DIFFERENCES OF SETS AND A PROBLEM OF GRAHAM

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R. L. Graham has posed the following question:

Given n positive integers $a_1 < a_2 < \dots < a_n$, does there exists a pair of indices i, j such that $a_i/(a_i, a_j) \geq n$? ((a_i, a_j) = g.c.d. of a_i and a_j).

The answer would be yes if it were possible to prove the stronger property:

(i) there exist n different ratios $a_i/(a_i, a_j)$.

However, this is not true in general as shown by a counter-example of Levin and Szemeredy; namely, the set of all non trivial divisors of 36. There are 7 divisors but only 5 distinct ratios. [This example was described in written communications from M. Levin and P. Erdös].

The following theorem is the combinatorial analogue of (i) and has been conjectured by one of us [1]. The corollary shows the relation to Graham's problem.

THEOREM. *If F is a finite collection of sets then the number of distinct differences of members of F is at least as large as the number of members of F.*

COROLLARY. *If* $a_1 < \ldots < a_n$ *are squarefree integers then the number of distinct ratios* $a_i/(a_i, a_j)$ *is* $\geq n$, *so one is* $\geq n$.

In order to obtain the desired result we introduce the following notation. Let $F = \{F_i\}$ be a finite collection of sets; the cardinal of F_i is denoted by $|F_i|$ and the collection of differences of members of any collection G, by ΔG. Let $k = \min|F_i \cap F_j|$ for $F_i \neq F_j$ and let F_1, F_2 be two fixed sets for which this minimum is attained, $F_1 \cap F_2 = I$, $|I| = k$.

LEMMA. *If* F *is any finite non-empty collection of sets there is a partition of* F *into disjoint subcollections* A *and* D, *with* $A \neq \phi$, *satisfying* $|\Delta F| \geq |A| + |\Delta D|$.

Proof of Lemma. Divide F into three disjoint subcollections A, B, C according to the following criteria:

(i) $C = \{$members of F which do not contain $I\}$.

The rest of the sets do contain I and we write $F_i = F_i' + I$ where $F_i' = F_i - I$, for such sets. Then,

(ii) $B = \{F_i:$ for all $F_j \notin C, F_i' \cap F_j' \neq \phi\}$.

(iii) $A = \{F_i:$ for some $F_j \notin C, F_i' \cap F_j' = \phi\}$.

It is clear that $A \neq \phi$ since at least F_1, F_2 are in A. If $F_i \in A$ and F_j is as in (iii) then F_j is also in A, F_i' and F_j' are disjoint and so appear in ΔA, $(F_i - F_j = F_i')$. We can see that F_i', F_j' do not occur in $\Delta(B \cup C)$ as follows. That each set in B has a non-empty intersection with F_i' is immediate from the definition of B. No member Q of C can be disjoint from F_i'; for $|Q \cap F_i| \geq k$ and

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since $Q \cap F_i \neq I$ (from (i)) $Q \cap F'_i = Q \cap (F_i - I) \neq \phi$. If now X, $Y \in B \cup C$ then $X - Y \neq F'_i$ because $X - Y$ contains no element of Y while F'_i does contain some element of Y. This holds for any F_i in A.

We have found then, that for each member F_i of A there is a difference F'_i appearing in ΔA which does not appear in $\Delta(B \cup C)$. Clearly $F_i \neq F_j$ implies $F'_i \neq F'_j$, and the lemma is proved.

Proof of Theorem. (By induction). The theorem clearly holds for collections of 1 or 2 sets. If F were a collection of minimal cardinal for which it failed, then taking $F = A \cup D$ as above we would have $|\Delta F| \geqslant |A| + |\Delta D|$; but $A \neq \phi$ so $|D| < |F|$ and by induction $|\Delta D| \geqslant |D|$. Thus $|\Delta F| \geqslant |A| + |D| = |F|$, a contradiction.

Remark. Let $K(n,F)$ denote $|\Delta F|$ for F a collection of n sets. We have shown $K(n,F) \geqslant n$ and since $F_i \in F$ implies $F_i - F_i = \phi$ it is clear that $K(n,F) \leqslant n^2 - n + 1$. It can be shown that both of these bounds are attained for each n with a suitable F. However, one can still ask which restrictions can be imposed on F in order to yield more precise but still usefull results, e.g., ϕ and $\cup F_i \notin F$.

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REFERENCE

1. J. Schonheim, Unsolved problem. (W.T. Tutte, Recent Progress in Combinatorics. Proc. Third Waterloo Conference, Academic Press, to appear).

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