and the functions $b_k(f,a)$ depend only on the germ of f and the Riemannian metric of M near the points $a \in N$.

(ii) *The coefficients $b_k(f,a)$ are of the form $b_k(f,a) = |\det B| b_k'(f,a)$ where $b_k'(f,a)$ is an invariant polynomial in the components of $B = (I - A)^{-1}$ (where A denotes the endomorphism induced by f on the fiber of the normal bundle over $a \in N$) and the curvature tensor R and its covariant derivatives at a.*

In particular,

$$b_0(f,a) = |\det B|,$$
$$b_1(f,a) = |\det B|\Big(\frac{\tau}{6} + \frac{1}{6}\rho_{kk} + \frac{1}{3}R_{iksh}B_{ki}B_{hs} + \frac{1}{3}R_{ikth}B_{kt}B_{hi} - R_{k\alpha h\alpha}B_{ks}B_{hs}\Big).$$

In [DGGW] Donnelly's work is extended to general compact orbifolds, where the heat invariants are expressed in a form that clarifies the asymptotic contributions of each part of the singular set of the orbifold. Borrowing their notation, we will summarise the construction used in [DGGW] in the following remarks before stating their main theorem.

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Remarks and Notation

(i) An Orbifold O was identified with the orbit space $F(O)/O(n)$, where $F(O)$ (a smooth manifold) is the orthonormal frame bundle of O and $O(n)$ is the orthogonal group, acting smoothly on the right and preserving the fibers. It can be shown that the action of $O(n)$ on the frame bundle $F(O)$ gives rise to a (Whitney) stratification of O. The strata are connected components of the isotropy equivalence classes in O. The set of regular points of O intersects each connected component O_0 of O in a single stratum that constitutes an open dense submanifold of O_0. The strata of O are referred to as *O-strata*.

(ii) If $(\widetilde{U}, G_U, \pi_U)$ is an orbifold chart on O, then it can be shown that the action of G_U on $\widetilde{U}$ gives rise to stratifications both of $\widetilde{U}$ and of U. These are referred to as *$\widetilde{U}$-strata* and *U-strata*, respectively.

(iii) Let O be a Riemannian orbifold and let $(\widetilde{U}, G_U, \pi_U)$ be an orbifold chart. Let $\widetilde{N}$ be a $\widetilde{U}$-stratum in $\widetilde{U}$. Then it can be shown that all the points in $\widetilde{N}$ have the same isotropy group in G_U; this group is referred to as the *isotropy group* of $\widetilde{N}$, denoted $\mathrm{Iso}(\widetilde{N})$.

(iv) Given a $\widetilde{U}$-stratum $\widetilde{N}$, denote by $\mathrm{Iso}^{\max}(\widetilde{N})$ the set of all $\gamma \in \mathrm{Iso}(\widetilde{N})$ such that $\widetilde{N}$ is open in the fixed point set $\mathrm{Fix}(\gamma)$ of γ. For $\gamma \in G_U$, it can be shown that each component W of the fixed point set $\mathrm{Fix}(\gamma)$ of γ (equivalently, the fixed point set of the cyclic group generated by γ) is a manifold stratified by a collection of $\widetilde{U}$-strata, and the strata in W of maximal dimension are open and their union has full measure in W. In particular, the union of those $\widetilde{U}$-strata $\widetilde{N}$ for which $\gamma \in \mathrm{Iso}^{\max}(\widetilde{N})$ has full measure in $\mathrm{Fix}(\gamma)$.

(v) Let γ be an isometry of a Riemannian manifold M and let $\Omega(\gamma)$ denote the set of components of the fixed point set of γ. Each element of $\Omega(\gamma)$ is a submanifold of M. For each non-negative integer k, Donnelly [D] defined a real-valued function (cited above), which we temporarily denote $b_k((M,\gamma),\cdot)$, on the fixed point set of γ. For each $W \in \Omega(\gamma)$, the restriction of $b_k((M,\gamma),\cdot)$ to W is smooth. Two key properties of the b_k are:

(a) *Locality*. For $a \in W$, $b_k((M,\gamma),a)$ depends only on the germs at a of the Riemannian metric of M and of the isometry γ. In particular, if U is a γ-invariant neighborhood of a in M, then $b_k((M,\gamma),a) = b_k((U,\gamma),a)$.

(b) *Universality*. If M and M' are Riemannian manifolds admitting the respective isometries γ and γ', and if $\sigma : M \to M'$ is an isometry satisfying $\sigma \circ \gamma = \gamma' \circ \sigma$, then $b_k((M,\gamma),x) = b_k((M',\gamma'),\sigma(x))$ for all $x \in \mathrm{Fix}(\gamma)$.

In view of the locality property, we will usually delete the explicit reference to M and rewrite these functions as $b_k(\gamma,\cdot)$, as they are written in [D].

(vi) Let O be an orbifold and let $(\widetilde{U}, G_U, \pi_U)$ be an orbifold chart. Let $\widetilde{N}$ be a $\widetilde{U}$-stratum and let $\gamma \in \mathrm{Iso}^{\max}(\widetilde{N})$. Then $\widetilde{N}$ is an open subset of a component of $\mathrm{Fix}(\gamma)$ and thus, $b_k(\gamma,\cdot)(= b_k((\widetilde{U},\gamma),\cdot))$ is smooth on $\widetilde{N}$ for each nonnegative integer k. Define a function $b_k(\widetilde{N},\cdot)$ on $\widetilde{N}$ by

$$b_k(\widetilde{N},x) = \sum_{\gamma \in \mathrm{Iso}^{\max}(\widetilde{N})} b_k(\gamma,x).$$

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Definition 6.3 Let O be a Riemannian orbifold and let N be an O-stratum.

(i) For each nonnegative integer k, define a real-valued function $b_k(N, \cdot)$ by setting $b_k(N, p) = b_k(\widetilde{N}, \widetilde{p})$ where $(\widetilde{U}, G_U, \pi_U)$ is any orbifold chart about p, $\widetilde{p} \in {\pi_U}^{-1}(p)$, and $\widetilde{N}$ is the $\widetilde{U}$-stratum through $\widetilde{p}$.

(ii) The Riemannian metric on O induces a Riemannian metric, and thus a volume element, on the manifold N. Set

$$I_N := (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N, x) d\,\mathrm{vol}_N(x),$$

where $d\,\mathrm{vol}_N$ is the Riemannian volume element.

(iii) Set

$$I_0 = (4\pi t)^{-\dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k,$$

where the $a_k(O)$ (which we will usually write simply as a_k) are the familiar heat invariants. In particular, $a_0 = \mathrm{vol}(O)$, $a_1 = \frac{1}{6}\int_O \tau(x) d\,\mathrm{vol}\, O(x)$, and so forth. Observe that if O is finitely covered by a Riemannian manifold M (say, $O = G\backslash M$), then $a_k(O) = \frac{1}{|G|} a_k(M)$.

We now state the main theorem that was proved in [DGGW].

Theorem 6.4 *Let O be a Riemannian orbifold and let $\lambda_1 \le \lambda_2 \le \cdots$ be the spectrum of the associated Laplacian acting on smooth functions on O. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of O is asymptotic as $t \to 0^+$ to*

$$I_0 + \sum_{N \in S(O)} \frac{I_N}{|\mathrm{Iso}(N)|},$$

where $S(O)$ is the set of all O-strata, $|\mathrm{Iso}(N)|$ is the order of the isotropy at each $p \in N$, and $\mathrm{Iso}(p)$ is the conjugacy class of subgroups of $O(n)$. This asymptotic expansion is of the form

$$(4\pi t)^{-\dim(O)/2} \sum_{j=0}^{\infty} c_j t^{j/2}$$

for some constants c_j .

We will be using this theorem later to calculate the first few coefficients of the asymptotic expansion of the heat trace.

6.2 Heat Kernel For 3-Dimensional Lens Spaces

Using the notation from [Iv], we define the normal coordinates for a three-sphere as follows. Consider a three-sphere of radius r,

$$\mathbb{S}^3(r) = \{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : (v_1)^2 + (v_2)^2 + (v_3)^2 + (v_4)^2 = r^2\},$$

and let (w, ψ, θ, ϕ) be the spherical coordinates in $\mathbb{R}^4$ where $w \in (0, \infty)$, $\psi \in (0, \pi]$, $\theta \in (0, \pi]$ and $\phi \in [0, 2\pi)$. These coordinates are connected with the standard

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coordinate system (u_1, u_2, u_3, u_4) in $\mathbb{R}^4$ by the following equations:

$$\begin{aligned} u_1 &= w \sin\psi \sin\theta \cos\phi, \\ u_2 &= w \sin\psi \sin\theta \sin\phi, \\ u_3 &= w \sin\psi \cos\theta, \\ u_4 &= w \cos\psi. \end{aligned} \tag{6.1}$$

The equation of $\mathbb{S}^3(r)$ in these coordinates is $w^2 = r^2$. The functions $x_1 = \psi$, $x_2 = \theta$, and $x_3 = \phi$ provide an internal coordinate system on $\mathbb{S}^3(r)$ (without one point) in which the metric g induced on $\mathbb{S}^3(r)$ from $\mathbb{E}^3$ has components g_{ij} such that

$$(g_{ij}) = \begin{pmatrix} r^2 & & 0 \\ & r^2 \sin^2\psi & \\ 0 & & r^2 \sin^2\psi \sin^2\theta \end{pmatrix}.$$

Then g induces on $\mathbb{S}^3(r)$ a Riemannian connection ∇. Using the formula

$$\Gamma_{ij}^m = \tfrac{1}{2} g^{ml} [\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}],$$

we can calculate the Christoffel symbols, which are as follows:

$$\begin{aligned} \Gamma_{21}^2 &= \Gamma_{12}^2 = \cot\psi, & \Gamma_{31}^3 &= \Gamma_{13}^3 = \cot\psi, \\ \Gamma_{32}^3 &= \Gamma_{23}^3 = \cot\theta, & \Gamma_{22}^1 &= -\sin\psi\cos\psi, \\ \Gamma_{33}^1 &= -\sin\psi\cos\psi\sin^2\theta, & \Gamma_{33}^2 &= -\sin\theta\cos\theta. \end{aligned}$$

All the other symbols are zero.

Now let $\gamma : [0, 2\pi] \to \mathbb{S}^3(r)$ be a path in $\mathbb{S}^3(r)$ such that $x_i \circ \gamma = \pi/2$ for $i = 1, 2$ and $x_3 \circ \gamma = id|_{[0,2\pi]}$. Since $\cos \pi/2 = \cot \pi/2 = 0$ and $\sin \pi/2 = 1$, we have $\Gamma_{jk}^i|_{\gamma([0,2\pi])} = 0$, and consequently, if we take $w = r = 1$, we get $g_{ij} = \delta_i^j$. Therefore, the coordinate system $\{x_1, x_2, x_3\}$ and the frame $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3\}$ are normal for ∇ along the path γ.

From equations (6.1) it is clear that the set $\gamma([0, 2\pi])$ is a circle obtained by intersecting $\mathbb{S}^3(r)$ with the (v_1, v_2)-plane $\{v \in \mathbb{R}^4 : v_i(p) = 0 \text{ for } i \geq 3\}$ in $\mathbb{R}^4$. In fact, we have

$$\gamma\big([0, 2\pi]\big) = \big\{ (v_1, v_2, 0, 0) \in \mathbb{R}^4 : v_1^2 + v_2^2 = r^2 \big\} = \mathbb{S}^1(r) \times (0, 0).$$

It is clear if C is a circle on $\mathbb{S}^3(r)$ obtained by intersecting $\mathbb{S}^3(r)$ by a 2-plane through its origin, then there are coordinates on $\mathbb{S}^3(r)$ normal along C for the Riemannian connection considered above.

We will assume $r = 1$. Then, using the above normal coordinate system and the formulas

$$\begin{aligned} R_{jlm}^i &= \partial_l \Gamma_{mj}^i - \partial_m \Gamma_{lj}^i + \Gamma_{mj}^k \Gamma_{lk}^i - \Gamma_{lj}^k \Gamma_{km}^i, \\ R_{abcd} &= g_{aj} R_{bcd}^j, \end{aligned}$$

we calculate the values of the curvature as follows:

$$\begin{aligned} R_{1212} &= R_{\psi\theta\psi\theta} = \sin^2\psi, \\ R_{1313} &= R_{\psi\phi\psi\phi} = \sin^2\psi \sin^2\theta, \\ R_{2323} &= R_{\theta\phi\theta\phi} = \sin^4\psi \sin^2\theta. \end{aligned}$$

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All other values are zero. The values of the Ricci tensor, calculated by $\rho_{ab} = R^c_{acb}$, are as follows:

$$\begin{aligned}\rho_{11} &= \rho_{\psi\psi} = 2,\\ \rho_{22} &= \rho_{\theta\theta} = 2\sin^2\psi,\\ \rho_{33} &= \rho_{\phi\phi} = 2\sin^2\psi\sin^2\theta.\end{aligned}$$

All other values are zero. We then calculate the scalar curvature as follows:

$$\tau = g^{\psi\psi}\rho_{\psi\psi} + g^{\theta\theta}\rho_{\theta\theta} + g^{\phi\phi}\rho_{\phi\phi} = 6.$$

Since τ is constant, all its covariant derivatives $\tau_{;j}$ are zero. Using $\rho_{ab;m} = \partial_m\rho_{ab} - \rho_{lb}\Gamma^l_{ma} - \rho_{al}\Gamma^l_{mb}$, we also calculate all the covariant derivatives of the Ricci tensor, which turn out to be zero as well.

Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, and $e_4 = (0,0,0,1)$ be the standard basis in $\mathbb{R}^4$. We define the following two subsets:

$$\begin{aligned}N_a &= \left\{(x,y,0,0) : x^2 + y^2 = 1\right\} \subset \mathbb{R}^4,\\ N_b &= \left\{(0,0,z,w) : z^2 + w^2 = 1\right\} \subset \mathbb{R}^4.\end{aligned}$$

The tangent space $T_{e_1}\mathbb{S}^3$, has basis vectors $\{e_2, e_3, e_4\}$ such that $\{e_2\}$ is a basis for $T_{e_1}N_a$ and $\{e_3, e_4\}$ is a basis for $T_{e_1}N_a^\perp$. Similarly, the tangent space $T_{e_4}\mathbb{S}^3$, has basis vectors $\{e_1, e_2, e_3\}$ such that $\{e_3\}$ is a basis for $T_{e_4}N_b$ and $\{e_1, e_2\}$ is a basis for $T_{e_4}N_b^\perp$. We will now calculate the values for $b_0(f,a)$ and $b_1(f,a)$.

Suppose $O = \mathbb{S}^3/G$ is an orbifold lens space where $G = \langle\gamma\rangle$ and

$$\gamma = \begin{pmatrix} M(\frac{\widehat{p_1}}{q}) & 0 \\ 0 & M(\frac{\widehat{p_2}}{q})\end{pmatrix},$$

where

$$M(\theta) = \begin{pmatrix}\cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}$$

is the rotation matrix and $\widehat{p_1} \not\equiv \pm\widehat{p_2} \pmod q$. Suppose $\gcd(\widehat{p_1}, q) = q_1$ and $\gcd(\widehat{p_2}, q) = q_2$, so that $\widehat{p_1} = p_1q_1$, $\widehat{p_2} = p_2q_2$, and $q = \widehat{\alpha}q_1 = \widehat{\beta}q_2$. Suppose $\gcd(\widehat{\alpha}, \widehat{\beta}) = g$ so that $\widehat{\alpha} = \alpha g$, $\widehat{\beta} = \beta g$ and $\gcd(\alpha, \beta) = 1$. This means we can write γ as

$$\gamma = \begin{pmatrix} M(\frac{p_1}{\alpha g}) & 0 \\ \\ 0 & M(\frac{p_2}{\beta g})\end{pmatrix}.$$

Now

$$\gamma^{\widehat{\alpha}} = \begin{pmatrix} I_2 & 0 \\ \\ 0 & M(\frac{p_2\alpha}{\beta})\end{pmatrix}$$

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fixes N_a, and

$$\gamma^{\widehat{\beta}} = \begin{pmatrix} M(\frac{p_1\beta}{\alpha}) & 0 \\ 0 & I_2 \end{pmatrix}$$

fixes N_b, where I_2 is the 2×2 identity matrix.

Note that since the group action is transitive and the fixed point sets are $\mathbb{S}^1$, the functions $b_k(\cdot, \cdot)$ are constant along these fixed circles. Therefore, it suffices to consider just a single point in these fixed point sets to calculate the values of the functions. We will choose the points $e_1 \in N_a$ and $e_4 \in N_b$ to calculate the values of functions.

In the notation of the Theorem 6.4, $\widetilde{N_a} \cong \mathbb{S}^1 \times \{(0,0)\}$ and $\widetilde{N_b} \cong \{(0,0)\} \times \mathbb{S}^1$. Also,

$$\mathrm{Iso}(N_a) = \{1, \gamma^{\widehat{\alpha}}, \gamma^{2\widehat{\alpha}}, \dots \gamma^{(\beta-1)\widehat{\alpha}}\}, \quad |\mathrm{Iso}(N_a)| = \beta,$$
$$\mathrm{Iso}(N_b) = \{1, \gamma^{\widehat{\beta}}, \gamma^{2\widehat{\beta}}, \dots \gamma^{(\alpha-1)\widehat{\beta}}\}, \quad |\mathrm{Iso}(N_b)| = \alpha.$$

We now use Theorem 6.4 to calculate the heat trace asymptotic for O using the formula $I_0 + \frac{I_{N_a}}{\beta} + \frac{I_{N_b}}{\alpha}$, where

$$I_0 = (4\pi t)^{-\dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k = (4\pi t)^{-\dim(O)/2} \sum_{k=0}^{\infty} \frac{1}{|G|} a_k(\mathbb{S}^3) t^k$$
$$= \frac{(4\pi t)^{-3/2}}{q} \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{4k!} t^k = \frac{(4t)^{-3/2}}{4q\pi} \sum_{k=0}^{\infty} \frac{t^k}{k!} = \frac{t^{-3/2}}{32q\pi} e^t,$$

and for $i \in a, b$,

$$I_{N_i} = (4\pi t)^{-\dim(N_i)/2} \sum_{k=0}^{\infty} t^k \int_{N_i} b_k(N_i, x) d\,\mathrm{vol}_{N_i}(x)$$
$$= \frac{(\pi t)^{-1/2}}{2} \sum_{k=0}^{\infty} t^k \int_{\widetilde{N}_i} b_k(\widetilde{N_i}, x) d\,\mathrm{vol}_{\widetilde{N}_i}(x),$$
$$\text{since } \widetilde{N_i} \to N_i \text{ is trivial in this case}$$
$$= \frac{(\pi t)^{-1/2}}{2} \sum_{k=0}^{\infty} t^k 2\pi b_k(\widetilde{N_i}, x) \text{ (for any choice of } x \text{ by homogeneity)}$$
$$= \sqrt{\pi} t^{-1/2} \sum_{k=0}^{\infty} t^k b_k(\widetilde{N_i}, x) \text{ , where } b_k(\widetilde{N_i}, x) = \sum_{\gamma \in \mathrm{Iso}^{\max} \widetilde{N}_i} b_k(\gamma, x).$$

Now for $a = e_1$ and $r \in \{1, 2, \dots (\beta - 1)\}$,

$$B_{\gamma^{r\widehat{\alpha}}}(a) = (I - A_{\gamma^{r\widehat{\alpha}}}(a))^{-1} = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}} \begin{pmatrix} 1 - \cos\frac{2p_2\pi\alpha r}{\beta} & -\sin\frac{2p_2\pi\alpha r}{\beta} \\ \sin\frac{2p_2\pi\alpha r}{\beta} & 1 - \cos\frac{2p_2\pi\alpha r}{\beta} \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & -\cot\frac{p_2\pi\alpha r}{\beta} \\ \cot\frac{p_2\pi\alpha r}{\beta} & 1 \end{pmatrix}.$$

So $|\det B_{\gamma^{r\widehat{\alpha}}}(a)| = \frac{1}{4}(1 + \cot^2 \frac{p_2\pi\alpha r}{\beta}) = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}}$.

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Similarly we can show that for $b = e_4$ and $r \in \{1, 2, \dots (\alpha - 1)\}$,

$$B_{\gamma^{r\widehat{\beta}}}(b) = \frac{1}{2}\begin{pmatrix} 1 & -\cot\frac{p_1\pi\beta r}{\alpha} \\ \cot\frac{p_1\pi\beta r}{\alpha} & 1 \end{pmatrix},$$

and $|\det B_{\gamma^{r\widehat{\beta}}}(b)| = \frac{1}{4}(1 + \cot^2\frac{p_1\pi\beta r}{\alpha}) = \frac{1}{4\sin^2\frac{p_1\pi\beta r}{\alpha}}$.

We will now calculate $b_i(\widetilde{N_j}, \cdot\,)$ for $i = 0, 1$ and $j = a, b$:

$$b_0(\gamma^{r\widehat{\alpha}}, a) = |\det B_{\gamma^{r\widehat{\alpha}}}(a)| = \frac{1}{4}\Big(1 + \cot^2\frac{p_2\pi\alpha r}{\beta}\Big) = \frac{1}{4\sin^2\frac{p_2\pi\alpha r}{\beta}}.$$

So

$$\begin{aligned} b_0(\widetilde{N_a}, a) &= \sum_{f\in \mathrm{Iso}^{\max}\widetilde{N_a}} b_0(f, a) \\ &= \sum_{r=1}^{\beta-1} b_0(\gamma^{r\widehat{\alpha}}, a) = \sum_{r=1}^{\beta-1}\frac{1}{4}\Big(1 + \cot^2\frac{p_2\pi\alpha r}{\beta}\Big) \\ &= \sum_{r=1}^{\beta-1}\frac{1}{4}\Big(1 + \cot^2\frac{\pi r}{\beta}\Big), \text{ since } \gcd(p_2\alpha, \beta) = 1 \\ &= \sum_{r=1}^{\beta-1}\frac{1}{4\sin^2\frac{\pi r}{\beta}} \\ &= \frac{\beta^2 - 1}{12}, \quad \text{by [DGGW, lemma 5.4].} \end{aligned}$$

We can similarly show that

$$b_0(\widetilde{N_b}, b) = \sum_{r=1}^{\alpha-1}\frac{1}{4}\Big(1 + \cot^2\frac{\pi r}{\alpha}\Big) = \frac{\alpha^2 - 1}{12}.$$

We will now calculate $b_1(\widetilde{N_a}, a)$ and $b_1(\widetilde{N_b}, b)$. Note that for both $B_{\gamma^{r\widehat{\alpha}}}(a)$ and $B_{\gamma^{r\widehat{\beta}}}(b)$, $B_{13} = B_{23} = B_{31} = B_{32} = B_{33} = 0$. Using the formula in Theorem 6.2, we get

$$\begin{aligned} b_1(\gamma^{r\widehat{\alpha}}, a) = {} & \frac{|\det(B_{\gamma^{r\widehat{\alpha}}}(a))|}{3} \\ & \times \Big\{ R_{1212}\Big[2 - (B_{12} + B_{21})^2 - (B_{11} + B_{22})^2 - 2({B_{11}}^2 + {B_{22}}^2)\Big] \\ & + R_{1313}\Big[2 - (B_{11} + B_{33})^2 - 2({B_{11}}^2 + {B_{33}}^2) - 3({B_{12}}^2 + {B_{32}}^2)\Big] \\ & + R_{2323}\Big[2 - (B_{22} + B_{33})^2 - 2({B_{22}}^2 + {B_{33}}^2) - 3({B_{21}}^2 + {B_{31}}^2)\Big]\Big\}. \end{aligned}$$

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This gives

$$\begin{aligned} b_1(\gamma^{r\widehat{\alpha}}, a) = {} & \frac{|\det(B_{\gamma^{r\widehat{\alpha}}}(a))|}{3} \\ & \times \Big\{ R_{1212}\Big[2 - \frac{1}{4}(\cot\theta_r - \cot\theta_r)^2 - \Big(\frac{1}{2}+\frac{1}{2}\Big)^2 - 2\Big(\frac{1}{4}+\frac{1}{4}\Big)\Big] \\ & + R_{1313}\Big[2 - \Big(\frac{1}{2}+0\Big)^2 - 2\Big(\frac{1}{4}+0\Big) - 3\Big(\frac{1}{4}\cot^2\theta_r + 0\Big)\Big] \\ & + R_{2323}\Big[2 - \Big(\frac{1}{2}+0\Big)^2 - 2\Big(\frac{1}{4}+0\Big) - 3\Big(\frac{1}{4}\cot^2\theta_r + 0\Big)\Big]\Big\}, \end{aligned}$$

which gives

$$\begin{aligned} b_1(\gamma^{r\widehat{\alpha}}, a) &= \frac{1}{12}(1+\cot^2\theta_r)\Big\{R_{1313}\Big(2 - \frac{3}{4} - \frac{3}{4}\cot^2\theta_r\Big) + R_{2323}\Big(2 - \frac{3}{4} - \frac{3}{4}\cot^2\theta_r)\Big)\Big\} \\ &= \frac{1}{12}(1+\cot^2\theta_r)(R_{1313}+R_{2323})[2 - \frac{3}{4}(1+\cot^2\theta_r)] \\ &= (R_{1313}+R_{2323})\Big[\frac{1}{6}(1+\cot^2\theta_r) - \frac{1}{16}(1+\cot^2\theta_r)^2\Big] \\ &= (R_{1313}+R_{2323})\Big[\frac{1}{6\sin^2\theta_r} - \frac{1}{16\sin^4\theta_r}\Big], \end{aligned}$$

where $\theta_r = \frac{p_2\pi\alpha r}{\beta}$.

So

$$\begin{aligned} b_1(\widetilde{N_a}, a) &= \sum_{r=1}^{\beta-1} b_1(\gamma^{r\widehat{\alpha}}, a) \\ &= \sum_{r=1}^{\beta-1}(R_{1313}+R_{2323})\Big[\frac{1}{6\sin^2\frac{p_2\pi\alpha r}{\beta}} - \frac{1}{16\sin^4\frac{p_2\pi\alpha r}{\beta}}\Big] \\ &= (R_{1313}+R_{2323})\Big[\frac{1}{6}\sum_{r=1}^{\beta-1}\frac{1}{\sin^2\frac{\pi r}{\beta}} - \frac{1}{16}\sum_{r=1}^{\beta-1}\frac{1}{\sin^4\frac{\pi r}{\beta}}\Big], \end{aligned}$$

since $\gcd(p_2\alpha, \beta) = 1$.

Also, $\sum_{r=1}^{\beta-1}\frac{1}{\sin^2\frac{\pi r}{\beta}} = \frac{\beta^2-1}{3}$ and $\sum_{r=1}^{\beta-1}\frac{1}{\sin^4\frac{\pi r}{\beta}} = \frac{\beta^4+10\beta^2-11}{45}$ (see [DGGW] for the simplification of their expression (5.9), which uses this result for evaluation of this geometric sum). So we get

$$\begin{aligned} b_1(\widetilde{N_a}, a) &= (R_{1313}+R_{2323})\Big(\frac{\beta^2-1}{18} - \frac{\beta^4+10\beta^2-11}{720}\Big) \\ &= -(R_{1313}+R_{2323})\frac{(\beta^2-29)(\beta^2-1)}{720}. \end{aligned}$$

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We can similarly show that

$$b_1(\widetilde{N_b}, b) = (R_{1313} + R_{2323})\Big(\frac{\alpha^2-1}{18} - \frac{\alpha^4+10\alpha^2-11}{720}\Big)$$
$$= -(R_{1313} + R_{2323})\frac{(\alpha^2-29)(\alpha^2-1)}{720}.$$

Using Theorem 6.4 we now calculate the first few coefficients of the asymptotic expansion as follows:

$$\begin{aligned} I_0 &+ \frac{I_{N_a}}{|\mathrm{Iso}(N_a)|} + \frac{I_{N_b}}{|\mathrm{Iso}(N_b)|} \\ &= \frac{t^{-3/2}}{32q\pi}e^t + \frac{(\pi t)^{-1/2}}{\beta}\Big[\, t^0\pi b_0(\widetilde{N_a}, a) + t^1\pi b_1(\widetilde{N_a}, a) + \cdots\Big] \\ &\quad + \frac{(\pi t)^{-1/2}}{\alpha}\Big[\, t^0\pi b_0(\widetilde{N_b}, b) + t^1\pi b_1(\widetilde{N_b}, b) + \cdots\Big] \\ &= \frac{t^{-3/2}}{32q\pi}\Big(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots\Big) + \Big(\frac{b_0(\widetilde{N_a}, a)}{\beta} + \frac{b_0(\widetilde{N_b}, b)}{\alpha}\Big)\sqrt{\pi}t^{-1/2} \\ &\quad + \Big(\frac{b_1(\widetilde{N_a}, a)}{\beta} + \frac{b_1(\widetilde{N_b}, b)}{\alpha}\Big)\sqrt{\pi}t^{1/2} + \cdots. \end{aligned}$$

From this, the coefficient of $t^{-3/2}$ is $\frac{1}{32q\pi}$; the coefficient of $t^{-1/2}$ is

$$\frac{1}{32q\pi} + \frac{b_0(\widetilde{N_a}, a)}{\beta}\sqrt{\pi} + \frac{b_0(\widetilde{N_b}, b)}{\alpha}\sqrt{\pi} = \frac{1}{32q\pi} + \frac{\sqrt{\pi}}{12\beta}(\beta^2-1) + \frac{\sqrt{\pi}}{12\alpha}(\alpha^2-1);$$

and the coefficient of $t^{1/2}$ is

$$\frac{1}{64q\pi} - \frac{\sqrt{\pi}(R_{1313} + R_{2323})[\alpha(\beta^2-29)(\beta^2-1) + \beta(\alpha^2-29)(\alpha^2-1)]}{720\alpha\beta}.$$

The above results show that the coefficients are dependent on q, α, β, the curvature tensor, and its covariant derivatives. Since all lens spaces are finitely covered by $\mathbb{S}^3$, the parts of the coefficients that consist of the curvature tensor and its covariant derivatives will be the same for all lens spaces. The only difference will therefore be in the terms containing α and β. We can rewrite

$$b_0(\widetilde{N_a}, a) = \sum_{r=1}^{\beta-1}\frac{1}{4}\Big(1 + \cot^2\frac{p_2\pi\alpha r}{\beta}\Big) = \sum_{r=1}^{\beta-1}\frac{1}{4} + \sum_{r=1}^{\beta-1}\frac{1}{4}\cot^2\frac{p_2\pi\alpha r}{\beta},$$
$$b_0(\widetilde{N_b}, b) = \sum_{r=1}^{\alpha-1}\frac{1}{4}\Big(1 + \cot^2\frac{p_1\pi\beta r}{\alpha}\Big) = \sum_{r=1}^{\alpha-1}\frac{1}{4} + \sum_{r=1}^{\alpha-1}\frac{1}{4}\cot^2\frac{p_1\pi\beta r}{\alpha},$$

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$$b_1(\widetilde{N_a}, a) = \sum_{r=1}^{\beta-1}(R_{1313} + R_{2323})\Big[\frac{1}{6}\Big(1 + \cot^2\frac{p_2\alpha\pi r}{\beta}\Big) - \frac{1}{16}\Big(1 + \cot^2\frac{p_2\alpha\pi r}{\beta}\Big)^2\Big]$$
$$= \sum_{r=1}^{\beta-1}\frac{5(R_{1313} + R_{2323})}{48} + \sum_{r=1}^{\beta-1}\Big(\frac{R_{1313} + R_{2323}}{24}\Big)\cot^2\frac{p_2\alpha\pi r}{\beta}$$
$$- \sum_{r=1}^{\beta-1}\Big(\frac{R_{1313} + R_{2323}}{16}\Big)\cot^4\frac{p_2\alpha\pi r}{\beta},$$
$$b_1(\widetilde{N_b}, b) = \sum_{r=1}^{\alpha-1}(R_{1313} + R_{2323})\Big[\frac{1}{6}\Big(1 + \cot^2\frac{p_1\beta\pi r}{\alpha}\Big) - \frac{1}{16}\Big(1 + \cot^2\frac{p_1\beta\pi r}{\alpha}\Big)^2\Big]$$
$$= \sum_{r=1}^{\alpha-1}\frac{5(R_{1313} + R_{2323})}{48} + \sum_{r=1}^{\alpha-1}\Big(\frac{R_{1313} + R_{2323}}{24}\Big)\cot^2\frac{p_1\beta\pi r}{\alpha}$$
$$- \sum_{r=1}^{\alpha-1}\Big(\frac{R_{1313} + R_{2323}}{16}\Big)\cot^4\frac{p_1\beta\pi r}{\alpha}.$$

Note that each $b_j(\widetilde{N_a}, a)$, $(j = 0, 1)$ is of the form

$$b_j(\widetilde{N_a}, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_j} C_{ij}^a(R)\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta},$$

where A_j is the finite number of monomials in the powers of $\cot\frac{p_2\alpha\pi r}{\beta}$, and for each i, $C_{ij}^a(R)$ are constant functions in terms of the curvature tensor and its covariant derivatives of the covering space, *i.e.*, the sphere. Since $\gcd(p_2\alpha, \beta) = 1$, and we are summing over r as it ranges from 1 to $\beta - 1$, we can write

$$b_j(\widetilde{N_a}, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_j} C_{ij}^a(R)\cot^{\lambda_i}\frac{\pi r}{\beta}.$$

Similarly, since $\gcd(\alpha, p_1\beta) = 1$, we can write

$$b_j(\widetilde{N_b}, b) = \sum_{r=1}^{\alpha-1}\sum_{i=1}^{A_j} C_{ij}^b(R)\cot^{\lambda_i}\frac{\pi r}{\alpha}.$$

More generally, for any k, the functions $b_k(\gamma^{r\widehat{\alpha}}, a)$ and $b_k(\gamma^{r\widehat{\beta}}, a)$ are universal polynomials in the components of the curvature tensor, its covariant derivatives and the elements of $B_{\gamma^{r\widehat{\alpha}}}(a)$ and $B_{\gamma^{r\widehat{\beta}}}(b)$, respectively. Since the elements of $B_{\gamma^{r\widehat{\alpha}}}(a)$ are $B_{11} = B_{22} = 1/2$, $B_{12} = -\frac{1}{2}\cot\frac{p_2\alpha\pi r}{\beta}$ and $B_{21} = \frac{1}{2}\cot\frac{p_2\alpha\pi r}{\beta}$, every $b_k(\gamma^{r\widehat{\alpha}}, a)$ will be of the form $\sum_{i=1}^{A_j} C_{ij}^a(R)\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$. This means that for each k, we will have

$$b_k(\widetilde{N_a}, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_k} C_{ik}^a(R)\cot^{\lambda_i}\frac{\pi r}{\beta},$$

and similarly,

$$b_k(\widetilde{N_b}, b) = \sum_{r=1}^{\alpha-1}\sum_{i=1}^{A_k} C_{ik}^b(R)\cot^{\lambda_i}\frac{\pi r}{\alpha}.$$

This observation gives us the following lemma for three-dimensional orbifold lens spaces.

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Lemma 6.5 *Given two orbifold lens spaces $O_1 = \mathbb{S}^3/G_1$ and $O_2 = \mathbb{S}^3/G_2$, such that $G_1 = \langle \gamma_1 \rangle$ and $G_2 = \langle \gamma_2 \rangle$ where*

$$\gamma_1 = \begin{pmatrix} M(\frac{\widehat{p_1}}{q}) & 0 \\ 0 & M(\frac{\widehat{p_2}}{q}) \end{pmatrix}$$

with $\widehat{p_1} \not\equiv \pm\widehat{p_2} \pmod q$, $\gcd(\widehat{p_1}, q) = q_{11}$, $\gcd(\widehat{p_2}, q) = q_{21}$, $\widehat{p_1} = p_1 q_{11}$, $\widehat{p_2} = p_2 q_{21}$, $q = \widehat{\alpha_1} q_{11} = \widehat{\beta_1} q_{21}$, $\gcd(\widehat{\alpha_1}, \widehat{\beta_1}) = g_1$, $\widehat{\alpha_1} = \alpha_1 g_1$, $\widehat{\beta_1} = \beta_1 g_1$, and

$$\gamma_2 = \begin{pmatrix} M(\frac{\widehat{s_1}}{q}) & 0 \\ 0 & M(\frac{\widehat{s_2}}{q}) \end{pmatrix},$$

with $\widehat{s_1} \not\equiv \pm\widehat{s_2} \pmod q$, $\gcd(\widehat{s_1}, q) = q_{12}$, $\gcd(\widehat{s_2}, q) = q_{22}$, $\widehat{s_1} = s_1 q_{12}$, $\widehat{s_2} = s_2 q_{22}$, $q = \widehat{\alpha_2} q_{12} = \widehat{\beta_2} q_{22}$, $\gcd(\widehat{\alpha_2}, \widehat{\beta_2}) = g_2$, $\widehat{\alpha_2} = \alpha_2 g_2$, $\widehat{\beta_2} = \beta_2 g_2$. Then $O_1 = \mathbb{S}^3/G_1$ and $O_2 = \mathbb{S}^3/G_2$ will have the exact same asymptotic expansion of the heat kernel if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

This lemma gives us a tool to find examples of 3-dimensional orbifold lens spaces that are non-isometric (hence non-isospectral) but have the exact same asymptotic expansion of the heat kernel.

Example 6.6 Suppose $q = 195$, and consider the two lens spaces $O_1 = L(195 : 3, 5)$ and $O_2 = L(195 : 6, 35)$. Since there is no integer l coprime to 195 and no $e_i \in \{1, -1\}$ such that $\{e_1 l 3, e_2 l 5\}$ is a permutation of $\{6, 35\} \pmod q$, O_1 and O_2 are not isometric (and hence non-isospectral). However, in the notation of the lemma above, $\widehat{p_1} = 3$, $\widehat{p_2} = 5$, $\widehat{s_1} = 6$, $\widehat{s_2} = 35$, $\gcd(\widehat{p_1}, q) = 3 = \gcd(\widehat{s_1}, q)$, $\gcd(\widehat{p_2}, q) = 5 = \gcd(\widehat{s_2}, q)$, and $q = 195 = 3 \times 65 = 5 \times 39$. So $\widehat{\alpha_1} = \widehat{\alpha_2} = 65$ and $\widehat{\beta_1} = \widehat{\beta_2} = 39$, with $\gcd(\widehat{\alpha_i}, \widehat{\beta_i}) = 13$ (for $i = 1, 2$) giving $\alpha_1 = \alpha_2 = 5$ and $\beta_1 = \beta_2 = 3$. Therefore, $O_1 = L(195 : 3, 5)$ and $O_2 = L(195 : 6, 35)$ have the exact same asymptotic expansion.

6.3 Heat Kernel For 4-Dimensional Lens Spaces

Similar to the three-dimensional case, we can show the construction of examples in four-dimensional lens spaces where the lens spaces will not be isospectral but will have the exact same asymptotic expansion of the trace of the heat kernel. Again, borrowing the notation from [Iv], we define the normal coordinates for a four-sphere as follows. Consider a four-sphere of radius r,

$$\mathbb{S}^4(r) = \left\{ (v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5 : (v_1)^2 + (v_2)^2 + (v_3)^2 + (v_4)^2 + (v_5)^2 = r^2 \right\},$$

and let $(w, \psi, \theta, \phi, t)$ be the spherical coordinates in $\mathbb{R}^5$ where $w \in (0, \infty)$, $\psi \in (0, \pi]$, $\theta \in (0, \pi]$, $\phi \in (0, \pi]$, and $t \in [0, 2\pi]$. These coordinates are connected with the

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standard coordinate system $(u_1, u_2, u_3, u_4, u_5)$ in $\mathbb{R}^5$ by the following equations:

$$\begin{aligned} u_1 &= w \sin\psi \sin\theta \sin\phi \cos t, \\ u_2 &= w \sin\psi \sin\theta \sin\phi \sin t, \\ u_3 &= w \sin\psi \sin\theta \cos\phi, \\ u_4 &= w \sin\psi \cos\theta, \\ u_5 &= w \cos\psi. \end{aligned}$$

The equation of $\mathbb{S}^4(r)$ in these coordinates is $w^2 = r^2$. The functions $x_1 = \psi$, $x_2 = \theta$, $x_3 = \phi$ and $x_4 = t$ provide an internal coordinate system on $\mathbb{S}^4(r)$ (without one point) in which the metric g induced on $\mathbb{S}^4(r)$ from $\mathbb{E}^4$ has components g_{ij} such that

$$(g_{ij}) = \begin{pmatrix} r^2 & & & 0 \\ & r^2 \sin^2\psi & & \\ & & r^2 \sin^2\psi \sin^2\theta & \\ 0 & & & r^2 \sin^2\psi \sin^2\theta \sin^2\phi \end{pmatrix}.$$

As before, we calculate the values of the curvature tensor as follows:

$$\begin{aligned} R_{1212} &= R_{\psi\theta\psi\theta} = \sin^2\psi, \\ R_{1313} &= R_{\psi\phi\psi\phi} = \sin^2\psi \sin^2\theta, \\ R_{1414} &= R_{\psi t\psi t} = \sin^2\psi \sin^2\theta \sin^2\phi, \\ R_{2323} &= R_{\theta\phi\theta\phi} = \sin^4\psi \sin^2\theta, \\ R_{2424} &= R_{\theta t\theta t} = \sin^4\psi \sin^2\theta \sin^2\phi, \\ R_{3434} &= R_{\phi t\phi t} = \sin^4\psi \sin^4\theta \sin^2\phi. \end{aligned}$$

All other values are zero. The values of the Ricci tensor, calculated by $\rho_{ab} = R^c_{acb}$, are as follows:

$$\begin{aligned} \rho_{11} &= \rho_{\psi\psi} = 3, \\ \rho_{22} &= \rho_{\theta\theta} = 3\sin^2\psi, \\ \rho_{33} &= \rho_{\phi\phi} = 3\sin^2\psi \sin^2\theta, \\ \rho_{44} &= \rho_{tt} = 3\sin^2\psi \sin^2\theta \sin^2\phi. \end{aligned}$$

All other values are zero. We then calculate the scalar curvature as follows:

$$\tau = g^{\psi\psi}\rho_{\psi\psi} + g^{\theta\theta}\rho_{\theta\theta} + g^{\phi\phi}\rho_{\phi\phi} + g^{tt}\rho_{tt} = 12.$$

Now, let $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$, $e_3 = (0,0,1,0,0)$, $e_4 = (0,0,0,1,0)$, $e_5 = (0,0,0,0,1)$ be the standard basis in $\mathbb{R}^5$. We can then define the following two subsets:

$$\begin{aligned} N_a &= \left\{ (x, y, 0, 0, v) : x^2 + y^2 + v^2 = 1 \right\} \subset \mathbb{R}^5, \\ N_b &= \left\{ (0, 0, z, w, v) : z^2 + w^2 + v^2 = 1 \right\} \subset \mathbb{R}^5. \end{aligned}$$

The tangent space $T_{e_1}\mathbb{S}^4$, has basis vectors $\{e_2, e_3, e_4, e_5\}$ such that $\{e_2, e_5\}$ is a basis for $T_{e_1}N_a$ and $\{e_3, e_4\}$ is a basis for $T_{e_1}N_a^{\perp}$. Similarly, the tangent space $T_{e_4}\mathbb{S}^4$,

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has basis vectors $\{e_1, e_2, e_3, e_5\}$ such that $\{e_3, e_5\}$ is a basis for $T_{e_4}N_b$ and $\{e_1, e_2\}$ is a basis for $T_{e_4}N_b^{\perp}$.

Suppose $O = \mathbb{S}^4/G$ is an orbifold lens space where $G = \langle \gamma \rangle$ and

$$\gamma = \begin{pmatrix} M(\frac{\widehat{p_1}}{q}) & & 0 \\ & M(\frac{\widehat{p_2}}{q}) & \\ 0 & & 1 \end{pmatrix},$$

where $\widehat{p_1} \not\equiv \pm\widehat{p_2} \pmod{q}$. Suppose $\gcd(\widehat{p_1}, q) = q_1$ and $\gcd(\widehat{p_2}, q) = q_2$, so that $\widehat{p_1} = p_1 q_1$, $\widehat{p_2} = p_2 q_2$ and $q = \widehat{\alpha} q_1 = \widehat{\beta} q_2$. Suppose $\gcd(\widehat{\alpha}, \widehat{\beta}) = g$ so that $\widehat{\alpha} = \alpha g$, $\widehat{\beta} = \beta g$ and $\gcd(\alpha, \beta) = 1$. This means we can write γ as

$$\gamma = \begin{pmatrix} M(\frac{p_1}{\alpha g}) & & 0 \\ & M(\frac{p_2}{\beta g}) & \\ 0 & & 1 \end{pmatrix}.$$

Now

$$\gamma^{\widehat{\alpha}} = \begin{pmatrix} I_2 & & 0 \\ & M(\frac{p_2 \alpha}{\beta}) & \\ 0 & & 1 \end{pmatrix}$$

fixes N_a, and

$$\gamma^{\widehat{\beta}} = \begin{pmatrix} M(\frac{p_1 \beta}{\alpha}) & 0 \\ 0 & I_3 \end{pmatrix}$$

fixes N_b. Here I_2 and I_3 are the 2×2 and 3×3 identity matrices respectively.

As before, it suffices to consider just a single point in these fixed point sets to calculate the values of the functions. We will choose the points $e_1 \in N_a$ and $e_4 \in N_b$ to calculate the values of functions.

We have, in the notation of Theorem 6.4, $\widetilde{N_a} \cong \mathbb{S}^2 \times \{(0,0)\}$ and $\widetilde{N_b} \cong \{(0,0)\} \times \mathbb{S}^2$. Also, $\mathrm{Iso}_{N_a} = \{1, \gamma^{\widehat{\alpha}}, \gamma^{2\widehat{\alpha}}, \ldots, \gamma^{(\beta-1)\widehat{\alpha}}\}$, $|\mathrm{Iso}_{N_a}| = \beta$, $\mathrm{Iso}_{N_b} = \{1, \gamma^{\widehat{\beta}}, \gamma^{2\widehat{\beta}}, \ldots, \gamma^{(\alpha-1)\widehat{\beta}}\}$, and $|\mathrm{Iso}_{N_b}| = \alpha$.

Now, as in the case of three-dimensional lens spaces, we have for $a = e_1$ and $r \in \{1, 2, \ldots, (\beta - 1)\}$,

$$B_{\gamma^{r\widehat{\alpha}}}(a) = \frac{1}{2} \begin{pmatrix} 1 & -\cot \frac{p_2 \pi \alpha r}{\beta} \\ \cot \frac{p_2 \pi \alpha r}{\beta} & 1 \end{pmatrix}.$$

So $|\det B_{\gamma^{r\widehat{\alpha}}}(a)| = \frac{1}{4}(1 + \cot^2 \frac{p_2 \pi \alpha r}{\beta}) = \frac{1}{4 \sin^2 \frac{p_2 \pi \alpha r}{\beta}}$.

Similarly we can show that for $b = e_4$ and $r \in \{1, 2, \ldots, (\alpha - 1)\}$,

$$B_{\gamma^{r\widehat{\beta}}}(b) = \frac{1}{2} \begin{pmatrix} 1 & -\cot \frac{p_1 \pi \beta r}{\alpha} \\ \cot \frac{p_1 \pi \beta r}{\alpha} & 1 \end{pmatrix},$$

and $|\det B_{\gamma^{r\widehat{\beta}}}(b)| = \frac{1}{4}(1 + \cot^2 \frac{p_1 \pi \beta r}{\alpha}) = \frac{1}{4 \sin^2 \frac{p_1 \pi \beta r}{\alpha}}$. Note again that for both $B_{\gamma^{r\widehat{\alpha}}}(a)$ and $B_{\gamma^{r\widehat{\beta}}}(b)$, $B_{13} = B_{23} = B_{31} = B_{32} = B_{33} = B_{41} = B_{14} = B_{42} = B_{24} = B_{43} = B_{34} = B_{44} = 0$.

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This means that, just as in the case of three-dimensional lens spaces, for each k, we will have,

$$b_k(\widetilde{N_a}, a) = \sum_{r=1}^{\beta-1} \sum_{i=1}^{A_k} C_{ik}^a(R) \cot^{\lambda_i} \frac{\pi r}{\beta},$$
$$b_k(\widetilde{N_b}, b) = \sum_{r=1}^{\alpha-1} \sum_{i=1}^{A_k} C_{ik}^b(R) \cot^{\lambda_i} \frac{\pi r}{\alpha}.$$

Similar to the three-dimensional case, this observation gives us the following lemma.

Lemma 6.7 *Given two orbifold lens spaces $O_1 = \mathbb{S}^4/G_1$ and $O_2 = \mathbb{S}^4/G_2$, such that $G_1 = \langle \gamma_1 \rangle$ and $G_2 = \langle \gamma_2 \rangle$ where*

$$\gamma_1 = \begin{pmatrix} M(\frac{\widehat{p_1}}{q}) & & 0 \\ & M(\frac{\widehat{p_2}}{q}) & \\ 0 & & 1 \end{pmatrix}$$

with $\widehat{p_1} \not\equiv \pm\widehat{p_2} \pmod q$, $\gcd(\widehat{p_1}, q) = q_{11}$, $\gcd(\widehat{p_2}, q) = q_{21}$, $\widehat{p_1} = p_1 q_{11}$, $\widehat{p_2} = p_2 q_{21}$, $q = \widehat{\alpha_1} q_{11} = \widehat{\beta_1} q_{21}$, $\gcd(\widehat{\alpha_1}, \widehat{\beta_1}) = g_1$, $\widehat{\alpha_1} = \alpha_1 g_1$, $\widehat{\beta_1} = \beta_1 g_1$, and

$$\gamma_2 = \begin{pmatrix} M(\frac{\widehat{s_1}}{q}) & & 0 \\ & M(\frac{\widehat{s_2}}{q}) & \\ 0 & & 1 \end{pmatrix},$$

with $\widehat{s_1} \not\equiv \pm\widehat{s_2} \pmod q$, $\gcd(\widehat{s_1}, q) = q_{12}$, $\gcd(\widehat{s_2}, q) = q_{22}$, $\widehat{s_1} = s_1 q_{12}$, $\widehat{s_2} = s_2 q_{22}$, $q = \widehat{\alpha_2} q_{12} = \widehat{\beta_2} q_{22}$, $\gcd(\widehat{\alpha_2}, \widehat{\beta_2}) = g_2$, $\widehat{\alpha_2} = \alpha_2 g_2$, $\widehat{\beta_2} = \beta_2 g_2$.

Then $O_1 = \mathbb{S}^4/G_1$ and $O_2 = \mathbb{S}^4/G_2$ will have the exact same asymptotic expansion of the heat kernel if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

This lemma gives us a tool to find examples of 4-dimensional orbifold lens spaces that are non-isometric (hence non-isospectral) but have the exact same asymptotic expansion of the heat kernel.

Example 6.8 Suppose $q = 195$, and consider the two lens spaces $O_1 = \widetilde{L}_{1+} = L(195 : 3, 5, 0)$ and $O_2 = \widetilde{L}'_{1+} = L(195 : 6, 35, 0)$ (using the notation from Lemma 4.1). Since there is no integer l coprime to 195 and no $e_i \in \{1, -1\}$ such that $\{e_1 l 3, e_2 l 5\}$ is a permutation of $\{6, 35\} \pmod q$, O_1 and O_2 are not isometric (and hence non-isospectral). However, in the notation of the lemma above, $\widehat{p_1} = 3$, $\widehat{p_2} = 5$, $\widehat{s_1} = 6$, $\widehat{s_2} = 35$, $\gcd(\widehat{p_1}, q) = 3 = \gcd(\widehat{s_1}, q)$, $\gcd(\widehat{p_2}, q) = 5 = \gcd(\widehat{s_2}, q)$ and $q = 195 = 3 \times 65 = 5 \times 39$. So $\widehat{\alpha_1} = \widehat{\alpha_2} = 65$ and $\widehat{\beta_1} = \widehat{\beta_2} = 39$, with $\gcd(\widehat{\alpha_i}, \widehat{\beta_i}) = 13$ (for $i = 1, 2$) giving $\alpha_1 = \alpha_2 = 5$, and $\beta_1 = \beta_2 = 3$. Therefore, O_1 and O_2 have the exact same asymptotic expansion.

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