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QUANTIZATION OF THE 4-DIMENSIONAL NILPOTENT ORBIT OF SL(3, $\mathbb{R}$)

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ABSTRACT. We give a new geometric model for the quantization of the 4-dimensional conical (nilpotent) adjoint orbit $O_{\mathbb{R}}$ of SL(3, $\mathbb{R}$). The space of quantization is the space of holomorphic functions on $\mathbb{C}^2 - \{0\}$) which are square integrable with respect to a signed measure defined by a Meijer G-function. We construct the quantization out a non-flat Kaehler structure on $\mathbb{C}^2 - \{0\}$) (the universal cover of $O_{\mathbb{R}}$) with Kaehler potential $\rho = |z|^4$.

1. **Introduction.** It is well-known that the universal cover $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ admits a unique faithful unitary representation, which we will call $H^{[1]}$, whose decomposition under the maximal compact subgroup SU(2) is multiplicity-free. The irreducible representations of SU(2) which occur in $H^{[1]}$ have spins $1/2, 5/2, 9/2\ldots$ etc. Furthermore $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ admits no multiplicity-free unitary representation with spin ladder $3/2, 7/2, 11/2, \ldots$ but does have two continuous families parameterized by $\mathbb{R}$ of multiplicity-free unitary representations with spin ladders $0, 2, 4, \ldots$ and $1, 3, 5, \ldots$ respectively (these then descend to SL(3, $\mathbb{R}$)). See [J], [Si], [R-S], and [T], as well as [B-C-H-W] for proposed physical applications.

The geometric construction of $H^{[1]}$, in the spirit of geometric quantization, was accomplished by Rawnsley and Sternberg [R-S] who gave a Fock space type model and by Torasso [T] who gave a Schroedinger type model. Indeed, in the method of orbits (originated by Kirillov on the geometric side and Dixmier on the algebraic side), $H^{[1]}$ corresponds to the minimal nilpotent (*i.e.*, conical) coadjoint orbit $O_{\mathbb{R}}$ of SL(3, $\mathbb{R}$) equipped with its natural K-K-S symplectic form ω. So $O_{\mathbb{R}}$ is the orbit of 3×3 matrices of rank 1 and square zero and $O_{\mathbb{R}}$ is a 4-dimensional real symplectic manifold. A natural symplectic model of $O_{\mathbb{R}}$ is obtained by reduction of the cotangent bundle phase space $T^*\mathbb{R}^3$ at the zero value of the moment map $T^*\mathbb{R}^3 \longrightarrow \mathbb{R}$, $(p,q)\longmapsto p\cdot q$, for the Hamiltonian $\mathbb{R}^*$-action $(p,q)\longmapsto (p\lambda^{-1},\lambda q)$. Then the Hamiltonian action of SL(3, $\mathbb{R}$) and the (non-Hamiltonian) fiberwise scaling action of $\mathbb{R}^*$ on $T^*\mathbb{R}^3$ survive the reduction to give the conical reduced phase space $O_{\mathbb{R}}$ and moment map (embedding) $\nu\colon O_{\mathbb{R}} \longrightarrow \mathfrak{sl}(3,\mathbb{R})$. As a manifold, $O_{\mathbb{R}}$ is diffeomorphic to the quotient of $\mathbb{R}^4 - \{0\}$ by a free action of $\mathbb{Z}_4$. The component functions of the map ν then give a Lie algebra $\mathfrak{g}$ of observables on $O_{\mathbb{R}}$ isomorphic to $\mathfrak{sl}(3,\mathbb{R})$.

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The geometric quantization problem on $O_{\mathbb{R}}$ is to construct geometric models of the irreducible unitary representations of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ attached to $O_{\mathbb{R}}$. In particular, the quantization process will produce, for each representation, a Hilbert space $\mathcal{H}$ of (possibly twisted or generalized) functions on $O_{\mathbb{R}}$ and will convert the observables in $\mathfrak{g}$ into self-adjoint operators on $\mathcal{H}$ consistent with Dirac's rule that Poisson bracket of observables goes over into commutator of operators. The construction of $\mathcal{H}$ along with its inner product is equally as important as the conversion of observables into operators.

In this paper we present a new approach to the geometric quantization of $O_{\mathbb{R}}$. We construct in Sections 4, 5 explicit Fock space type models $\mathcal{H}^{[m]}$, $m = 0, 1, 2$, of the three irreducible unitary representations attached to $O_{\mathbb{R}}$ with spin ladders $m/2, 2 + m/2, 4 + m/2 \ldots$. To do this we begin in Section 2 by fixing an SO(3)-invariant complex structure $\mathbf{J}$ on $O_{\mathbb{R}}$ which is polarizing and in fact makes $O_{\mathbb{R}}$ into a (positive) Kaehler manifold. Our models are explicit in that $\mathcal{H}^{[m]}$ consists of holomorphic sections of a homogeneous holomorphic line bundle over $O_{\mathbb{R}}$ and the positive-definite inner product is given by integration over $O_{\mathbb{R}}$ with respect to an explicit (signed) measure $\gamma(z,\overline{z})\omega^2$ where ω^2 is the Liouville volume form. Then $\mathcal{H}^{[m]}$ is exactly the space of holomorphic sections on $O_{\mathbb{R}}$ which are square integrable with respect to the positive measure $|\gamma(z,\overline{z})|\omega^2$ obtained by taking the absolute value of the weight function $\gamma(z,\overline{z})$ (see Section 5). The three irreducible Hilbert spaces $\mathcal{H}^{[0]}, \mathcal{H}^{[1]}, \mathcal{H}^{[2]}$ have reproducing kernels $\Psi^{[0]}, \Psi^{[1]}, \Psi^{[2]}$ which we compute in Section 4 and find are given in terms of hypergeometric functions of type ${}_1F_2$. To simplify matters, we pass to the universal 4-fold cover $\widetilde{O_{\mathbb{R}}}$ of $O_{\mathbb{R}}$ (but this is only for technical convenience).

In our models the observables in $\mathfrak{g}$ are converted into explicit (pseudo-differential) operators on the subspaces $H^{[m]} \subset \mathcal{H}^{[m]}$, $m \in \{0, 1, 2\}$, of SU(2)-finite vectors. These operators give the infinitesimal $\mathfrak{sl}(3,\mathbb{R})$-representations which then define the unitary $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-representations on $\mathcal{H}^{[m]}$ by exponentiation. The ladder decomposition under SU(2) implies that the representation on $\mathcal{H}^{[1]}$ (but not on $\mathcal{H}^{[0]}$ or $\mathcal{H}^{[2]}$) is *genuine* in that it does descend to a representation of $\mathrm{SL}(3,\mathbb{R})$. The construction of the operators (and the occurrence of hypergeometric functions in computing the inner product) is an instance of the more general construction we made with Bert Kostant in [B-K1,2,4].

The explicit quantization, the construction and properties of our signed measure, its expression as a linear combination of the three reproducing kernels, and the resulting realizations of the Hilbert spaces are our main results. In particular the weight function $\gamma(z,\overline{z})$ of our measure is given by a Meijer G-function G so that $\gamma(z,\overline{z}) = G(\rho^2/2)$ where ρ is the SO(3)-invariant homogeneous linear Kaehler potential on $O_{\mathbb{R}}$. Our function $\gamma(z,\overline{z})$ plays the role of the weight factor $e^{-|z|/2}$ of Fock space. The appearance of these G-functions in quantization of conical phase space seems quite significant to us. It should also be noted that these G-functions arise in what is called "fractional calculus"—I thank Aravind Asok for telling me about this theory.

In Section 6 we explain the relationship between our work here and the quantization by Rawnsley and Sternberg [R-S]. In particular we recover their quantization and write down the explicit operators giving their $\mathfrak{sl}(3,\mathbb{R})$-representations. It turns out (see Section 6) that there is a nice family of SO(3)-invariant complex polarizations of $O_{\mathbb{R}}$ each of which gives rise to a Kaehler manifold structure on $O_{\mathbb{R}}$ with Kaehler form ω. In each case the universal cover of $\widetilde{O_{\mathbb{R}}}$ is SO(3)-equivariantly biholomorphic to $\mathbb{C}^2 - \{0\}$ and the Kaehler potential ρ is a multiple of $|z|^{4k}$ for some k. Corresponding to each of these polarizations we construct a quantization (our "correspondence" here is at least heuristic) where $\mathfrak{sl}(3,\mathbb{R})$ acts by skew-adjoint operators. In each case, the skew-symmetric matrices, which form the Lie subalgebra $\mathfrak{k} = \mathfrak{so}(3)$, quantize into the corresponding Hamiltonian vector fields; this reflects the SO(3)-equivariance of the polarization. The heart of the matter then is the quantization of the symmetric matrices—these form a 5-dimensional subspace $\mathfrak{p}$ in $\mathfrak{sl}(3,\mathbb{R})$. The Rawnsley-Sternberg case is then the case where $\rho = |z|^2$ so that $\widetilde{O_{\mathbb{R}}}$ is flat and the Hilbert space of quantization is the classical Fock space constructed out of the measure $e^{-|z|^2|}|dz\,d\overline{z}|$. Their quantization operators for the symmetric matrices are given by an unexpected kernel. Our case is the one where $\rho = 2|z|^4$—the factor 2 is inessential but the exponent 4 is important as it corresponds to ρ being linear on $O_{\mathbb{R}}$ (*cf.* [B2]).

Our quantization generalizes a different aspect of the oscillator representation of the metaplectic group on classical Fock space. In our approach the symmetric matrices correspond on the classical level to real parts $f + \overline{f} = 2\mathrm{Re}\, f$ of holomorphic functions and on the quantum level to operators $i(f + T_{\overline{f}})$ where $T_{\overline{f}}$ is the adjoint of multiplication by the holomorphic function f.

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In [B2], [B3] and subsequent papers we will show how the picture of quantization developed in this paper applies to the quantization of arbitrary conical (*i.e.*, nilpotent) orbits of any real semisimple Lie group.

It would be extremely interesting to construct intertwining operators between our $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-representation on $H^{[1]}$ and the other known models. The intertwining operators over to the Rawnsley-Sternberg model should follow from our work in Section 6, but the construction of intertwinors with Torasso's model seems a challenging problem. Torasso has raised this question as well.

2. Kaehler Polarization and the Algebra of Classical Observables.

We consider $\mathbb{C}^2$ with homogeneous linear complex coordinate functions z_0 and z_1 and standard Hermitian inner product with norm $|z|^2 = |z_0|^2 + |z_1|^2$. Let $\mathbf{J}$ be the standard complex structure on $\mathbb{C}^2$ and set

$$\rho = 2|z|^4 = 2(|z_0|^2 + |z_1|^2)^2.$$

Then $\omega = i\partial\overline{\partial}\rho$ is a smooth closed differential 2-form on $\mathbb{C}^2$. Explicitly we have $\omega = \sum_{jk} i(\partial_j\overline{\partial}_k\rho)\,dz_j\,d\overline{z_k}$ with

$$\begin{pmatrix} \partial_0\overline{\partial}_0\rho & \partial_0\overline{\partial}_1\rho \\ \partial_1\overline{\partial}_0\rho & \partial_1\overline{\partial}_1\rho \end{pmatrix} = 4\begin{pmatrix} 2|z_0|^2 + |z_1|^2 & z_1\overline{z}_0 \\ z_0\overline{z}_1 & |z_0|^2 + 2|z_1|^2 \end{pmatrix}. \tag{2.1}$$

Then $\det(\partial_j\overline{\partial}_k\rho) = 16\rho$ and

$$\omega \wedge \omega = -16\rho\, dz_0\, d\overline{z_0}\, dz_1\, d\overline{z_1}. \tag{2.2}$$

This implies that ω is non-degenerate on

$$Z = \mathbb{C}^2 - \{0\}$$

while ω is singular at the origin. Furthermore the Hermitian matrix in (2.1) is positive definite. Thus we get a Kaehler manifold structure $(Z, \mathbf{J}, \omega)$ with $\rho_Z = \rho$ being the Kaehler potential.

A real or complex *observable* on Z is a smooth $\mathbb{R}$-valued or $\mathbb{C}$-valued function on Z. The symplectic form ω induces a Poisson bracket on the algebra $C^\infty(Z)$ of real observables which then extends in a complex bilinear way to the algebra $C^\infty(Z,\mathbb{C})$ of complex observables. In this way $C^\infty(Z)$ becomes a real form of the complex Lie algebra $C^\infty(Z,\mathbb{C})$. There is a natural operation $\phi \longmapsto \overline{\phi}$ of complex conjugation on $C^\infty(Z,\mathbb{C})$. If $\mathfrak{s}$ is any complex Lie subalgebra of $C^\infty(Z,\mathbb{C})$ such that $\mathfrak{s}$ is stable under complex conjugation, then the set $\{\mathrm{Re}\,\phi \mid \phi \in \mathfrak{s}\}$ is a real form of $\mathfrak{s}$.

Next we compute the Poisson brackets among the four coordinate functions z_0, $\overline{z_0}$, z_1, $\overline{z_1}$. We have $\{z_j, z_k\} = \{\overline{z_j}, \overline{z_k}\} = 0$ and inverting (2.1) we find

$$\begin{pmatrix} \{z_0,\overline{z_0}\} & \{z_1,\overline{z_0}\} \\ \{z_0,\overline{z_1}\} & \{z_1,\overline{z_1}\} \end{pmatrix} = \frac{-i}{4\rho}\begin{pmatrix} |z_0|^2 + 2|z_1|^2 & -z_1\overline{z_0} \\ -z_0\overline{z_1} & 2|z_0|^2 + |z_1|^2 \end{pmatrix}. \tag{2.3}$$

(Hence the Poisson tensor on Z does not extend to $\mathbb{C}^2$, as it blows up at the origin.)

We write $z_j = p_j + iq_j$ where p_j and q_j are real observables. It follows from (2.3) that the algebra $\mathbb{C}[z_0, \overline{z_0}, z_1, \overline{z_1}] = \mathbb{C}[p_0, q_0, p_1, q_1]$ of complex polynomial observables does *not* form a Lie algebra under Poisson bracket.

Now Z is a complex cone inside $\mathbb{C}^2$ in that it is stable under the natural scaling action of $\mathbb{C}^*$ on $\mathbb{C}^2$. This gives the induced linear representation of $\mathbb{C}^*$ on the observables. We have the product decomposition $\mathbb{C}^* = \mathbb{R}^+ \times S^1$ corresponding to the polar representation $s = re^{i\theta}$ of complex numbers, where $\mathbb{R}^+$ is the group of positive reals. A (real or complex) observable ϕ on Z is *homogeneous of degree d* if $\phi(rm) = r^d m$ for all $r \in \mathbb{R}^+$ and $m \in Z$. The potential ρ, and hence the Kaehler form ω, is homogeneous of degree 4. Consequently, the Poisson bracket of two homogeneous observables of degrees k and l is homogeneous of degree $k + l - 4$. Thus

LEMMA 2.1. *The homogeneous quartic real (or complex) observables on Z form an infinite-dimensional real (or complex) Lie algebra under Poisson bracket.*

REMARK 2.2. It is useful to compare our setup with the flat case. The flat Kaehler structure on $\mathbb{C}^2$ has Kaehler form $\omega_{\mathrm{flat}} = (i/2)(dz_0\, d\overline{z_0} + dz_1\, d\overline{z_1}) = dp_0\, dq_0 + dp_1\, dq_1$ with Kaehler potential $\rho_{\mathrm{flat}} = \frac{1}{2}|z|^2$ and Poisson brackets

$$\{z_j, z_k\}_{\mathrm{flat}} = \{\overline{z_j}, \overline{z_k}\}_{\mathrm{flat}} = 0 \qquad \{z_j, \overline{z_k}\}_{\mathrm{flat}} = -2i\delta_{jk}. \tag{2.4}$$

The form ω is homogeneous of degree 2. The homogeneous quadratic real (or complex) polynomial observables form a maximal finite-dimensional Lie subalgebra which is 10-dimensional and isomorphic to $\mathfrak{sp}(2,\mathbb{R})$ (or $\mathfrak{sp}(2,\mathbb{C})$).

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PROPOSITION 2.3. *Inside the Lie algebra of homogeneous quartic complex observables on Z we have a complex finite-dimensional Lie subalgebra* $\mathfrak{g}_{\mathbb{C}}$ *isomorphic to* $\mathfrak{sl}(3,\mathbb{C})$ *with basis*

$$
(2.5)\qquad \begin{aligned} x_1 &= z_0\bar{z}_1\sqrt{8\rho} \\ x_0 &= -i(|z_0|^2-|z_1|^2)\sqrt{2\rho} \\ \bar{x}_1 &= z_1\bar{z}_0\sqrt{8\rho} \end{aligned} \qquad \begin{aligned} \upsilon_2 &= z_0^4+\bar{z}_1^4 \\ \upsilon_1 &= z_0^3z_1-\bar{z}_0\bar{z}_1^3 \\ \upsilon_0 &= z_0^2z_1^2+\bar{z}_0^2\bar{z}_1^2 \\ \bar{\upsilon}_1 &= z_0z_1^3-\bar{z}_0^3\bar{z}_1 \\ \bar{\upsilon}_2 &= z_1^4+\bar{z}_0^4. \end{aligned}
$$

These observables satisfy the bracket relations $\{x_0,x_j\} = jx_j$, $\{x_1,\bar{x}_1\} = -2x_0$, $\{x_0,\upsilon_j\} = j\upsilon_j$, $\{\upsilon_2,\bar{\upsilon}_2\} = x_0$ *where we put* $x_{-k} = \bar{x}_k$, $\upsilon_{-k} = \bar{\upsilon}_k$ *for k negative.*

Thus we have a complex Cartan decomposition

$$
(2.6)\qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}
$$

where $\mathfrak{k}_{\mathbb{C}} = \mathbb{C}x_1 \oplus \mathbb{C}x_0 \oplus \mathbb{C}\bar{x}_1 \simeq \mathfrak{so}(3,\mathbb{C})$ *and* $\mathfrak{p}_{\mathbb{C}} = \mathbb{C}v_2 \oplus \mathbb{C}\upsilon_1 \oplus \mathbb{C}\upsilon_0 \oplus \mathbb{C}\bar{\upsilon}_1 \oplus \mathbb{C}\bar{\upsilon}_2$ *carries the irreducible* 5*-dimensional representation of* $\mathfrak{k}_{\mathbb{C}}$.

PROOF. Consider $\mathbb{C}^n$ with its flat Kaehler structure as in Remark 1.2. We have a natural symplectomorphism of $\mathbb{C}^n$ with $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$ where $w = u + iv$ with $u, v \in \mathbb{R}^n$ corresponds to (u, v^t). Then $\{v_j, v_k\} = \{u_j, u_k\} = 0$ and $\{v_j, u_k\} = \delta_{jk}$. It follows that the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{C}^n$ given by $a \cdot w = au + iav$ is Hamiltonian with moment map $\nu\colon \mathbb{C}^n \longrightarrow \mathfrak{gl}(n,\mathbb{R})$ defined by $\nu(w) = uv^t$. Then ν is a $\mathrm{GL}(n,\mathbb{R})$-equivariant Poisson map and the pullback of the matrix coordinate function e_{jk} on $\mathfrak{gl}(n,\mathbb{R})$ is the function $\nu^*(e_{jk}) = u_jv_k$ on $\mathbb{C}^n$. Thus the functions u_jv_k give a basis of a Lie algebra of real observables isomorphic to $\mathfrak{gl}(n,\mathbb{R})$.

Next we restrict ν to the complex quadric hypersurface $Q \subset \mathbb{C}^n$ defined by $w \cdot w = 0$ where $w \neq 0$. Then $w \in Q$ iff u and v are orthogonal vectors with common non-zero length. It follows that Q is an $\mathrm{SL}(n,\mathbb{R})$-orbit in $\mathbb{C}^n$ and a (locally closed) complex submanifold. Furthermore $\nu(Q)$ is the $\mathrm{SL}(n,\mathbb{R})$-orbit $O_{\mathbb{R}} \subset \mathfrak{sl}(n,\mathbb{R})$ consisting of non-zero rank 1 matrices of square zero and ν gives a 2-to-1 covering map $\nu_Q\colon Q \longrightarrow O_{\mathbb{R}}$. Notice that Q, being a complex submanifold of $\mathbb{C}^n$, inherits a Kaehler submanifold structure with Kaehler potential $\rho_Q = \frac{1}{2}|w|^2$ from $\mathbb{C}^n$.

Now ν_Q is the symplectic moment map for the transitive Hamiltonian $\mathrm{SL}(n,\mathbb{R})$-action on Q; *i.e.*, the restrictions of the functions u_jv_k to Q give a Lie algebra $\mathfrak{s}$ of observables isomorphic to $\mathfrak{sl}(n,\mathbb{R})$. The Cartan decomposition $\mathfrak{s} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup $\mathrm{SO}(n)$ is as follows: $\mathfrak{k}$, which corresponds to the skew-symmetric matrices, is the span of $u_jv_k - u_kv_j = \mathrm{Im}\,\bar{w}_jw_k$ and $\mathfrak{p}$, which corresponds to the symmetric matrices, is the span of $u_jv_k + u_kv_j = \mathrm{Im}\,w_jw_k$.

Now put $n = 3$. Identifying $\mathbb{C}^3 = S^2\mathbb{C}^2$ we have the "squaring" map $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ defined by

$$
(2.7)\qquad (z_0, z_1) \longmapsto w = \sqrt{2i}\left(\frac{z_0^2+z_1^2}{i}, 2z_0z_1, z_0^2 - z_1^2\right)
$$

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where $\sqrt{i} = e^{\pi i/4}$. This gives a holomorphic 2-to-1 covering map $\pi\colon Z \longrightarrow Q$ so that we now have a 4-fold covering

$$Z \xrightarrow{\pi} Q \xrightarrow{\nu_Q} O_{\mathbb{R}}. \tag{2.8}$$

Then $\pi^*|w|^2 = 4|z|^2$ and so $\pi^*\rho_Q = \rho_Z$. Thus π is symplectic and $\mathfrak{g} = \pi^*\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. We get the desired observables now by taking

$$\begin{aligned} x_1 &= \operatorname{Im} \bar{w}_1 w_2 - i \operatorname{Im} \bar{w}_3 w_2 = u_1 v_2 - u_2 v_1 - i(u_3 v_2 - u_2 v_3) \\ x_0 &= i \operatorname{Im} \bar{w}_1 w_3 = i(u_1 v_3 - u_3 v_1) \\ \upsilon_2 &= \frac{1}{4}\big(\operatorname{Im}(w_3^2 - w_1^2) + 2i \operatorname{Im} w_1 w_3\big) = \frac{1}{2}\big(u_3 v_3 - u_1 v_1 + i(u_1 v_3 + u_3 v_1)\big) \end{aligned}$$

and so on. ∎

Plainly (and by construction) the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is stable under complex conjugation. Notice that the span of $\{ix_0, x_1 + \bar{x}_1, i(x_1 - \bar{x}_1)\}$ is isomorphic to $\mathfrak{su}(2)$ while the span of $\{ix_0, \upsilon_2 + \bar{\upsilon}_2, i(\upsilon_2 - \bar{\upsilon}_2)\}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Recall that a set of observables is called *complete* if the differentials everywhere span the tangent spaces.

COROLLARY 2.4. *The real form* $\mathfrak{g} = \{\operatorname{Re}\phi \mid \phi \in \mathfrak{g}_{\mathbb{C}}\}$ *of* $\mathfrak{g}_{\mathbb{C}}$ *is isomorphic to* $\mathfrak{sl}(3,\mathbb{R})$ *and (2.6) induces the following Cartan decomposition where* $\mathfrak{k} = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g} \simeq \mathfrak{so}(3)$ *and* $\mathfrak{p} = \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{g}$*:*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \tag{2.9}$$

If $\upsilon \in \mathfrak{p}$ *then* υ *is the real part of a holomorphic function* f *on* Z *so that*

$$\upsilon = \operatorname{Re} f = \frac{1}{2}(f + \bar{f}). \tag{2.10}$$

We may choose f *to be homogeneous of degree* 1, *and then* υ *determines* f *uniquely.*

Finally any basis of $\mathfrak{g}$ *is a complete set of eight observables on* Z*.*

Let $O_{\mathbb{R}} \subset \mathfrak{sl}(3,\mathbb{R})$ be the set of rank 1 non-zero matrices with square zero; this is the unique 4-dimensional nilpotent orbit of the adjoint (*i.e.*, conjugation) action of $SL(3,\mathbb{R})$. (The term "nilpotent orbit" is used to indicate an orbit of nilpotent matrices.) Then $O_{\mathbb{R}}$, being equivalent to a coadjoint orbit of $SL(3,\mathbb{R})$, carries the $SL(3,\mathbb{R})$-equivariant K-K-S symplectic form $\omega_{O_{\mathbb{R}}}$. Let $\widetilde{SL}(3,\mathbb{R})$ be the simply-connected (double) covering group of $SL(3,\mathbb{R})$. Then we have the diagram

$$\begin{array}{ccc} \widetilde{SL}(3,\mathbb{R}) & \supset & SU(2) \\ \downarrow & & \downarrow \\ SL(3,\mathbb{R}) & \supset & SO(3) \end{array}$$

where the vertical arrows are double covers and the inclusions give maximal compact subgroups.

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COROLLARY 2.5. *The infinitesimal Lie algebra action of the Hamiltonian vector fields* ξ_ϕ, $\phi \in \mathfrak{g}$, *integrates to a transitive symplectic action of* $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ *on Z. The action of the subgroup* SU(2) *gives the Hamiltonian flow of* $\mathfrak{k}$, *is Kaehler, preserves the Kaehler potential* ρ_Z *and identifies with the standard matrix action of* SU(2) *on* $Z = \mathbb{C}^2 - \{0\}$.

The $\widetilde{\mathrm{SL}}(3,\mathbb{R})$*-action is Hamiltonian with moment map* $\nu_Z \colon Z \longrightarrow \mathfrak{sl}(3,\mathbb{R})$ *given by* (2.8) *so that* ν_Z *gives a* 4*-to-*1 $\widetilde{\mathrm{SL}}(3,\mathbb{R})$*-equivariant symplectic covering*

$$\nu_Z \colon Z \longrightarrow O_{\mathbb{R}}. \tag{2.11}$$

Thus Z is realized, in an $\widetilde{\mathrm{SL}}(3,\mathbb{R})$*-equivariant symplectic fashion, as the universal covering space of* $O_{\mathbb{R}}$.

We call a (real or) complex vector field ξ on a complex manifold $(X, \mathbf{I})$ **I**-*polarized* if ξ preserves the complex structure **I**. In this case, ξ preserves the algebras of holomorphic and anti-holomorphic functions and there is a unique *holomorphic* vector field $\hat{\xi}$ on X such $\xi(f) = \hat{\xi}(f)$ for every holomorphic function f. The map $\xi \longmapsto \hat{\xi}$ is complex linear and preserves commutators.

COROLLARY 2.6. *The Hamiltonian vector fields* ξ_ϕ, $\phi \in \mathfrak{k}_{\mathbb{C}}$, *are* **J**-*polarized and we have*

$$\hat{\xi}_{x_1} = iz_0 \frac{\partial}{\partial z_1} \qquad \hat{\xi}_{x_0} = \frac{1}{2}\left(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1}\right) \qquad \hat{\xi}_{\bar{x}_1} = iz_1 \frac{\partial}{\partial z_0}. \tag{2.12}$$

The infinitesimal Hamiltonian action of $\mathfrak{k}_{\mathbb{C}}$ *integrates to a transitive holomorphic action of* $\mathrm{SL}(2,\mathbb{C})$ *on Z which then commutes with the holomorphic scaling action of* $\mathbb{C}^*$. *This* $\mathrm{SL}(2,\mathbb{C})$*-action is the complexification of the* SU(2)*-action and identifies with the standard matrix action.*

$\mathrm{SU}(2) \times S^1$ *is the subgroup of* $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^*$ *which acts on Z by Kaehler automorphisms.*

The last assertion follows because $\mathrm{SU}(2) \times S^1$ is the subgroup preserving the Kaehler potential, and the Kaehler potential here is determined uniquely by the added condition that it is $\mathbb{R}^+$-homogeneous (see [B2]).

While the whole S^1 subgroup of $\mathbb{C}^*$ preserves $|z|$ and each observable x_j in (2.5), only the subgroup $\mathbb{Z}_4$ preserves the observables υ_j. Here we identify $\mathbb{Z}_n$ with the subgroup of nth roots of unity in S^1. We conclude

COROLLARY 2.7. $\mathbb{R}^+ \times \mathbb{Z}_4$ *is the full subgroup of* $\mathbb{C}^* = \mathbb{R}^+ \times S^1$ *whose scaling action on Z commutes with the* $\widetilde{\mathrm{SL}}(3,\mathbb{R})$*-action. Furthermore the action of the* $\mathbb{Z}_4$ *factor is Kaehler and gives the group of deck transformations of (2.11).*

Thus (2.11) realizes $O_{\mathbb{R}}$ as the quotient of the Kaehler manifold Z by a Kaehler action of $\mathbb{Z}_4$. In this way, $O_{\mathbb{R}}$ acquires a SU(2)-invariant Kaehler structure $(O_{\mathbb{R}}, \mathbf{J}, \omega_{O_{\mathbb{R}}})$ where $\omega_{O_{\mathbb{R}}}$ is the $\mathrm{SL}(3,\mathbb{R})$-invariant K-K-S symplectic form.

We note that the homogeneous quartic Kaehler potential ρ on Z descends through the covering (2.11) to a homogeneous linear Kaehler potential on $O_{\mathbb{R}}$.

The $\mathbb{Z}_4$-action induces a decomposition $R^{\mathrm{hol}}(Z) = \oplus_{m=1}^{4} R^{\mathrm{hol}}_{\chi^m}(Z)$ where $R^{\mathrm{hol}}_{\chi^m}(Z)$ is the space of holomorphic functions on Z which transform by the character χ^m where $\chi(e^{i\pi/2}) = e^{i\pi/2}$. Then $R^{\mathrm{hol}}_{\chi^m}(Z)$ identifies with the space of holomorphic sections of a holomorphic complex line bundle over $O_{\mathbb{R}}$.

3. **The Complex Cotangent Bundle and Holomorphic Symbols.** The complex cotangent bundle T^*Z has a canonical holomorphic symplectic form Ω since Z is a complex manifold. Then the real part $\mathrm{Re}\,\Omega$ is the canonical real symplectic form on T^*Z. Locally the picture looks as follows. T^*Z admits local holomorphic coordinates $(z_1,\ldots,z_n,\zeta_1,\ldots,\zeta_n)$ where $(z_1,\ldots,z_n)$ are local holomorphic coordinates on Z and $\Omega = \sum_{k=1}^{n} d\zeta_k \wedge dz_k$. Changing over to real coordinates we can write $z_k = q_{2k-1} + iq_{2k}$ and $\zeta_k = p_{2k-1} - ip_{2k}$ and then $\mathrm{Re}\,\Omega = \sum_{k=1}^{2n} dp_k \wedge dq_k$. The forms Ω and $\mathrm{Re}\,\Omega$ define respectively Poisson brackets $\{\ ,\ \}_\Omega$ and $\{\ ,\ \}_{\mathrm{Re}\,\Omega}$ on the algebra $R^{\mathrm{hol}}(T^*Z)$ of holomorphic functions on T^*Z and the algebra $C^\infty(T^*Z,\mathbb{C})$ of complex observables on T^*Z. Then $\{f,g\}_\Omega = \frac{1}{2}\{f,g\}_{\mathrm{Re}\,\Omega}$ for $f,g \in R^{\mathrm{hol}}(T^*Z)$. This follows, *e.g.*, by calculating in local coordinates.

We identify $T^*Z = Z \times \mathbb{C}^2$ in the obvious way and then $\Omega = d\zeta_0 \wedge dz_0 + d\zeta_1 \wedge dz_1$ where ζ_0, ζ_1 are the standard holomorphic coordinates on $\mathbb{C}^2$. Then

$$\{z_j, z_k\}_\Omega = \{\zeta_j, \zeta_k\}_\Omega = 0 \quad \text{and} \quad \{\zeta_j, z_k\}_\Omega = \delta_{jk}.$$

The holomorphic function

$$\Lambda = z_0\zeta_0 + z_1\zeta_1 \tag{3.1}$$

on T^*Z is the symbol of the holomorphic Euler vector field on Z

$$E = z_0\frac{\partial}{\partial z_0} + z_1\frac{\partial}{\partial z_1}. \tag{3.2}$$

Let T^+Z be the open set of T^*Z where Λ is non-vanishing.

The 1-form $\theta = \frac{i}{2}(\overline{\partial} - \partial)\rho$ is a symplectic potential on (Z,ω), *i.e.*, θ is a smooth real 1-form such that $\omega = d\theta$. Let

$$b\colon Z \longrightarrow T^*Z \tag{3.3}$$

be the section of the cotangent bundle defined by θ. Let $\mathcal{H}(Z) = b^*R^{\mathrm{hol}}(T^+Z)$ be the algebra of complex observables on Z obtained by pullback of holomorphic functions from T^+Z.

PROPOSITION 3.1. *We have $b(Z) \subset T^+Z$ and b is an embedding of Z into T^+Z as a totally real symplectic submanifold so that $\omega = b^*(\mathrm{Re}\,\Omega)$. Then $\mathcal{H}(Z)$ is a complex Poisson subalgebra of $C^\infty(Z,\mathbb{C})$ and each observable $\psi \in \mathcal{H}(Z)$ has a unique extension to a holomorphic function $\Phi(\psi)$ on T^+Z. The resulting map*

$$\Phi\colon \mathcal{H}(Z) \longrightarrow R^{\mathrm{hol}}(T^+Z) \tag{3.4}$$

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is an isomorphism of complex Poisson algebras where $\{\phi,\psi\}_\omega = \big\{\Phi(\phi),\Phi(\psi)\big\}_\Omega$. *The observables* z_k, $\bar{z}_k\sqrt{\rho}$, $\bar{z}_j\bar{z}_k$ *and* ρ *all lie in* $\mathcal{H}(Z)$ *and their holomorphic extensions are given by*

$$\Phi(z_k) = z_k, \qquad \Phi(\bar{z}_k\sqrt{8\rho}) = i\zeta_k$$
$$\Phi(\rho) = \frac{i}{2}\Lambda, \qquad \Phi(\bar{z}_j\bar{z}_k) = \frac{i\zeta_j\zeta_k}{4\Lambda}. \tag{3.5}$$

PROOF. The first paragraph is proven in [B2]. To prove the rest, we first compute

$$\theta = \frac{i}{2}(\bar\partial - \partial)(2|z|^4) = 2i|z|^2\Big(\sum_{k=0}^{1} z_k\, d\bar{z}_k - \bar{z}_k\, dz_k\Big). \tag{3.6}$$

It follows (see [B2]) that $b^*(\zeta_k) = -4i|z|^2\bar{z}_k$ and so $b^*(\zeta_k) = -i\bar{z}_k\sqrt{8\rho}$. This gives $b^*(\frac{i}{2}\Lambda) = \rho$ and $b^*\big(i\zeta_j\zeta_k/(4\Lambda)\big) = \bar{z}_j\bar{z}_k$. ∎

COROLLARY 3.2. *Every observable in the Lie algebra* $\mathfrak{g}_\mathbb{C}$ *(constructed in Proposition 2.3) lies in* $\mathcal{H}(Z)$ *and hence extends uniquely to a holomorphic function on* T^+Z. *Explicitly we have, where* $j = 0,1,2,3,4$ *and* $k = 4-j$,

$$\Phi(x_1) = iz_0\zeta_1$$
$$\Phi(x_0) = \tfrac{1}{2}(z_0\zeta_0 - z_1\zeta_1) \qquad \Phi(v_{j-2}) = \frac{z_0^j z_1^k - (-1)^k\zeta_0^k\zeta_1^j}{16\Lambda^2}. \tag{3.7}$$
$$\Phi(\bar{x}_1) = iz_1\zeta_0$$

Thus $\mathfrak{r}_\mathbb{C} = \Phi(\mathfrak{g}_\mathbb{C})$ *is a complex Lie subalgebra of* $R^{\mathrm{hol}}(T^+Z)$ *with respect to* $\{\ ,\ \}_\Omega$ *and* $\mathfrak{r}_\mathbb{C} \simeq \mathfrak{g}_\mathbb{C} \simeq \mathfrak{sl}(3,\mathbb{C})$.

Let $\bar{Z}$ be the complex conjugate manifold $(Z, -\mathbf{J})$. We may regard Z as a totally real submanifold of $Z\times\bar{Z}$ by means of the diagonal embedding $\Delta\colon Z \longrightarrow Z\times\bar{Z}$, $\Delta(z) = (z,\bar{z})$. Then all the polynomial observables on Z extend uniquely to holomorphic functions on $Z\times\bar{Z}$. We denote the extensions of z_j and $\bar{z}_j$ respectively by z_j (again) and $\bar{w}_j$. Then $z_0, z_1, \bar{w}_0, \bar{w}_1$ are holomorphic coordinate functions on $Z\times\bar{Z}$. The holomorphic extension of $|z|^2$ is

$$\lambda(z,\bar{w}) = z_0\bar{w}_0 + z_1\bar{w}_1. \tag{3.8}$$

Let $(Z\times\bar{Z})^o$ be the open set of $Z\times\bar{Z}$ where λ is non-vanishing.

COROLLARY 3.3. *The section* $b\colon Z \longrightarrow T^+Z$ *extends uniquely to a holomorphic map* $B\colon Z\times\bar{Z} \longrightarrow T^*Z$ *where* $B(u,\bar{v}) = \big(u, -4i\lambda(u,\bar{v})\bar{v}\big)$. *Then* $B^*z_k = z_k$ *and* $B^*\zeta_k = -4i\lambda\bar{w}_k$. *Also*

$$B^*\Phi\Big(\frac{\rho}{2}\Big) = \lambda^2. \tag{3.9}$$

We have $B(Z\times\bar{Z})^o = T^+Z$. *The restricted map*

$$B\colon (Z\times\bar{Z})^o \longrightarrow T^+Z \tag{3.10}$$

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is a holomorphic 2*-to-*1 *covering map of complex manifolds. Thus we may pull back the canonical holomorphic symplectic form* Ω *on* T^*Z *to obtain the holomorphic symplectic form* $\Omega^* = B^*\Omega$ *on* $(Z \times \bar{Z})^o$.

PROOF. The first paragraph is immediate from Proposition 3.1. Then $B(Z \times \bar{Z})^o \subset T^+Z$ because of (3.9). To prove the rest we essentially need to construct an inverse map to (3.10). There is an obstruction to this as only the square of λ, not λ itself, is the pullback of a function on T^+Z. To remedy this, we construct a complex manifold M which double covers T^+Z by "extracting a square root of Λ". A model for M is the codimension 1 complex submanifold

$$M = \left\{(w,t) \mid \Lambda(w) = -4it^2\right\} \subset T^+Z \times \mathbb{C}^*$$

and then the natural projection $\tau: M \longrightarrow T^+Z$ is a holomorphic 2-to-1 covering. Now we can lift B to a holomorphic map $\tilde{B}: (Z \times \bar{Z})^o \longrightarrow M$ where $\tilde{B}(m) = \bigl(B(m), \lambda(m)\bigr)$ since $B^*(\Lambda) = -4i\lambda^2$ by (3.5) and (3.9). Then $\tilde{B}$ clearly has an inverse and so is a biholomorphic isomorphism. This gives our result as $B = \tau \circ \tilde{B}$. ∎

4. **Reproducing Kernels and Quantization of Observables.** First we briefly discuss reproducing kernels and Wick-Berezin symbols. Let H be a separable pre-Hilbert space with inner product $\langle\ ,\ \rangle$; let $\mathcal{H}$ be the Hilbert space completion of H. Assume that H consists of holomorphic functions on a complex manifold $(X, \mathbf{I})$. Let $\bar{X}$ be the complex conjugate manifold $(X, -\mathbf{I})$. Pick some (countable) orthonormal basis $\{s_n\}$ of H. If the sum $\sum_n s_n(z)\overline{s_n(w)}$ converges (absolutely and uniformly on compact sets) to a holomorphic function $\Psi(z, \bar{w})$ on $X \times \bar{X}$, then we say that $\mathcal{H}$ is a holomorphic reproducing kernel Hilbert space with reproducing kernel Ψ. It follows that (i) $\mathcal{H}$ consists of holomorphic functions on X, (ii) $\Psi(z, \bar{w}) = \sum_n t_n(z)\overline{t_n(w)}$ for any orthonormal basis $\{t_n\}$ of $\mathcal{H}$, (iii) the function $\Psi_w = \Psi(z, \bar{w})$ lies in $\mathcal{H}$ for any $w \in X$, and we have the reproducing property (iv) $\langle s, \Psi_w\rangle = s(w)$ for every $s \in \mathcal{H}$. The functions Ψ_w are called the *coherent states*. We may call Ψ the reproducing kernel of $\bigl(H, \langle\ ,\ \rangle\bigr)$ and write (H, Ψ) for this pre-Hilbert space.

Let us explain (i) in more detail as this is a key point for us (*cf.* Corollary 4.4 below). The (abstract) Hilbert space completion of H is the space $\mathcal{H}$ of series $f = \sum_n c_n s_n$, $c_n \in \mathbb{C}$, such that $\sum_n |c_n|^2$ converges. The content of (i) is that the holomorphicity of Ψ implies that the series $f = \sum_n c_n s_n$ defines (by absolute and uniform convergence on compact sets) a holomorphic function f. To see this we use the coherent states. Indeed, $\Psi_w = \sum_n \overline{s_n(w)} s_n$ is a holomorphic function on X since Ψ is already holomorphic on $X \times \bar{X}$. Also Ψ_w has finite norm as $\|\Psi_w\|^2 = \sum_n |s_n(w)|^2 = \Psi(w, \bar{w})$. But then for $f = \sum_n c_n s_n \in \mathcal{H}$, $\langle f, \Psi_w\rangle$ is finite and given by $\langle f, \Psi_w\rangle = \sum_n c_n s_n(w)$ where the series converges absolutely. We conclude that the series $f = \sum_n c_n s_n$ defines f as a function on X and $f(w) = \langle f, \Psi_w\rangle$. Hence $\mathcal{H}$ consists of functions on X. Now suppose a sequence f_p in H converges to f with respect to $\|\ \|$. The Schwarz inequality gives

$$|f_p(w) - f(w)| \le \|f_p - f\| \|\Psi_w\| = \|f_p - f\| \Psi(w, \bar{w}).$$

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Hence f_p converges to f pointwise and uniformly on compact sets. Consequently the holomorphicity of the f_p implies that f is holomorphic. This argument was adapted from [F-K, proof of Proposition IX.2.7, pp. 171–2]).

Let $(X \times \overline{X})^*$ be the open set where Ψ is non-vanishing. Then $(X \times \overline{X})^*$ contains the diagonal $\Delta_X = \{(z, \overline{z}) \mid z \in X\}$ since Ψ is positive on Δ_X. Notice that Ψ corresponds to the identity operator $H \longrightarrow H$ in the obvious way.

We will say that a complex linear operator $A\colon H \longrightarrow H$ is Ψ*-admissible* if the quantity $A\Psi_w(z) = \sum_n (At_n)(z)\overline{t_n(w)}$ defines a holomorphic function on $X \times \overline{X}$. Clearly such operators may be unbounded. From now on we consider only Ψ-admissible operators on H.

The Wick-Berezin symbol $\phi_A = \phi$ of A is given by the formula

$$(A\Psi_w)(z) = \phi(z, \overline{w})\Psi_w(z) \tag{4.1}$$

so that ϕ_A is a holomorphic function on $(X \times \overline{X})^*$. The symbol of the adjoint operator A^* is

$$\phi_{A^*}(z, \overline{w}) = \overline{\phi_A(w, \overline{z})}. \tag{4.2}$$

If A is multiplication by a holomorphic function f, *i.e.*, if $A = f$, then $\phi_A(z, \overline{w}) = f(z)$.

We can reverse our perspective and observe that via (4.1) a holomorphic function $\phi(z, \overline{w})$ *defines* a Ψ-admissible complex linear operator $A = A^\phi\colon H \longrightarrow H$ with symbol ϕ. If $\phi \in C^\infty(X, \mathbb{C})$ extends to a holomorphic function $\tilde{\phi}$ on $X \times \overline{X}$ so that $\tilde{\phi}(z, \overline{w}) = \phi(z, \overline{z})$, then we will write $\mathcal{T}(\phi) = \mathcal{T}_\phi$ for $A^{\tilde{\phi}}$. Then (4.2) gives $\mathcal{T}_\phi^* = \mathcal{T}_{\overline{\phi}}$. Notice that the operator $\mathcal{T}_\phi$ is unchanged if we replace $\langle\ ,\ \rangle$, or equivalently Ψ, by any positive multiple.

Now we return to our situation. Let $\mathbb{C}[Z] = \mathbb{C}[z_0, z_1]$ be the algebra of holomorphic polynomial functions on Z, with $\mathbb{C}_n[Z]$ the subspace of homogeneous degree n polynomials. The Kaehler action of SU(2) on Z (see Corollaries 2.5 and 2.6) gives a corresponding SU(2) representation on $\mathbb{C}[Z]$. This representation is completely reducible and multiplicity-free. In fact $\mathbb{C}_n[Z]$ carries the irreducible $n+1$-dimensional representation of SU(2). It follows that for any SU(2)-invariant Hermitian non-degenerate pairing on $\mathbb{C}[Z]$, the spaces $\mathbb{C}_n[Z]$, being inequivalent SU(2)-representations, are orthogonal. Furthermore (by Schur's Lemma) the pairing is unique on each space $\mathbb{C}_n[Z]$ up to a scalar in $\mathbb{R}^+$. We conclude that any reproducing kernel Ψ on $\mathbb{C}[Z]$ is a series in the function λ defined in (3.8) so that

$$\Psi(z, \overline{w}) = \sum_{n \in \mathbb{Z}_+} \frac{d_n}{n!}\lambda^n. \tag{4.3}$$

The linear action of $\mathbb{Z}_4$ on $\mathbb{C}[Z]$ (see Corollary 2.7) breaks up into a direct sum of joint eigenspaces

$$\mathbb{C}[Z] = H^{[0]} \oplus H^{[1]} \oplus H^{[2]} \oplus H^{[3]}$$

where $H^{[m]} = \mathbb{C}[Z] \cap R^{\mathrm{hol}}_{\chi^m}(Z)$. Then $H^{[m]} = \bigoplus_{n \in \mathbb{Z}_+} \mathbb{C}_{4n+m}[Z]$ where $\mathbb{Z}_+$ is the set of non-negative integers.

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For each $m = 0, 1, 2$ or 3, our goal is (if possible) to "quantize" our real Lie algebra $\mathfrak{g}$, or equivalently our complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, of observables into an algebra of operators on $H^{[m]}$. Precisely, this means that we wish to construct a positive definite-Hermitian inner product $\langle\ ,\ \rangle$ on $H^{[m]}$ and a complex linear *quantization map*

$$Q : \mathfrak{g}_{\mathbb{C}} \longrightarrow \operatorname{End} H^{[m]}$$

which together satisfy the Dirac axioms (*cf.* [Ki]):

(i) $Q(1)$ is the identity operator
(ii) The operators $Q(\phi)$ and $Q(\overline{\phi})$ are adjoint
(iii) The operators satisfy

$$Q\big(\{\phi,\psi\}\big) = i\big[Q(\phi), Q(\psi)\big] \tag{4.4}$$

(iv) $Q(\mathfrak{g})$ is a complete set of operators on $H^{[m]}$ (since $\mathfrak{g}$ is complete by Corollary 2.4).

We will add three more axioms to the list:

(v) If the Hamiltonian flow of ϕ is Kaehler, (*i.e.*, if ξ_ϕ preserves $\mathbf{J}$) then $Q(\phi) = -i\hat{\xi}_\phi$
(vi) If f is holomorphic then $Q(f)$ is multiplication by f, *i.e.*, $Q(f) = f$
(vii) $\big(H^{[m]}, \langle\ ,\ \rangle\big)$ has a holomorphic reproducing kernel $\Psi^{[m]}$.

The new axioms (v)–(vii), as well as our original choice of $H^{[m]}$ as the pre-Hilbert space of quantization, of course depend on the choice of polarization. There are many physical motivations for (v)–(vii). In fact (v) is basic to polarization theory, (vi) is natural in twistor theory, and (vii) means that the quantization has coherent states. These axioms arose in the context of quantizing the whole algebra $C^\infty(Z, \mathbb{C})$, but also make sense for Lie subalgebras of observables.

Now axioms (ii), (vi) and (vii) determine the quantization of antiholomorphic observables. In terms of the Wick-Berezin operators $T(\phi)$ discussed above we conclude that if f is holomorphic then

$$Q(\overline{f}) = T(\overline{f}). \tag{4.5}$$

But then the axioms and the value of $\Psi^{[m]}$ completely determine the quantization of our Lie algebra $\mathfrak{g}_{\mathbb{C}}$ since the observables in $\mathfrak{g}$ have such a simple form. Indeed by Corollaries 2.4 and 2.5, we have the Cartan decomposition (2.9) and if the Hamiltonian flow of $x \in \mathfrak{k}$ is Kaehler while every $v \in \mathfrak{p}$ is given by (2.10).

To summarize this discussion, we make

DEFINITION 4.1. An SU(2)-invariant Hermitian inner product $\langle\ ,\ \rangle$ on $H^{[m]}$ with reproducing kernel $\Psi^{[m]}$ is *quantum* if the corresponding complex linear map $Q : \mathfrak{g}_{\mathbb{C}} \longrightarrow \operatorname{End} H^{[m]}$ defined by

$$\begin{aligned} Q(x) &= -i\hat{\xi}_x && \text{if } x \in \mathfrak{k} \\ Q(v) &= f + T(\overline{f}) && \text{if } v = f + \overline{f} \in \mathfrak{p} \text{ and } f \in \mathbb{C}[Z] \end{aligned} \tag{4.6}$$

is a *quantization* in that $(H^{[m]}, \Psi^{[m]}, Q)$ satisfies axioms (i)–(vii).

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It follows easily now that $\Psi^{[m]}$ is quantum iff Q as defined by (4.6) satisfies (4.4). Moreover the definition (4.6) automatically ensures that (4.4) holds if $\phi, \psi \in \mathfrak{k}$ or if $\phi \in \mathfrak{k}, \psi \in \mathfrak{p}$. Thus Q is a quantization iff (4.4) holds for all $\phi, \psi \in \mathfrak{p}$, so iff for all $\upsilon, \upsilon' \in \mathfrak{p}$ we have

$$-\xi_{\{\upsilon,\upsilon'\}} = \big[T(\upsilon), T(\upsilon')\big]. \tag{4.7}$$

This is a condition on $\langle\ ,\ \rangle$, or equivalently on $\Psi^{[m]}$, which we completely analyze in our next result.

THEOREM 4.2. *For $m = 0,1$ or 2, $H^{[m]}$ admits a quantum* SU(2)*-invariant Hermitian inner product $\langle\ ,\ \rangle$. This inner product is unique up to multiplication by a positive scalar. The corresponding reproducing kernels are functions of λ and given, up to positive multiples, by*

$$\Psi^{[0]}(z,\overline{w}) = \Psi^{[0]}(\lambda) = {}_1F_2\left(\frac{5}{4}; \frac{3}{4}, \frac{1}{2}; \lambda^4\right)$$

$$\Psi^{[1]}(z,\overline{w}) = \Psi^{[1]}(\lambda) = {}_1F_2\left(\frac{3}{2}; \frac{5}{4}, \frac{3}{4}; \lambda^4\right)\lambda$$

$$\Psi^{[2]}(z,\overline{w}) = \Psi^{[2]}(\lambda) = {}_1F_2\left(\frac{7}{4}; \frac{3}{2}, \frac{5}{4}; \lambda^4\right)\lambda^2.$$

However on $H^{[3]}$, no SU(2)*-invariant Hermitian inner product is quantum.*

Here and throughout the paper we use the standard hypergeometric function notation so that

$${}_pF_q(a_1,\ldots,a_p; b_1,\ldots,b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{n!\,(b_1)_n \cdots (b_q)_n} x^n$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

PROOF. At the outset, we may consider all cases simultaneously by considering an SU(2)-invariant Hermitian inner product $\langle\ ,\ \rangle$ on $\mathbb{C}[Z]$ with reproducing kernel Ψ. Then Ψ must be of the form (4.3) and

$$\|z_0^n\|^2 = \|z_1^n\|^2 = \frac{n!}{d_n}. \tag{4.8}$$

We want to determine what condition Ψ must satisfy in order that (4.7) holds for all $\upsilon, \upsilon' \in \mathfrak{p}$, or equivalently for all $\upsilon, \upsilon' \in \mathfrak{p}_{\mathbb{C}}$. It suffices (see [B-K3, Lemma 3.6]) to verify (4.7) in the one instance

$$-\hat{\xi}_{x_0} = \big[T(\upsilon_2), T(\overline{\upsilon_2})\big]. \tag{4.9}$$

Let $f_a = z_a^4$ and $T_a = T(\overline{f_a})$ for $a = 0,1$. By (2.5) we have $T(\upsilon_2) = f_0 + T_1$ and $T(\overline{\upsilon_2}) = f_1 + T_0$ and (2.12) computes $\hat{\xi}_{x_0}$. Clearly $[f_0, f_1] = [T_0, T_1] = 0$, and so (4.9) becomes

$$[f_0, T_0] - [f_1, T_1] = -\frac{1}{2}\left(z_0\frac{\partial}{\partial z_0} - z_1\frac{\partial}{\partial z_1}\right). \tag{4.10}$$

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In order to analyze (4.10) we will derive a formula for T_a. Indeed (4.1) gives

$$T_a \Psi_w = \bar{w}_a^4 \Psi_w \tag{4.11}$$

which, by homogeneity in the $\bar{w}_0, \bar{w}_1$ variables, implies

$$T_a\left(\frac{d_{n+4}}{(n+4)!}\lambda^{n+4}\right) = \bar{w}_a^4 \frac{d_n}{n!}\lambda^n$$

But $\frac{\partial}{\partial z_a}\lambda = \bar{w}_a$ and we easily find:

$$T_a = D\frac{\partial^4}{\partial z_a^4} \tag{4.12}$$

where

$$D = \sum_{n\in\mathbb{Z}_+} D_n \varepsilon_n \quad \text{and} \quad D_n = \frac{d_n}{d_{n+4}}. \tag{4.13}$$

Here ε_n is the linear operator on $\mathbb{C}[Z]$ such that $\varepsilon_n f = \delta_{n,p} f$ if $f \in \mathbb{C}_p[Z]$. Set $D_n = 0$ for $n < 0$.

To find Ψ, we compute out the relation (4.10) on each test function $s_{jk} = z_0^j z_1^k \in \mathbb{C}[Z]$ with $n = j + k$. Then (4.10) reduces to the numerical equation

$$D_{n-4}\big([j]_4 - [k]_4\big) - D_n\big([j+4]_4 - [k+4]_4\big) = \frac{k-j}{2} \tag{4.14}$$

where we put $[p]_4 = p(p-1)(p-2)(p-3)$. The problem of determining Ψ is now the problem of solving this equation for the scalars D_n, $n \in \mathbb{Z}_+$. To analyze (4.14) we start off inductively. For $n \in \{0, 1, 2, 3\}$, (4.14) becomes

$$-D_n\big([j+4]_4 - [k+4]_4\big) = \frac{k-j}{2}. \tag{4.15}$$

We need to analyze all cases $j + k = n$ where $j > k$. For $n = 0$, there are no such cases, and so (4.15) gives no restriction on D_0. For $n = 1$, there is $(j, k) = (1, 0)$ and this gives $D_1 = 1/(16 \cdot 12)$. For $n = 2$, there is $(j, k) = (2, 0)$ and this gives $D_2 = 1/(16 \cdot 21)$. For $n = 3$, there are two cases $(j, k) = (3, 0), (2, 1)$ and these give the *different* values $D_3 = 1/(16 \cdot 34)$ and $D_3 = 5/24$. Hence there is no solution for D_3. This proves that $H^{[3]}$ admits no quantum inner product.

Next we go back to (4.14) and consider $n = 4$. Then the two cases $(j, k) = (4, 0), (3, 1)$ give two independent equations $D_0 - 69D_4 = -1/12$ and $30D_4 = 1/24$ with unique solution $D_0 = 1/80$ and $D_4 = 1/(30 \cdot 24)$.

At this point, we see that for each of $m = 0, 1, 2$, there are two possibilities. Either (4.15) uniquely determines the sequence $(D_m, D_{m+4}, \ldots)$, or (4.15) becomes unsolvable for some value $n = m + 4k$ as we check through all cases $j + k = n$. In the former case, the sequence $(D_m, D_{m+4}, \ldots)$ determines the unique (up to multiple) quantum inner product on $H^{[m]}$, while in the latter case $H^{[m]}$ admits no quantum inner product. We

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claim that the former case is the one that actually happens. To verify this, we will not continue the inductive process, but instead simply guess a formula for D_n and prove that it solves (4.14).

Motivated by the form of the pseudo-differential symbols in (3.7), we make the guess: D is the quantization of $1/(16\Lambda^2)$ and so D should be the inverse of an operator of the form $16(E+c_1)(E+c_2)$ where c_1 and c_2 are constants. This means that D_n is the inverse of $16(n+c_1)(n+c_2)$. Comparing with the four values D_0, D_1, D_2, D_4 we have already computed, we guess that

$$D_n = \frac{1}{16(n+1)(n+5)}. \tag{4.16}$$

To check that (4.16) solves (4.14), we can simply substitute into the formal identity from [B-K3, Lemma 4.8]. Let us recall the statement: if

$$J(a_i; b) = J(a_0, a_1, a_2, a_3; b) = \frac{a_0a_1a_2a_3}{b(b+1)}$$

and $a'_m = b - a_m$ where a_0, a_1, a_2, a_3 are five indeterminates then

$$\begin{aligned} J(a_i; b) - J(a'_i; b) - J(a_i+1; b+1) + J(a'_i+1; b+1) \\ = 2b - (a_0+a_1+a_2+a_3) \end{aligned}. \tag{4.17}$$

To use this, we put $b = (n-3)/4$ and $a_m = (j-m)/4$ for $m = 0, 1, 2, 3$. Then $a'_m = (k-3+m)/4$ where $j+k = n$ and $2b - (a_0+a_1+a_2+a_3) = (k-j)/2$. Then (4.17) gives the identity

$$\frac{[j]_4 - [k]_4}{16(n-3)(n+1)} - \frac{[j+4]_4 - [k+4]_4}{16(n+1)(n+5)} = \frac{k-j}{2}. \tag{4.18}$$

Notice that division by $n-3$ is allowed since we have excluded the case $n = 3+4m$. Finally (4.18) says that our guess (4.16) solves (4.14). Clearly the coefficients of the corresponding series $\Psi^{[m]}$ are all positive once we choose d_m positive. Thus we get a unique (up to multiple) quantum inner product on $H^{[m]}$ for $m = 0, 1, 2$.

It is easy now to compute the reproducing kernels $\Psi^{[m]}$ for $m = 0, 1, 2$. Indeed (4.13) and (4.16) give the recursion relation $d_{n+4} = 16(n+1)(n+5)d_n$ where $n \in \mathbb{Z}_+$. Let $t_n = d_n/n!$ so that $\Psi^{[m]}(\lambda) = \sum_{n\in 4\mathbb{Z}_++m} t_n\lambda^n$. Then we get the recursion

$$t_{n+4} = 16\frac{n+5}{(n+2)(n+3)(n+4)}\, t_n. \tag{4.19}$$

Now we write $n = 4k+m$ where $m = 0, 1, 2$ and set $m' = m/4$ and $k' = k+m'$. Then

$$\begin{aligned} t_{4k+m+4} &= \frac{k'+\frac{5}{4}}{(k'+1)(k'+\frac{3}{4})(k'+\frac{1}{2})}\, t_{4k+m} \\ &= \frac{(m'+\frac{5}{4})_k}{(m'+1)_k(m'+\frac{3}{4})_k(m'+\frac{1}{2})_k}\, t_m. \end{aligned} \tag{4.20}$$

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We may set $t_0 = t_1 = t_2 = 1$. Notice that our three values of m are precisely the values which make $m'+1$, $m'+\frac{3}{4}$ or $m'+\frac{5}{4}$ equal to 1, hence which make one of the denominator factors equal to $k!$. Thus for each of $m = 0, 1, 2$, we get a hypergeometric function. This proves our formulas for the reproducing kernels. ∎

REMARK 4.3. We can revise the proof by analyzing the relation (4.10) on a suitable series $W(z_0, z_1)$ instead of on the monomials $z_0^j z_1^k$. If we choose $W(z) = e^\lambda$ then we recover many of the calculations in [R-S]. A more natural approach for us is to choose F to be the unknown reproducing kernel $\Psi(z, \overline{w})$, but we are not yet able to carry out this approach in a nice manner. The ideal proof here would produce directly the differential equation (5.5) below.

For use in Section 5 (see Corollary 5.3) we construct the hypergeometric functions

$$(4.21) \qquad P^{[m]}(x) = {}_2F_3\left(m' + \frac{5}{4}, m' + \frac{1}{4}; \widehat{m' + 1}, \widehat{m' + \frac{3}{4}}, \widehat{m' + \frac{1}{2}}, \widehat{m' + \frac{1}{4}}; x^4\right) x^m$$

where $m \in \{0, 1, 2, 3\}$, $m' = m/4$ and the hats mean that we omit the term if it is equal to 1. Clearly $P^{[m]}(\lambda) = \Psi^{[m]}(\lambda)$ for $m = 0, 1, 2$.

The theorem and its proof give several corollaries. First the holomorphicity of $\Psi^{[m]}$ implies

COROLLARY 4.4. *Let $m \in \{0, 1, 2\}$. Then the Hilbert space completion of $(H^{[m]}, \Psi^{[m]})$ is a Hilbert space $\mathcal{H}^{[m]}$ of holomorphic functions on Z. We have $\mathcal{H} \subset R^{\mathrm{hol}}_{\lambda^m}(Z)$.*

The proof gives an explicit formula for the operators corresponding to $\mathfrak{p}$. Let

$$(4.22) \qquad \mathcal{E} = (E+1)(E+5).$$

COROLLARY 4.5. *Let $m \in \{0, 1, 2\}$. Let $f = z_0^k z_1^j \in \mathbb{C}_4[Z]$. Then*

$$(4.23) \qquad \mathcal{T}(\bar{f}) = \frac{1}{16\mathcal{E}} \frac{\partial^4}{\partial z_0^k \partial z_1^j}.$$

For $j = 0, 1, 2, 3, 4$ and $k = 4 - j$, we have

$$(4.24) \qquad \mathcal{Q}(v_{j-2}) = \mathcal{Q}\left(z_0^j z_1^k + (-1)^k \bar{z}_0^k \bar{z}_1^j\right) = z_0^j z_1^k + (-1)^k \frac{1}{16\mathcal{E}} \frac{\partial^4}{\partial z_0^k \partial z_1^j}.$$

The proof of [B-K3, Theorem 5.2] applies equally well in this setting and gives

COROLLARY 4.5. *Let $m \in \{0, 1, 2\}$. The skew-adjoint operators $\pi(\phi) = i\mathcal{Q}(\phi)$, $\phi \in \mathfrak{g}$ give an irreducible $\mathfrak{g}$-representation*

$$(4.25) \qquad \pi \colon \mathfrak{g} \longrightarrow \operatorname{End} H^{[m]}.$$

Furthermore the operators $i\mathcal{Q}(\phi)$ exponentiate to give an irreducible unitary representation

$$(4.26) \qquad \widetilde{\mathrm{SL}}(3, \mathbb{R}) \longrightarrow \operatorname{Unit} \mathcal{H}^{[m]}.$$

For $m = 0$ or 2, but not for $m = 1$, this representation of $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ descends to a representation of $\mathrm{SL}(3, \mathbb{R})$.

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5. **The** SU(2)**-Invariant Measure.** Any real observable $\mu(z,\overline{z})$ on $Z = \mathbb{C}^2 - \{0\}$ defines a distribution

$$M(\phi) = \int_Z \phi(z,\overline{z})\,\mu(z,\overline{z})|dz\,d\overline{z}| \tag{5.1}$$

where ϕ is a continuous function on Z of compact support. In fact M is a (signed) measure and $M(\phi)$ is defined as long as the integral in (5.1) converges absolutely. Here Z is orientable and integrating with respect to the density $|dz\,d\overline{z}|$ is the same as integration against the volume form $dz\,d\overline{z} = dz_0\,dz_1\,d\overline{z_0}\,d\overline{z_1}$.

We recall the notion of absolute convergence. Let $\alpha(z,\overline{z})$ be a continuous $\mathbb{R}$-valued function on Z. We say that the integral $\int_Z \alpha\,|dz\,d\overline{z}|$ *converges absolutely* iff $\int_Z |\alpha|\,|dz\,d\overline{z}|$ converges, and then the value of $\int_Z \alpha\,|dz\,d\overline{z}|$ is $\int_Z \alpha^+\,|dz\,d\overline{z}| - \int_Z \alpha^-\,|dz\,d\overline{z}|$ where $\alpha = \alpha^+ - \alpha^-$ with α^+ and α^- non-negative and $\alpha^+\alpha^- = 0$. Then $\int_Z \alpha\,|dz\,d\overline{z}|$ can also be computed as a sum of a series of integrals over bounded sets of Z. If we take any partition of unity $\{u_n(z,\overline{z})\}$ then $\int_Z \alpha\,|dz\,d\overline{z}| = \sum_n \int_Z u_n\alpha|dz\,d\overline{z}|$. Also $\int_Z \alpha|dz\,d\overline{z}| = \lim_{r\to\infty}\int_{\rho<r}\alpha|dz\,d\overline{z}|$. This discussion extends to the case when α is $\mathbb{C}$-valued for then we put $\int_Z \alpha\,|dz\,d\overline{z}| = \int_Z \alpha_1\,|dz\,d\overline{z}| + i\int_Z \alpha_2\,|dz\,d\overline{z}|$ where α_1 and α_2 are the real and imaginary parts of α.

Given μ, we will say that a holomorphic function f on Z is *square integrable* with respect to $\mu|dz\,d\overline{z}|$ if the integral $\int_Z |f|^2\mu|dz\,d\overline{z}|$ converges absolutely. Notice that holomorphicity implies that $|f|^2$ is bounded as $\rho \longrightarrow 0$.

We will say that $\mu|dz\,d\overline{z}|$ is an *admissible measure* if the following growth conditions are satisfied: $\mu(z,\overline{z})$ and all its partial derivatives (of all orders) are (i) bounded as $\rho \longrightarrow 0$ and (ii) tend to zero faster than any algebraic power of ρ as $\rho \longrightarrow \infty$. In this case $M(f\overline{g})$ is defined for any $f, g \in \mathbb{C}[Z]$ and the formula

$$\langle f, g\rangle = \int_Z f(z)\overline{g(\overline{z})}\mu(z,\overline{z})|dz\,d\overline{z}| \tag{5.2}$$

defines a Hermitian pairing on $\mathbb{C}[Z]$. We do *not* require that μ is positive on Z.

Our goal in this section is to find an admissible measure $\mu|dz\,d\overline{z}|$ such that, for $m \in \{0,1,2\}$, our Hilbert space $H^{[m]}$ is exactly the space of functions in $R^{\mathrm{hol}}_{\chi^m}(Z)$ which are square integrable w.r.t. μ and furthermore (5.2) gives the positive-definite inner product on $H^{[m]}$ with reproducing kernel $\Psi^{[m]}$ computed in Theorem 4.2. We will see (Theorem 5.5) that this problem is impossible to solve for μ positive, but has a unique solution where μ assumes both positive and negative values. Then μ is a function of ρ and μ is positive outside some finite ball $\rho \le r$. We compute μ explicitly. We argue that $\mu(z,\overline{z})|dz\,d\overline{z}|$ plays the same role as the measure $e^{-|z|^2}|dz\,d\overline{z}|$ on $\mathbb{C}^n$ which gives rise to Fock space.

REMARK 5.1. A better formulation of (5.2) comes by using half-forms. Cf. [B-K1] and [B2]. For later comparison we sketch how to translate our results here into the half-form language. First (2.2) gives $\sqrt{\omega\wedge\omega} = \sqrt{16\rho}\sqrt{|dz\,d\overline{z}|}$. Next a function $f(z) \in H^{[m]}$ is replaced by the half-form $s(z) = f(z)\sqrt{dz}$. Then $|s(z)|^2 = s(z)\overline{s(z)} = |f(z)|^2\sqrt{|dz\,d\overline{z}|}$. Finally $\mu|dz\,d\overline{z}|$ is replaced by the half-form $\check{\mu} = \mu\sqrt{|dz\,d\overline{z}|} = \gamma\sqrt{\omega\wedge\omega}$. Then the quantity to be integrated over Z is $|s(z)|^2\check{\mu} = |f(z)|^2\gamma(z,\overline{z})\sqrt{16\rho}|dz\,d\overline{z}|$.

PROPOSITION 5.2. *Choose* $m \in \{0, 1, 2\}$. *Suppose that* $\mu|dz\, d\overline{z}|$ *is an admissible measure on Z and consider the corresponding Hermitian pairing on* $H^{[m]}$ *defined by (5.2). Then the following conditions are equivalent:*

(i) A multiple of the pairing is positive definite with reproducing kernel $\Psi^{[m]}$.

(ii) The pairing is non-degenerate and $\mathfrak{g}$*-invariant with respect to the representation* (4.25).

(iii) $\mu(z, \overline{z})$ *is a smooth non-zero function of* ρ *and* μ *satisfies*

$$\frac{1}{16E}\frac{\partial^4}{\partial z_0^4}\mu(z,\overline{z}) = \overline{z}_0^4\mu(z,\overline{z}). \tag{5.3}$$

(iv) There is a smooth non-zero function $F(x)$ *on* $(0,\infty)$ *such that*

$$\mu(z,\overline{z}) = F\left(\sqrt{\frac{\rho}{2}}\right), \tag{5.4}$$

and $F(x)$ *satisfies the order* 4 *differential equation*

$$16\left(x\frac{d}{dx}+1\right)\left(x\frac{d}{dx}+5\right)F = \frac{d^4}{dx^4}F. \tag{5.5}$$

PROOF. The equivalence of (i) and (ii), for any Hermitian pairing on $H^{[m]}$, follows from Theorem 4.2 and Corollary 4.5; notice in (i) that since $\Psi^{[m]}$ is SU(2)-invariant, it can be the reproducing kernel of only an SU(2)-invariant pairing. Furthermore (ii) is equivalent to the combination of the pairing being SU(2)-invariant and the operators $T(\overline{z}_0^4) = \frac{1}{16E}\frac{\partial^4}{\partial z_0^4}$ and z_0^4 being adjoint. Now $\langle\ ,\ \rangle$ as defined by (5.2) is SU(2)-invariant iff $\mu(z,\overline{z})$ is SU(2)-invariant. This follows as the SU(2)-action on $H^{[m]}$ is induced from the natural SU(2)-action on Z (see Section 2). Since the SU(2)-action on Z is free and its orbits are the level surfaces of $|z|$, any smooth SU(2)-invariant function is a smooth function of $|z|$. Thus $\langle\ ,\ \rangle$ is SU(2)-invariant iff μ is of the form (5.4).

Next we show that $T(\overline{z}_0^4)$ and z_0^4 are adjoint iff μ satisfies (5.3). Clearly $T(\overline{z}_0^4)$ and z_0^4 are adjoint iff $\frac{\partial^4}{\partial z_0^4}$ and $16z_0^4E$ are adjoint, so iff for all $f, g \in H^{[m]}$ we have $\langle \frac{\partial^4 f}{\partial z_0^4}, g\rangle = \langle f, 16z_0^4Eg\rangle$. In terms of (5.2), this means

$$\int_Z \frac{\partial^4 f}{\partial z_0^4}\,\overline{g}\,\mu|dz\,d\overline{z}| = 16\int_Z f\,(\overline{z_0^4Eg})\,\mu|dz\,d\overline{z}|. \tag{5.6}$$

We will rewrite both of these integrals using the formula

$$\int_Z \frac{\partial f}{\partial z_0}\overline{g}\mu|dz\,d\overline{z}| = -\int_Z f\frac{\partial(\overline{g}\mu)}{\partial z_0}|dz\,d\overline{z}| = -\int_Z f\overline{g}\frac{\partial\mu}{\partial z_0}|dz\,d\overline{z}|. \tag{5.7}$$

To get the first equality we integrate by parts and check that the boundary term vanishes. Indeed the boundary term is $\lim_{r\to\infty}\int_{\rho=r}\eta - \lim_{r\to 0}\int_{\rho=r}\eta$ where $\eta = f\overline{g}\mu\, dz_0\, d\overline{z}_0\, d\overline{z}_1$ is

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the boundary three-form on Z. But our growth assumptions on μ imply that both limits are zero.

Now by (5.7) (iterated four times), the left hand side of (5.6) is

$$\int_Z \frac{\partial^4 f}{\partial z_0^4}\overline{g}\,\mu|dz\,d\overline{z}| = \int_Z f\overline{g}\,\frac{\partial^4\mu}{\partial z_0^4}\,|dz\,d\overline{z}|.$$

On the other hand, it follows from (5.7) that

$$\int_Z f(\overline{Eg})\mu|dz\,d\overline{z}| = -\int_Z f\overline{g}\big((E+2)\mu\big)|dz\,d\overline{z}|. \tag{5.8}$$

Since the substitution $E \longrightarrow -E-2$ transforms $z_0^4\mathcal{E} = (E-3)(E+1)z_0^4$ into $\mathcal{E}z_0^4$, we find that the right hand side of (5.6) is

$$16\int_Z f\,(z_0^4\overline{\mathcal{E}g})\,\mu|dz\,d\overline{z}| = 16\int_Z f\,\overline{g}\,(\overline{z}_0^4\mathcal{E}\mu)|dz\,d\overline{z}|.$$

Noting that $\overline{z}_0^4\mathcal{E} = \mathcal{E}\overline{z}_0^4$ since E is holomorphic, we conclude that (5.6) is equivalent to (5.3). This proves that (ii)⇔(iii).

Next we show (iii)⇔(iv). The problem is to show that equations (5.5) and (5.3) are equivalent where we take $\mu = F(|z|)$ because of (5.4). The equivalence is easy since $\frac{\partial\mu}{\partial z_a} = \overline{z}_a F'(x)$ and $E\mu = xF'(x)$ where $x = |z|$. ∎

Our application of integration by parts in the proof above was modeled after the standard calculations (see, *e.g.*, [Fo]) for the oscillator representation.

Recall the definition of $P^{[m]}(x)$ from (4.21).

COROLLARY 5.3. *$F(x)$ is a solution to (5.5) iff F is a solution to*

$$\frac{1}{16\mathcal{E}}\frac{\partial^4}{\partial z_0^4}F(\lambda) = \overline{w}_0^4 F(\lambda). \tag{5.9}$$

The solutions of (5.5) are the functions $F(x)$ of the form, for $r_j \in \mathbb{R}$,

$$F(x) = r_0\Psi^{[0]}(x) + r_1\Psi^{[1]}(x) + r_2\Psi^{[2]}(x) + r_3P^{[3]}(x). \tag{5.10}$$

In particular every solution $F(x)$ extends to a holomorphic function on the complex plane.

PROOF. The first statement is clear from the previous proof. To see the rest, we go back to the proof of Theorem 4.2. In the last step, we deduced the recursive formula (4.20) from (4.13) and (4.16). From this and (4.11), we see the following is true: the function $F(\lambda) = \sum_{n\in\mathbb{Z}_+} t_n\lambda^n$ is a solution to (5.9) iff the t_n satisfy (4.20). The four functions $P^{[m]}(x)$, $m = 0, 1, 2, 3$ are linearly independent and their coefficients satisfy the recursion (4.20). The result follows. ∎

The following gives a holomorphic version of (5.3).

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COROLLARY 5.4. *Suppose $m \in \{0,1,2\}$ and $\mu(z,\overline{z})$ satisfies the equivalent conditions in Theorem 5.2. Then $\mu(z,\overline{z})$ extends to a holomorphic function $\tilde{\mu}$ on $Z \times \overline{Z}$ and $\tilde{\mu}(z,\overline{w}) = \mu(\lambda)$ is a holomorphic function of λ.*

Furthermore $\mu(\lambda)$, like $\Psi^{[m]}(\lambda])$ for $m = 0,1,2$, is a solution to (5.9).

The last statement says that the measure function μ and the three reproducing kernels satisfy the same order 4 differential equation. This is a key result and leads to a formula for μ in Theorem 5.5 below.

The Meijer G-function of type $\binom{3,0}{1,3}$

$$G(u) = G^{3,0}_{1,3}\Big(u\Big|^{\alpha}_{\beta_1,\beta_2,\beta_3}\Big) \tag{5.11}$$

is defined by the contour integral

$$G(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\sigma+\beta_1)\Gamma(\sigma+\beta_2)\Gamma(\sigma+\beta_3)}{\Gamma(\sigma+\alpha_1)} u^{-\sigma}\, d\sigma$$

where σ is a complex parameter and $c > 1/4$. The Mellin transform of G is given by (see, *e.g.*, [Ma], [P-W])

$$\int_0^\infty u^{\sigma-1} G(u)\, du = \frac{\Gamma(\sigma+\beta_1)\Gamma(\sigma+\beta_2)\Gamma(\sigma+\beta_3)}{\Gamma(\sigma+\alpha_1)}. \tag{5.12}$$

THEOREM 5.5. *Choose $m \in \{0,1,2\}$. Then there is a unique admissible measure $\mu(z,\overline{z})|dz\, d\overline{z}|$ on Z such that (5.2) gives the Hermitian inner product on $H^{[m]}$ with reproducing kernel $\Psi^{[m]}$. Furthermore, up to positive multiple, $\mu(z,\overline{z})$ is independent of m and is given by a Meijer G-function so that*

$$\mu(z,\overline{z}) = G\left(\frac{\rho^2}{4}\right) \quad \textit{where} \quad G(u) = G^{3,0}_{1,3}\left(u\Bigg|^{-\frac{1}{4}}_{0,\frac{1}{4},\frac{1}{2}}\right). \tag{5.13}$$

We have

$$\mu(\lambda) = r_0\Psi^{[0]}(\lambda) + r_1\Psi^{[1]}(\lambda) + r_2\Psi^{[2]}(\lambda) \tag{5.14}$$

where $r_0, r_2 < 0$, $r_1 > 0$ and

$$r_0 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(-\frac{1}{4})}, \qquad r_1 = \frac{\Gamma(\frac{1}{4})\Gamma(-\frac{1}{4})}{\Gamma(-\frac{1}{2})}, \qquad r_2 = \frac{\Gamma(-\frac{1}{4})\Gamma(-\frac{1}{2})}{\Gamma(-\frac{3}{4})}. \tag{5.15}$$

Finally the asymptotics of $\mu(z,\overline{z})$ are

$$\begin{aligned} &\textit{as } \rho \longrightarrow 0, && \mu(z,\overline{z}) \longrightarrow r_0 \\ &\textit{as } \rho \longrightarrow \infty, && \mu(z,\overline{z}) \sim \frac{\sqrt{\pi}}{2}\,\rho e^{-\rho}. \end{aligned}$$

In particular, there exist $a, b \in \mathbb{R}^+$ such that $\mu(z,\overline{z})$ is negative if $\rho < a$ while $\mu(z,\overline{z})$ is positive for $\rho > b$.

PROOF. Because of Proposition 5.2, we write $\mu(z,\overline{z}) = F(x)$ where $x = \sqrt{\rho/2}$ and the problem is to find solutions $F(x)$ of (5.5) which exhibit appropriate growth behavior as $x \longrightarrow 0$ and as $x \longrightarrow \infty$. Corollary 5.3 implies that every solution is bounded at $x \longrightarrow 0$. So the problem is to find solutions which decay, and decay fast enough, as $x \longrightarrow \infty$. However the asymptotic theory of differential equations of the type of (5.5) is well-known (see [P-W, Chapter 3]). It turns out that one can construct (using Meijer G-functions) a fundamental system of solutions of (5.5) which, as $x \longrightarrow \infty$, are asymptotic to e^{-2x^2}, e^{2x^2}, x^{-1}, and x^{-5} respectively. It follows that there is a unique solution $F(x)$ asymptotic to e^{-2x^2}, and this is the unique solution up to scaling that decays fast enough at infinity for us. This solution $F(x)$ is given by (see [P-W, p. 85])

$$F(x) = G^{4,0}_{2,4}\Big(x^4\Big|^{-\frac14,\frac34}_{0,\frac14,\frac12,\frac34}\Big) = G^{3,0}_{1,3}\Big(x^4\Big|^{-\frac14}_{0,\frac14,\frac12}\Big). \tag{5.16}$$

There is a well-known formula (see [Ma, th. 2.3, p. 98]) that expresses a Meijer G-function $G(u)$ of type $\binom{r,s}{p,q}$ as a linear combination of r terms of the type ${}_pF_{q-1}(\pm u)u^b$. Applied to $G^{3,0}_{1,3}$ with $\beta_1, \beta_2, \beta_3$ distinct this gives

$$G^{3,0}_{1,3}\Big(u\Big|^{\alpha}_{\beta_1,\beta_2,\beta_3}\Big) = \sum_{k=1}^{3} \frac{\Gamma(\beta_i-\beta_k)\Gamma(\beta_j-\beta_k)}{\Gamma(\alpha-\beta_k)}\, {}_1F_2(1+\beta_k-\alpha; 1+\beta_k-\beta_i, 1+\beta_k-\beta_j; u)\, u^{\beta_k}$$

where in the summation $\{i,j,k\} = \{1,2,3\}$. This gives

$$F(x) = {}_1F_2\Big(\frac54; \frac34, \frac12; x^4\Big) r_0 + {}_1F_2\Big(\frac32; \frac54, \frac34; x^4\Big) r_1 x + {}_1F_2\Big(\frac74; \frac32, \frac54; x^4\Big) r_2 x^2$$

where (5.15) gives r_0, r_1, r_2. Clearly the constant term r_0 dominates as $x \longrightarrow 0$. Because of (5.4) this means $\mu \longrightarrow 0$ as $\rho \longrightarrow 0$.

The asymptotic of $G^{q,0}_{p,q}(u)$ as $u \longrightarrow \infty$ is computed in a formula due to Barnes (see [P-W, th. 3, p. 32]). This gives, putting $\beta = \beta_1 + \beta_2 + \beta_3$,

$$G^{3,0}_{1,3}\Big(u\Big|^{\alpha}_{\beta_1,\beta_2,\beta_3}\Big) \sim \sqrt{\pi}\, u^{\beta-\alpha-\frac12} e^{-2\sqrt{u}}.$$

This produces our asymptotic for μ as $\rho \longrightarrow \infty$. ∎

Notice that the Mellin transform formula (5.12) gives a negative value for $\int_0^\infty u^{\sigma-1}G(u)\,du$ when G is given by (5.13) and $\sigma \in (0, 1/4)$. This alone implies that $G(u)$ must assume negative values somewhere on $(0,\infty)$.

REMARK 5.6. (i) Returning to the discussion in Remark 5.1, we find $\gamma(z,\bar z) = \frac{1}{\sqrt{32}} x^{-1} F(x)$ for $x = \sqrt{\rho/2}$ and so (5.16) gives

$$\gamma(z,\bar z) = \frac{1}{\sqrt{32}} G^{3,0}_{1,3}\Big(\frac{\rho^2}{4}\Big|^{-\frac12}_{-\frac14,0,\frac14}\Big).$$

(ii) The expression (5.14) for $\mu(\lambda)$ in terms of the reproducing kernels marvellously seems to recognize the fact that in Section 4 we did not get a quantization on the fourth space $H^{[3]}$.

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THEOREM 5.7. *Choose $m \in \{0,1,2\}$. Let $\mathcal{H}^{[m]}$ be the Hilbert space of holomorphic functions with reproducing kernel $\Psi^{[m]}$ which we constructed in Section 4 as the formal completion of $H^{[m]}$. Define $\mu(z,\bar{z})$ by* (5.13).

Then $\mathcal{H}^{[m]}$ is the space of all holomorphic functions f on Z such that f has $\mathbb{Z}_4$-weight χ^m and f is square integrable with respect to the positive measure $|\mu\, dz\, d\bar{z}|$. The inner product on $\mathcal{H}^{[m]}$ is defined by the non-positive measure $\mu|dz\, d\bar{z}|$ by (5.2) so that the norm of $f \in \mathcal{H}^{[m]}$ is given by the absolutely convergent integral

$$\|f\|^2 = \int_Z |f(z)|^2 \mu(z,\bar{z}) |dz\, d\bar{z}|. \tag{5.17}$$

PROOF. If ν is any positive measure on Z, then the space of holomorphic functions on Z which are square integrable with respect to ν form a Hilbert space $\mathcal{L}^2(Z,\nu)$ with norm given by $\|f\|_\nu^2 = \int_Z |f|^2 \nu$. We may take $\nu = |\mu\, dz\, d\bar{z}|$. Then $\mathbb{C}[Z]$ lies in $\mathcal{L}^2(Z,\nu)$ as a dense subspace. Since ν is SU(2)-invariant, it follows that the spaces $\mathbb{C}_n[Z]$ are all orthogonal to each other in $\mathcal{L}^2(Z,\nu)$. For f holomorphic we may write $f = \sum_{n\in\mathbb{Z}_+} f_n$ where $f_n \in \mathbb{C}_n[Z]$. Then

$$\|f\|_\nu^2 = \sum_{n\in\mathbb{Z}_+} \int_Z |f_n|^2\, |\mu|\, |dz\, d\bar{z}|. \tag{5.18}$$

The $\mathbb{Z}_4$ action induces an orthogonal decomposition $\mathcal{L}^2(Z,\nu) = \oplus_{s=0}^3 \mathcal{K}^{[s]}$ where $\mathcal{K}^{[s]}$ is the subspace of functions of weight χ^s. On the other hand, by Corollary 4.4 and Theorem 5.5, $\mathcal{H}^{[m]}$ the space of holomorphic functions $f = \sum_{n\in 4\mathbb{Z}_+ + m} f_n$ such that $\|f\|^2 < \infty$ where

$$\|f\|^2 = \sum_{n\in\mathbb{Z}_+} \int_Z |f_n|^2\, \mu\, |dz\, d\bar{z}|. \tag{5.19}$$

(Each term of the series is non-negative by Theorem 5.5.)

Our aim is to show $\mathcal{K}^{[m]} = \mathcal{H}^{[m]}$ where $m \in \{0,1,2\}$. Clearly convergence of the series in (5.18) implies convergence of the series in (5.19); hence $\mathcal{K}^{[m]} \subset \mathcal{H}^{[m]}$. The problem is to prove the converse.

We have two different SU(2)-invariant Hermitian positive definite inner products on $\mathbb{C}_n[Z]$ ($n = 4k+m$) corresponding to $\mathcal{K}^{[m]}$ and $\mathcal{H}^{[m]}$, with respective norms $\|f_n\|_\nu$ and $\|f_n\|$. Consequently $\|f_n\|_\nu = c_n \|f_n\|$ for some positive scalar c_n. We claim that $c_n \longrightarrow 1$ as $n \longrightarrow \infty$. To demonstrate this, it suffices to consider $f_n = z_0^n$. To begin with, we choose τ so that μ is positive outside the ball where $\rho \le \tau$. Let $I_n = \int_{\rho\le\tau} |f_n|^2\, |\mu|\, |dz\, d\bar{z}|$. Then $2I_n > \|f_n\|_\nu^2 - \|f_n\|^2$. But also $I_n / \|f_n\|^2 \longrightarrow 0$ as $n \longrightarrow \infty$ follows since (i) $I_n \le A\tau^{2n}$ where A is a constant independent of n and (ii) $\|f_n\| >^2 n!$ for n large because (see the proof of Theorem 4.2) $\|z_0^n\|^2 = 1/t_n$ where t_n was given by (4.20). Consequently $\|f_n\|_\nu / \|f_n\| \longrightarrow 1$ as $n \longrightarrow \infty$ as claimed. But then, given (any) $\kappa > 1$, we can find N such that $\|f_n\|_\nu < \kappa \|f_n\|$ for $n > N$. It follows that convergence in (5.19) implies convergence in (5.18), and so $\mathcal{H}^{[m]} \subset \mathcal{K}^{[m]}$.

Finally we want to show that if $f \in \mathcal{H}^{[m]}$ then the integral in (5.17) computes the sum in (5.19) giving $\|f\|^2$. But $\int_Z |f|^2 \mu |dz\, d\bar{z}|$ defines a continuous function on $\mathcal{H}^{[m]}$ which coincides with $\|f\|^2$ on $H^{[m]}$ by Theorem 5.5. The continuity follows easily using the previous paragraph. Thus $\int_Z |f|^2 \mu |dz\, d\bar{z}| = \|f\|^2$. ∎

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COROLLARY 5.8. *Choose $m \in \{0,1,2\}$. Then $H^{[m]}$ is the space of all holomorphic functions $f \in R^{\mathrm{hol}}_{\lambda^m}(Z)$ such that the limit $\ell(f) = \lim_{r\to\infty} \int_{\rho<r} |f(z)|^2 \mu(z,\overline{z}) |dz\, d\overline{z}|$ exists. Furthermore $\|f\|^2 = \ell(f)$ whenever $\ell(f)$ exists.*

To round out this discussion, we compute the exact reproducing kernel for the inner product on $H^{[0]} \oplus H^{[1]} \oplus H^{[2]}$ defined by our non-positive measure. This fixes the scaling of the $\Psi^{[m]}$, $m = 0,1,2$, relative to each other.

In fact, Lemma 5.9 and Proposition 5.10 reprove the part of Theorem 5.5 which says that if μ is given by the G-function formula (5.13) then (5.2) gives the correct inner product on $H^{[m]}$ (*i.e.*, with reproducing kernel $\Psi^{[m]}$). Thus we get Theorem 5.5 minus the uniqueness of μ. The approach to finding μ given below is in fact the one we originally used to discover the existence of μ and the G-function formula.

LEMMA 5.9. *Let $f(z) = z_0^j z_1^k$ and let $n = j+k$. Suppose $\mu(z,\overline{z})|dz\,d\overline{z}|$ is an admissible measure on Z and $\mu(z,\overline{z})$ is given by (5.4). Then*

$$\int_Z |f(z)|^2 \mu(z,\overline{z}) |dz\, d\overline{z}| = \pi^2 \frac{j!\,k!}{(n+1)!} \int_0^\infty x^{\frac{n}{4}-\frac{1}{2}} F(x^{\frac{1}{4}})\,dx.$$

PROOF. In polar coordinates $z_a = r_a e^{i\theta_a}$ for $a = 0,1$. Then $|dz_a\, d\overline{z_a}| = 4 r_a\, dr_a\, d\theta_a$. Substituting and integrating out the θ_0, θ_1 variables we find

$$\int_Z |f|^2 \mu |dz\, d\overline{z}| = 16\pi^2 \int_0^\infty \int_0^\infty r_0^{2j+1} r_1^{2k+1} F(\sqrt{\rho/2})\, dr_0\, dr_1.$$

Next we construct a complex variable $r_0 + ir_1$ and take its polar representation $r_0 + ir_1 = re^{i\theta}$. Then $\rho = 2r^4$ and we find

$$\int_Z |f|^2 \mu |dz\, d\overline{z}| = 16\pi^2 \int_0^\infty r^{2j+2k+3} F(r^2)\, dr \int_0^{\frac{\pi}{2}} \cos^{2j+1}\theta \sin^{2k+1}\theta\, d\theta.$$

We evaluate the second integral as $\frac{1}{2}\Gamma(j+1)\Gamma(k+1)/\Gamma(j+k+2)$ ([W-W, p. 256]). Now substituting $x = r^8$ we get our result. ∎

PROPOSITION 5.10. *Choose $m \in \{0,1,2\}$. Suppose $\mu(z,\overline{z})|dz\,d\overline{z}|$ is an admissible measure on Z and $\mu(z,\overline{z})$ is given by (5.4). Then (5.2) defines a Hermitian inner product on $H^{[m]}$ with reproducing kernel $\Psi^{[m]}$ iff*

$$\int_0^\infty x^{\tau-1} F(x^{\frac{1}{4}})\, dx = \frac{\Gamma(\tau+\frac{1}{2})\Gamma(\tau+\frac{1}{4})\Gamma(\tau)}{w_m \Gamma(\tau-\frac{1}{2})} \tag{5.20}$$

for all $\tau = k + \frac{m}{4} + \frac{1}{2}$ with $k \in \mathbb{Z}_+$ where $(w_0, w_1, w_2) = \frac{\pi^3}{4}\big(-\frac{1}{r_0}, \frac{2}{r_1}, -\frac{1}{r_2}\big)$.

PROOF. Let $f = z_0^n$. Write $n = 4k'$ where $m' = m/4$ and $k' = k+m'$. Then Lemma 5.9 gives

$$\|s\|^2 = \frac{\pi^2}{4(k'+\frac{1}{4})} \int_0^\infty x^{k'-\frac{1}{2}} F(x^{\frac{1}{4}})\, dx.$$

On the other hand Theorem 4.2 gives

$$\|s\|^2 = \frac{\Gamma(k'+1)\Gamma(k'+\frac{3}{4})\Gamma(k'+\frac{1}{2})}{\Gamma(k'+\frac{5}{4})}c$$

where $c = \frac{\Gamma(m'+5/4)}{\Gamma(m'+1)\Gamma(m'+3/4)\Gamma(m'+1/2)}$. Then we find

$$\int_0^\infty x^{k'-\frac{1}{2}} F(x^{\frac{1}{4}})\,dx = \frac{\Gamma(k'+1)\Gamma(k'+\frac{3}{4})\Gamma(k'+\frac{1}{2})}{w_m\Gamma(k'+\frac{1}{4})}$$

where we used the relation $\Gamma(a+1) = a\Gamma(a)$ for $a = k' + 1/4$ and $w_m^{-1} = 4c/\pi^2$. Next we substitute $\tau = k' + \frac{1}{2}$. This gives (5.20) and some computation gives the formula for w_m. In particular, we use the relations $r_0 r_2 = (3/2)\pi$ and $r_1^2 = 8\pi$. ■

COROLLARY 5.11. *If $\mu(z,\bar{z})$ is given by (5.13) then the corresponding Hermitian inner product (5.2) on $H^{[m]}$ has reproducing kernel equal to $w_m\Psi^{[m]}$. Then $r_0w_0 + r_1w_1 + r_2w_2 = 0$.*

Finally, we show that our non-positive measure from Theorem 5.7 gives rise to a theory of integral transform operators in full analogy to the familiar theory for Fock space. In particular, our quantization operators constructed in Section 4 are all given by integration against a kernel. Choose $m \in \{0,1,2\}$.

We will say a holomorphic function $K(z,\bar{w})$ is the kernel of a $\Psi^{[m]}$-admissible operator $T: H^{[m]} \longrightarrow H^{[m]}$ with respect to $\mu(z,\bar{z})|\,dz\,d\bar{z}|$ if $K_w = K(z,\bar{w})$ lies in $R^{\mathrm{hol}}_{\lambda^m}(Z)$ and

$$(Tf)(w) = \int_Z f(z)K(w,\bar{z})\mu(z,\bar{z})|dz\,d\bar{z}| \tag{5.21}$$

for all $f \in H^{[m]}$. Then T has a unique kernel K and $K(w,\bar{z}) = (T^*\overline{\Psi_w})(z)$. Indeed uniqueness follows easily and we have

$$(Tf)(w) = \langle Tf, \Psi_w\rangle = \langle f, T^*\Psi_w\rangle = \int_Z f(z)(T^*\overline{\Psi_w})(z)\mu(z,\bar{z})|dz\,d\bar{z}|. \tag{5.22}$$

It follows using (4.2) that the kernel of T_ϕ (see Section 4) is $\phi(z,\bar{w})\Psi(z,\bar{w})$ if $\phi(z,\bar{z})$ extends to a holomorphic function $\phi(z,\bar{w})$ on $Z \times \bar{Z}$.

We can now find the kernels, w.r.t. $\mu|dz\,d\bar{z}|$, of our quantization operators $Q(\phi)$ for $\phi \in \mathfrak{g}_{\mathbb{C}}$. By (2.5) or (2.10), each observable $\upsilon \in \mathfrak{p}_{\mathbb{C}}$ extends to a holomorphic function $\upsilon(z,\bar{w})$. Then

$$\text{kernel of } Q(\upsilon) = \upsilon(z,\bar{w})\Psi^{[m]}(z,\bar{w}) \qquad \text{if } \upsilon \in \mathfrak{p}_{\mathbb{C}}. \tag{5.23}$$

For $x \in \mathfrak{k}_{\mathbb{C}}$, the operators $Q(x) = -i\hat{\xi}_x$ are vector fields given by (2.5) and we can just compute $(Q(x)^*\Psi_w^{[m]})(z)$ directly by differentiating $\Psi^{[m]} = \Psi^{[m]}(\lambda)$. We get

$$\text{kernel of } Q(x) = -i\hat{\xi}_x(\lambda)\frac{d\Psi^{[m]}(\lambda)}{d\lambda} \qquad \text{if } x \in \mathfrak{k}_{\mathbb{C}} \tag{5.24}$$

6. **Variation of Quantization and Kaehler Polarization.** In constructing our quantization of $\mathfrak{g}$ in Section 4 we found that the spaces $H^{[m]}$, $m \in \{0,1,2\}$, and the quantization of the observables in $\mathfrak{k}$ into operators on $H^{[m]}$ arose very naturally. Indeed, it is easy to axiomatize their construction, as the Hamiltonian flow of $\phi \in \mathfrak{k}$ preserves the Kaehler polarization of Z. The Hermitian inner product on $H^{[m]}$ was determined up to a then unknown positive scalar factor in each degree. The missing information was the quantization of the observables in $\mathfrak{p}$ and the complete determination of the inner product—in fact the self-adjointness condition made the former determine the latter. We determined the quantization of $\mathfrak{p}$ uniquely by adding to the Dirac axioms.

However having constructed the quantization, we find that it may be modified to produce equivalent quantizations (so equivalent unitary representations of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$). Indeed it follows easily from the proof of Theorem 4.2 that (4.4) still holds on $\mathfrak{g}$ if we replace the operators $Q(v)$ for $v \in \mathfrak{p}$ by the modified operators

$$Q(v_{j-2}) = z_0^j z_1^k \frac{1}{a(E)} + (-1)^k \frac{1}{b(E)} \frac{\partial^4}{\partial z_0^k \partial z_1^j} \tag{6.1}$$

where $a(E)$ and $b(E)$ are any pair functions of the Euler operator E which have positive spectrum on $H^{[m]}$ and satisfy $a(E)b(E) = 16E$. We of course keep the original $Q(\phi)$ for $\phi \in \mathfrak{k}$. Moreover there is a unique (up to one scalar factor) positive definite Hermitian inner product on $H^{[m]}$ such that the the quantization operators are self adjoint. Some care is necessary to insure we still get a holomorphic reproducing kernel.

For instance, if we choose

$$Q(v_{j-2}) = z_0^j z_1^k \frac{1}{(16E)^t} + (-1)^k \frac{1}{(16E)^{1-t}} \frac{\partial^4}{\partial z_0^k \partial z_1^j} \tag{6.2}$$

where $t > -1/2$ then the new reproducing kernel is $\Psi^{[m]} = \sum_{n \in 4\mathbb{Z}_+ + m} g_n \lambda^n$ where

$$g_{n+4} = \frac{\big[16(n+1)(n+5)\big]^{1-2t}}{(n+1)(n+2)(n+3)(n+4)} g_n. \tag{6.3}$$

If $t = 0$, this is our quantization from Section 4. If $t = 1/2$, then (6.3) implies that (up to scaling of $\Psi^{[m]}$) $g_n = 1/n!$. Hence $\Psi^{[m]}$ is equal to the subseries of $e^\lambda = \sum_{n \in \mathbb{Z}_+} \lambda^n / n!$ given by taking only terms $\lambda^n/n!$ where $n \in 4\mathbb{Z}_+ + m$. Consequently we find that the new inner product on $H^{[m]}$ corresponding to $\Psi^{[m]}$ is given by $\langle f, g\rangle = \int_Z f\overline{g} e^{-|z|^2} |dz\, d\overline{z}|$. Thus instead of our measure constructed in Section 5 we find the positive measure $e^{-|z|^2}|dz\, d\overline{z}|$ of Fock space. It would be interesting to work out the measure for every value of $t > -1/2$ (we expect it exists) and see for which values the measure becomes non-positive, and more importantly to see how (5.14) changes especially in connection with Remark 5.6(ii).

The case where $t = 1/2$ yields the quantization operators constructed by Rawnsley and Sternberg in [R-S].

A natural question is whether the variation of quantization we have just constructed corresponds to a variation of Kaehler polarization. The answer we claim is yes. We will

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make a case for this on geometric grounds by first constructing a family of Kaehler structures all with Kaehler form ω and then working out the analog of Corollary 3.2 with respect to the new complex structure. We will see that the new "pseudo-differential symbols look like symbols" of the operators in (6.2).

PROPOSITION 6.1. *Fix $s, c \in \mathbb{R}^+$. Then there is a unique complex structure $\mathbf{J}'$ on Z where the two functions $z'_a = c^{-1}|z|^{s-1}z_a$, $a = 0, 1$, are global holomorphic coordinates. Furthermore $(Z, \mathbf{J}', \omega)$ is a Kaehler manifold with $\rho' = s\rho$ being a global Kaehler potential. Then $\rho' = 2s|cz'|^{4/s}$.*

PROOF. We wish to construct a family of Kaehler structures on Z with Kaehler form ω. To begin with, we build something different—a family of Kaehler structures with complex structure $\mathbf{J}$. If $k \in \mathbb{R}^+$ then the function $|z|^k$ is a Kaehler potential on $(X, \mathbf{J})$; *i.e.*, $(X, \mathbf{J}, i\partial\overline{\partial}|z|^k)$ is a Kaehler manifold. This follows as in Section 2. The analog of (2.1) is

$$\begin{pmatrix} \partial_0\overline{\partial_0} & \partial_0\overline{\partial_1} \\ \partial_1\overline{\partial_0} & \partial_1\overline{\partial_1} \end{pmatrix}|z|^{2k} = k|z|^{2k-4}\begin{pmatrix} k|z_0|^2 + |z_1|^2 & (k-1)z_1\overline{z_0} \\ (k-1)z_0\overline{z_1} & |z_0|^2 + k|z_1|^2 \end{pmatrix} \tag{6.4}$$

and so the matrix is positive definite for $k > 0$.

Now dilation by $|z|^{s-1}$ defines an automorphism σ of Z. Then σ carries $(\mathbf{J}, i\partial\overline{\partial}|z|^k)$ to a new Kaehler structure $(\mathbf{J}', i\partial'\overline{\partial}'(|z'|^k)$. Here $\mathbf{J}'$ has holomorphic coordinates z'_0, z'_1 and the decomposition $d = \partial' + \overline{\partial}'$ is induced by $\mathbf{J}'$. Notice $|z'| = |z|^s$. Using (6.4) we find with some computation (made much shorter using invariant theory) that σ carries $i\partial\overline{\partial}|z|^k$ to $i\partial\overline{\partial}(|z|^{ks}/s)$. Thus $i\partial'\overline{\partial}'(s|z'|^k) = i\partial\overline{\partial}(|z|^{ks})$. Setting $ks = 4$ and inserting the parameter c we get our result. ∎

REMARK 6.2. Proposition 6.1 implies the following. If $r, c \in \mathbb{R}^+$, then $\rho' = c|z|^r$ is a Kaehler potential on $(Z, \mathbf{J})$, *i.e.*, $\omega' = i\partial\overline{\partial}\rho'$ is a Kaehler form. Moreover, $(Z, \mathbf{J}, \omega)$ is symplectomorphic to $(Z, \mathbf{J}, \omega')$ (but not Kaehler isomorphic unless $r = 4$). Consequently this symplectomorphism carries $\mathfrak{g}$ to an isomorphic algebra of observables which are homogeneous of degree r.

Now we rewrite our observables from (2.5) in terms of the new variables z'_0, z'_1 from Proposition 6.1. Throughout, we have $j = \pm 2, \pm 1, 0$ and $j + k = 4$. For convenience we drop the primes from our notation. We find

$$\begin{aligned} x_1 &= 4z_0\overline{z_1}c^{4/s}|z|^{-2+4/s} \\ x_0 &= -2i(|z_0|^2 - |z_1|^2)c^{4/s}|z|^{-2+4/s} \qquad \upsilon_{j-2} = \frac{{z_0}^j{z_1}^k + (-1)^k\overline{z_0}^k\overline{z_1}^j}{c^{-4/s}|z|^{4-4/s}}. \\ \overline{x_1} &= 4z_1\overline{z_0}c^{4/s}|z|^{-2+4/s} \end{aligned} \tag{6.5}$$

Notice that $\upsilon \in \mathfrak{p}$ are sums of holomorphic and antiholomorphic functions iff $s = 1$, *i.e.*, the value that corresponds to our original complex structure $\mathbf{J}$.

Next we apply the methods of Section 3 to extend our observables in $\mathfrak{g}$ to holomorphic functions on T^+Z. We continue to drop the primes. We find that $b^*(\zeta_a) = -4i|z|^{-2+4/s}c^{-1+4/s}\overline{z_a}$.

$$\Phi(x_1) = iz_0\zeta_1$$

(6.6) $$\Phi(x_0) = \tfrac{1}{2}(z_0\zeta_0 - z_1\zeta_1) \qquad \Phi(\upsilon_{j-2}) = \frac{z_0^j z_1^k}{c^{-4}\left(\frac{i\Lambda}{4}\right)^{s-1}} + \frac{(-1)^k \zeta_0^k \zeta_1^j}{256c^4\left(\frac{i\Lambda}{4}\right)^{3-s}}.$$

$$\Phi(\overline{x_1}) = iz_1\zeta_0$$

The fact that $\Phi(x)$ for $x \in \mathfrak{k}$ is independent of choice of $\mathbf{J}'$ is a manifestation of the fact that the Hamiltonian flow of x preserves $\mathbf{J}'$. On the other hand, the "symbol" of the operators $Q\,(\upsilon_{j-2})$ in (6.2) agree with (6.6) when we take $s = 2t + 1$ and $c = 4^{-t}$.

For $s = 2$ and $c = 1/2$ we get $\rho = |z|^2$ and $t = 1/2$. The formulas above reduce to

$$x_1 = z_0\overline{z_1}$$

(6.7) $$x_0 = -\tfrac{i}{2}(|z_0|^2 - |z_1|^2) \qquad \upsilon_{j-2} = \frac{{z_0}^j {z_1}^k + (-1)^k \overline{z_0}^k \overline{z_1}^j}{4|z|^2}$$

$$\overline{x_1} = z_1\overline{z_0}$$

(6.8) $$\Phi(\upsilon_{j-2}) = \frac{z_0^j z_1^k}{4i\Lambda} + (-1)^k \frac{\zeta_0^k \zeta_1^j}{4i\Lambda}.$$

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