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ON BASE RADICAL OPERATORS FOR CLASSES OF FINITE ASSOCIATIVE RINGS

R. G. MCDOUGALL and L. K. THORNTON✉

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Abstract

In this paper, class operators are used to give a complete listing of distinct base radical and semisimple classes for universal classes of finite associative rings. General relations between operators reveal that the maximum order of the semigroup formed is 46. In this setting, the homomorphically closed semisimple classes are precisely the hereditary radical classes and hence radical–semisimple classes, and the largest homomorphically closed subclass of a semisimple class is a radical–semisimple class.

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1. Introduction

A number of papers on radical theory begin by citing the work of Wedderburn [15] concerned with a class of finite-dimensional algebras over fields, where the collection of nilpotent algebras forms what we would now call a radical class. In the century since, the theory established in classes of associative rings [1, 7] has been extended using notions appropriate to the setting (for example, radical classes in categories [3] and connectedness and disconnectedness in topology [2]) and much of radical theory has been described in a very general way [8]. We have set this work in the universal class of finite associative rings to describe various compositions of class operators which have become essential for working in this area and to uncover more of the duality lost along the way [4]. Our main results include a complete listing of distinct class operators that generate base radical and semisimple classes for any subclass of the universal class of finite associative rings. In the universal class of all associative rings, it is well known that a homomorphically closed semisimple class is a radical class [16], but, among other results, we show that for the finite case, the homomorphically closed semisimple classes are precisely the hereditary radical classes and hence radical–semisimple classes (as is the case for classes of finite groups [6]) and that the largest homomorphically closed subclass of a semisimple class is a radical–semisimple class.

2. Background preliminaries

For background material on the radical theory of associative rings not mentioned here, we refer the reader to the text by Gardner and Wiegandt [5]. Classes are denoted by calligraphic font such as $\mathcal{X}$ and elements of a class by uppercase letters such as A. A class $\mathcal{X}$ is *hereditary* if, for all $A \in \mathcal{X}$, every ideal I of A (denoted $I \triangleleft A$) is in $\mathcal{X}$, and $\mathcal{X}$ is *homomorphically closed* if, for all $A \in \mathcal{X}$, $I \triangleleft A$ implies the homomorphic image $A/I \in \mathcal{X}$. A class $\mathcal{X}$ is *closed under extensions* if $\mathcal{X}$ has the property that whenever I is an ideal of a ring A and both I and A/I are elements of $\mathcal{X}$, then $A \in \mathcal{X}$. A universal class is hereditary and homomorphically closed. In what follows, 0 is sometimes used to represent the class $\{0\}$ when the context is clear.

If A is nonzero with no nonzero proper ideals, then A is *simple*. Define a subring J to be an *accessible subring* of a ring A if there is a finite chain $J = J_n \triangleleft J_{n-1} \triangleleft \cdots \triangleleft J_0 = A$. The class operators $\mathbf{U}$ and $\mathbf{S}$ acting on subclasses $\mathcal{X}$ of $\mathcal{A}$ are defined by

$$\mathbf{U}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero homomorphic image in } \mathcal{X}\}$$

and

$$\mathbf{S}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero accessible subring in } \mathcal{X}\}.$$

For the trivial classes in $\mathcal{A}$, $\mathbf{U}(0) = \mathbf{S}(0) = \mathcal{A}$ and $\mathbf{U}(\mathcal{A}) = \mathbf{S}(\mathcal{A}) = 0$ and, for all subclasses $\mathcal{X} \subseteq \mathcal{A}$, we have $\mathcal{X} \cap \mathbf{U}(\mathcal{X}) = \mathcal{X} \cap \mathbf{S}(\mathcal{X}) = 0$. We can often interchange the operators $\mathbf{U}$ and $\mathbf{S}$ to produce meaningful outcomes called *dual* results. The proof arguments are said to be *dualised* and are only sometimes included.

The *base radical operator* $\mathbf{US}$ and *base semisimple operator* $\mathbf{SU}$ are then $\mathbf{US}(\mathcal{X}) = \{A \in \mathcal{A} \mid \text{every nonzero homomorphic image of } A \text{ has a nonzero accessible subring in } \mathcal{X}\}$ and $\mathbf{SU}(\mathcal{X}) = \{A \in \mathcal{A} \mid \text{every nonzero accessible subring of } A \text{ has a nonzero homomorphic image in } \mathcal{X}\}$ [9]. A class $\mathcal{X} \subseteq \mathcal{A}$ is a *base radical class* if and only if $\mathcal{X} = \mathbf{US}(\mathcal{X})$ and, dually, $\mathcal{X}$ is a *base semisimple class* if and only if $\mathcal{X} = \mathbf{SU}(\mathcal{X})$ [10]. Some of the arguments to follow require the image to be nonzero but many comments still hold for the zero image and the zero accessible subring, so we omit this word when appropriate. A class $\mathcal{X}$ is called *radical–semisimple* if $\mathcal{X}$ is both a radical class and a semisimple class. For classes of associative rings, Kurosh–Amitsur radical classes are precisely base radical classes and, in this setting, Kurosh–Amitsur semisimple classes are hereditary, so their respective semisimple classes coincide [10].

The notion of generating an operator semigroup through repeated application of the $\mathbf{U}$ and $\mathbf{S}$ operators on all subclasses $\mathcal{X} \subseteq \mathcal{A}$ was introduced in [9] and some general properties of this semigroup and its possible orders in a more general setting, which includes finite associative rings, were detailed in [14]. The semigroup, denoted $RT_{\mathcal{A}}$ with order $|RT_{\mathcal{A}}|$, contains all distinct operators composed of $\mathbf{U}$ and $\mathbf{S}$ which can act on subclasses of a universal class $\mathcal{A}$. For all $\mathbf{P}, \mathbf{Q} \in RT_{\mathcal{A}}$, $\mathbf{P} = \mathbf{Q}$ if and only if $\mathbf{P}(\mathcal{X}) = \mathbf{Q}(\mathcal{X})$ for all $\mathcal{X} \subseteq \mathcal{A}$. For example, the universal class of finite associative rings $\mathcal{A}_1 = \{0, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2^0\}$ has associated operator semigroup $RT_{\mathcal{A}_1} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{USS}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{USSS}\}$ with $|RT_{\mathcal{A}_1}| = 12$ [14].

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We include an additional class operator $\neg$ for a richer treatment; it is defined for all subclasses $\mathcal{X} \subseteq \mathcal{A}$ by $\neg(\mathcal{X}) = \{A \in \mathcal{A} \mid A \notin \mathcal{X}\}$ and we denote the operator semigroup formed by operators $\mathbf{U}, \mathbf{S}$ and $\neg$ by $RT^*_{\mathcal{A}}$ using an equivalent definition for element equality. If the class $\mathcal{X}$ is homomorphically closed, then $\mathcal{X} = \mathbf{U}\neg(\mathcal{X}) \subseteq \mathbf{US}(\mathcal{X})$, $\mathbf{S}(\mathcal{X})$ is a semisimple class and hence $\mathbf{S}(\mathcal{X}) = \mathbf{SUS}(\mathcal{X})$. Dually, if $\mathcal{X}$ is hereditary, then $\mathcal{X} = \mathbf{S}\neg(\mathcal{X}) \subseteq \mathbf{SU}(\mathcal{X})$, $\mathbf{U}(\mathcal{X})$ is a radical class and so $\mathbf{U}(\mathcal{X}) = \mathbf{USU}(\mathcal{X})$.

From here on, $\mathcal{A}$ denotes the universal class of all finite associative rings and mention of radical and semisimple classes refers to base radical and base semisimple classes, respectively. For every nonzero $A \in \mathcal{A}$, A has a simple accessible subring and a simple homomorphic image in $\mathcal{A}$. Indeed, for all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{US}(\mathcal{X})$ is nonzero if and only if $\mathcal{X}$ contains a simple ring.

The following results are used extensively in the proofs to come.

THEOREM 2.1 [11, Corollary 1]. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{S}\neg\mathbf{US}(\mathcal{X})$ is a radical class. That is, the largest hereditary class contained in a radical class is a radical class.*

THEOREM 2.2 [9, Lemma 4.5, Theorems 2.1(4), 2.1(5) and 4.3]. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$:*

(i) $\mathbf{S}\neg\mathbf{U}\neg\mathbf{S}(\mathcal{X}) = \mathbf{U}\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{S}\neg\mathbf{U}(\mathcal{X})$;
(ii) $\mathbf{U}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}(\mathcal{X})$ *and* $\mathbf{S}\neg\mathbf{S}(\mathcal{X}) = \mathbf{S}(\mathcal{X})$;
(iii) $\mathbf{UUUU}(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$ *and* $\mathbf{SSSS}(\mathcal{X}) = \mathbf{SS}(\mathcal{X})$;
(iv) $\mathbf{S}\neg\mathbf{U}(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{S}(\mathcal{X})$.

When every homomorphic image of each accessible subring of $A \in \mathcal{A}$ is isomorphic to an accessible subring of a homomorphic image of A, the dual of Theorem 2.2(iv) holds and $\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}\neg\mathbf{S}(\mathcal{X})$ [9].

Adjusting from a more general setting [14] to universal classes of finite associative rings gives the results collected in the next two theorems.

THEOREM 2.3 [14, Theorem 1, Remark 6, Proposition 1 and Theorem 2]. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$:*

(i) *if $\mathcal{X}$ is hereditary, then $\mathbf{S}(\mathcal{X})$ is a semisimple class and, if $\mathcal{X}$ is homomorphically closed, then $\mathbf{U}(\mathcal{X})$ is a radical class;*
(ii) *for all $A \in \mathcal{A}$, $A \in \mathbf{UU}(\mathcal{X})$ if and only if $B \in \mathcal{X}$ for every simple homomorphic image B of A; dually, $A \in \mathbf{SS}(\mathcal{X})$ if and only if $C \in \mathcal{X}$ for every simple accessible subring C of A;*
(iii) $\mathbf{USSS}(\mathcal{X}) \subseteq \mathbf{US}(\mathcal{X}) \subseteq \mathbf{UU}(\mathcal{X})$ *and* $\mathbf{SUUU}(\mathcal{X}) \subseteq \mathbf{SU}(\mathcal{X}) \subseteq \mathbf{SS}(\mathcal{X})$;
(iv) $\mathbf{UUS}(\mathcal{X}) = \mathbf{UUU}(\mathcal{X})$ *and* $\mathbf{SSU}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$.

While $\mathbf{U}(\mathcal{X})$ is not always a radical class and $\mathbf{S}(\mathcal{X})$ is not always a semisimple class, it follows from Theorem 2.3(i) that classes of the form $\mathbf{UU}(\mathcal{X})$ and $\mathbf{UUU}(\mathcal{X})$ will be radical classes. Dually, $\mathbf{SS}(\mathcal{X})$ and $\mathbf{SSS}(\mathcal{X})$ are semisimple classes.

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Consider two properties a class $\mathcal{X} \subseteq \mathcal{A}$ may have:

(1*) if $A \in \mathcal{X}$ and $A \neq 0$, then A has a simple homomorphic image in $\mathcal{X}$;
(2*) if $A \in \mathcal{X}$ and $A \neq 0$, then A has a simple accessible subring in $\mathcal{X}$.

Theorem 2.4 [14, Proposition 8]. *A class* $\mathcal{X} \subseteq \mathcal{A}$ *satisfies* (1^*) *if and only if* $\mathbf{U}(\mathcal{X}) = \mathbf{UUU}(\mathcal{X})$. *Dually,* $\mathcal{X}$ *satisfies* (2^*) *if and only if* $\mathbf{S}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$.

3. On radical and semisimple operators

In this section a number of general relations between the elements of the semigroup $RT^*_{\mathcal{A}}$ are described for the universal class of finite associative rings. The relations are then used to give a complete listing of elements in the operator semigroup and identify the operators that always generate radical and semisimple classes. The results that follow rely on containment arguments and so work more generally for subclasses of the universal class $\mathcal{A}$ as well as in the semigroup itself.

Theorem 3.1. *For all subclasses* $\mathcal{X} \subseteq \mathcal{A}$:

(i) $\mathbf{UU}\neg\mathbf{S}(\mathcal{X}) = \mathbf{US}\neg\mathbf{U}(\mathcal{X}) = \mathbf{UUU}\neg(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$;
(ii) $\mathbf{SS}\neg\mathbf{U}(\mathcal{X}) = \mathbf{SU}\neg\mathbf{S}(\mathcal{X}) = \mathbf{SSS}\neg(\mathcal{X}) = \mathbf{SS}(\mathcal{X})$.

Proof. (i) Since $\mathcal{X} \subseteq \neg\mathbf{S}(\mathcal{X})$, $\mathbf{UU}(\mathcal{X}) \subseteq \mathbf{UU}\neg\mathbf{S}(\mathcal{X})$. Let $A \in \mathbf{UU}\neg\mathbf{S}(\mathcal{X})$. Then every simple homomorphic image of A is in $\neg\mathbf{S}(\mathcal{X})$ by Theorem 2.3(ii) and therefore in $\mathcal{X}$. Hence, $A \in \mathbf{UU}(\mathcal{X})$ and so $\mathbf{UU}\neg\mathbf{S}(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$.

From Theorem 2.2(iv), $\mathbf{U}(\mathbf{U}\neg\mathbf{S}(\mathcal{X})) \subseteq \mathbf{U}(\mathbf{S}\neg\mathbf{U}(\mathcal{X}))$ and so $\mathbf{UU}(\mathcal{X}) \subseteq \mathbf{US}\neg\mathbf{U}(\mathcal{X})$. Now, $\mathcal{X} \subseteq \neg\mathbf{U}(\mathcal{X})$ and so $\mathbf{US}(\mathcal{X}) \subseteq \mathbf{US}\neg\mathbf{U}(\mathcal{X})$. By Theorem 2.3(iii), $\mathbf{US}(\neg\mathbf{U}(\mathcal{X})) \subseteq \mathbf{UU}(\neg\mathbf{U}(\mathcal{X}))$ and $\mathbf{UU}\neg\mathbf{U}(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$ by Theorem 2.2(ii). Hence, $\mathbf{US}\neg\mathbf{U}(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$.

To prove the final equality, note that since $\mathbf{U}\neg(\mathcal{X}) \subseteq \mathcal{X}$, then $\mathbf{UUU}\neg(\mathcal{X}) \subseteq \mathbf{UU}(\mathcal{X})$. Suppose that $A \in \mathbf{UU}(\mathcal{X})$. By Theorem 2.3(ii), every simple homomorphic image B of A is in $\mathcal{X}$ and hence $B \in \mathbf{U}\neg(\mathcal{X})$. So, $A \in \mathbf{UU}(\mathbf{U}\neg(\mathcal{X}))$, $\mathbf{UUU}\neg(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$ and $\mathbf{UU}\neg\mathbf{S}(\mathcal{X}) = \mathbf{US}\neg\mathbf{U}(\mathcal{X}) = \mathbf{UUU}\neg(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$.

(ii) The dual arguments of Theorem 3.1(i) show that $\mathbf{SS}\neg\mathbf{U}(\mathcal{X}) = \mathbf{SSS}\neg(\mathcal{X}) = \mathbf{SS}(\mathcal{X})$. From Theorem 2.2(iv), $\mathbf{S}(\mathbf{U}\neg\mathbf{S}(\mathcal{X})) \subseteq \mathbf{S}(\mathbf{S}\neg\mathbf{U}(\mathcal{X})) = \mathbf{SS}(\mathcal{X})$. Since $\mathbf{SS}(\mathcal{X}) \subseteq \neg\mathbf{S}(\mathcal{X})$, it follows that $\mathbf{SU}(\mathbf{SS}(\mathcal{X})) \subseteq \mathbf{SU}(\neg\mathbf{S}(\mathcal{X}))$. Since $\mathbf{S}(\mathcal{X})$ is hereditary, then, by Theorem 2.3(i), $\mathbf{S}(\mathbf{S}(\mathcal{X})) = \mathbf{SUS}(\mathbf{S}(\mathcal{X}))$ and so $\mathbf{SS}(\mathcal{X}) \subseteq \mathbf{SU}\neg\mathbf{S}(\mathcal{X})$, showing that $\mathbf{SU}\neg\mathbf{S}(\mathcal{X}) = \mathbf{SS}(\mathcal{X})$, which completes the proof. □

The following lemmas identify conditions on a subclass $\mathcal{X}$ which ensure that the largest hereditary or homomorphically closed subclass is a semisimple class or a radical class, respectively.

Lemma 3.2. *If a subclass* $\mathcal{X} \subseteq \mathcal{A}$ *is closed under extensions, then* $\mathbf{SU}(\mathcal{X}) = \mathbf{S}\neg(\mathcal{X})$.

Proof. Since $\mathbf{U}(\mathcal{X}) \subseteq \neg(\mathcal{X})$ for all subclasses $\mathcal{X}$ of $\mathcal{A}$, then $\mathbf{S}\neg(\mathcal{X}) \subseteq \mathbf{SU}(\mathcal{X})$. Let $\mathcal{X}$ be closed under extensions and consider a nonzero $A \in \mathbf{SU}(\mathcal{X})$ but $A \notin \mathcal{X}$. Then A is not simple and it has an ideal $I_1 \neq A$ such that A/I_1 is in $\mathcal{X}$. As $\mathcal{X}$ is closed under

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extensions, I_1 is not in $\mathcal{X}$. But I_1 is in $\mathbf{SU}(\mathcal{X})$, so it in turn is neither simple nor zero and has an ideal $I_2 \notin \mathcal{X}$ with $I_1/I_2 \in \mathcal{X}$.

Repeating this argument yields $I_n \triangleleft I_{n-1} \triangleleft \cdots \triangleleft I_2 \triangleleft I_1 \triangleleft I_0 = A$, an arbitrarily long series with $I_0, I_1, \ldots, I_n$ not in $\mathcal{X}$, but all with a nonzero homomorphic image in $\mathcal{X}$. This means that $I_1, \ldots, I_n$ are not simple. But every chain of this kind descending from A must reach a simple ring. From this contradiction, we conclude that there is no such A and thus $\mathbf{SU}(\mathcal{X}) \subseteq \mathcal{X}$. Thus, $\mathbf{SU}(\mathcal{X})$ is a hereditary subclass of $\mathcal{X}$, so $\mathbf{SU}(\mathcal{X}) \subseteq \mathbf{S}\neg(\mathcal{X}) \subseteq \mathbf{SU}(\mathcal{X})$ and $\mathbf{SU}(\mathcal{X}) = \mathbf{S}\neg(\mathcal{X})$. □

As an accessible subring of a ring may not be an ideal of the ring, we adjust the definition of closed under extensions to obtain the dual result.

Lemma 3.3. *For a subclass $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{US}(\mathcal{X}) = \mathbf{U}\neg(\mathcal{X})$ whenever $\mathcal{X}$ satisfies the condition: if C is an accessible subring of an ideal I of A and both C and A/I are in $\mathcal{X}$, then $A \in \mathcal{X}$.*

Proof. Let a subclass $\mathcal{X}$ of $\mathcal{A}$ satisfy the condition described and A be a nonzero element of $\mathbf{US}(\mathcal{X})$. Then A has a nonzero accessible subring in $\mathcal{X}$, say C. If $C = A$, then $A \in \mathcal{X}$ and $\mathbf{US}(\mathcal{X}) \subseteq \mathcal{X}$. If $C \neq A$, then C is the accessible subring of a maximal ideal B of A such that $B \neq A$. The homomorphic image A/B is simple and in $\mathbf{US}(\mathcal{X})$ and hence in $\mathcal{X}$. Therefore, A must be in $\mathcal{X}$, showing that $\mathbf{US}(\mathcal{X}) \subseteq \mathcal{X}$. Now, $\mathbf{U}\neg(\mathcal{X}) \subseteq \mathbf{US}(\mathcal{X}) \subseteq \mathbf{U}\neg(\mathcal{X}) \subseteq \mathcal{X}$ and $\mathbf{US}(\mathcal{X}) = \mathbf{U}\neg(\mathcal{X})$. □

Necessary and sufficient conditions for a class of associative rings to be semisimple are that the class be hereditary, closed under subdirect sums and closed under extensions [12]. In this finite setting we need only two of these three conditions. If $\mathcal{X}$ is hereditary and closed under extensions, then, from Lemma 3.2, $\mathcal{X} = \mathbf{S}\neg(\mathcal{X}) = \mathbf{SU}(\mathcal{X})$ and $\mathcal{X}$ is a semisimple class. For classes closed under extensions, the dual of Lemma 3.2 holds when the property of being an ideal is transitive. If $\mathcal{X}$ is homomorphically closed and satisfies the condition in Lemma 3.3, then $\mathcal{X} = \mathbf{U}\neg(\mathcal{X}) = \mathbf{US}(\mathcal{X})$ and $\mathcal{X}$ is a radical class.

It has been observed that, in general, homomorphically closed semisimple classes are rare [17]. The following theorem shows that in this finite setting, every subclass $\mathcal{X} \subseteq \mathcal{A}$ can generate a homomorphically closed semisimple class.

Theorem 3.4. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{SUU}(\mathcal{X})$ is a radical–semisimple class and $\mathbf{SUU}(\mathcal{X}) = \mathbf{USS}(\mathcal{X})$.*

Proof. The class $\mathbf{SUU}(\mathcal{X})$ is semisimple by definition. By Theorem 2.3(i), $\mathbf{UUU}(\mathcal{X})$ is a radical class for all $\mathcal{X} \subseteq \mathcal{A}$ and hence closed under extensions. Lemma 3.2 gives $\mathbf{SU}(\mathbf{UUU}(\mathcal{X})) = \mathbf{S}\neg(\mathbf{UUU}(\mathcal{X}))$ and, by Theorem 2.2(iii), $\mathbf{SUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$. By Theorem 2.2(i), $\mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$ is homomorphically closed and, since any homomorphically closed semisimple class is a radical class [16], $\mathbf{SUU}(\mathcal{X})$ is a radical–semisimple class. It follows that $\mathbf{SUU}(\mathcal{X}) = \mathbf{US}(\mathbf{SUU}(\mathcal{X}))$ and, by Theorem 2.3(iv), $\mathbf{US}(\mathbf{SUU}(\mathcal{X})) = \mathbf{USSSU}(\mathcal{X}) = \mathbf{USSSS}(\mathcal{X})$. From Theorem 2.2(iii), $\mathbf{USSSS}(\mathcal{X}) = \mathbf{USS}(\mathcal{X})$, showing that $\mathbf{SUU}(\mathcal{X}) = \mathbf{USS}(\mathcal{X})$. □

Corollary 3.5. *If a subclass* $\mathcal{X} \subseteq \mathcal{A}$ *satisfies condition* (2^*)*, then* $\mathbf{US}(\mathcal{X})$ *is a radical–semisimple class. Dually, if* $\mathcal{X}$ *satisfies condition* (1^*)*, then* $\mathbf{SU}(\mathcal{X})$ *is radical–semisimple.*

Proof. Let a subclass $\mathcal{X}$ of $\mathcal{A}$ satisfy condition (2^*). By Theorem 2.4, $\mathbf{S}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$ and therefore $\mathbf{US}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$, which is a radical–semisimple class as a consequence of Theorem 3.4. The dual argument proves the second part. □

The following corollaries reveal a range of radical–semisimple classes with implications for the corresponding operators.

Corollary 3.6. *For all subclasses* $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{SUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{US}\neg(\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUS}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SU}\neg(\mathcal{X}) = \mathbf{U}\neg\mathbf{SSS}(\mathcal{X}) = \mathbf{USS}(\mathcal{X})$.

Proof. First note that for a subclass $\mathcal{X}$ of $\mathcal{A}$, if $A \in \mathbf{US}\neg(\mathcal{X})$, then A has no simple homomorphic image in $\mathcal{X}$ and so $A \in \mathbf{UUU}(\mathcal{X})$. Dually, $\mathbf{SU}\neg(\mathcal{X}) \subseteq \mathbf{SSS}(\mathcal{X})$. From Theorem 2.3(iii), $\mathbf{SU}(\mathcal{X}) \subseteq \mathbf{SS}(\mathcal{X})$ and therefore $\mathbf{USS}(\mathcal{X}) \subseteq \mathbf{USU}(\mathcal{X}) \subseteq \mathbf{US}\neg(\mathcal{X}) \subseteq \mathbf{UUU}(\mathcal{X})$ and similarly $\mathbf{SUU}(\mathcal{X}) \subseteq \mathbf{SUS}(\mathcal{X}) \subseteq \mathbf{SU}\neg(\mathcal{X}) \subseteq \mathbf{SSS}(\mathcal{X})$. Since $\mathbf{USS}(\mathcal{X})$ is hereditary as a consequence of Theorem 3.4, $\mathbf{USS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USS}(\mathcal{X}) \subseteq \mathbf{S}\neg\mathbf{USU}(\mathcal{X}) \subseteq \mathbf{S}\neg\mathbf{US}\neg(\mathcal{X}) \subseteq \mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$. From Theorem 2.2(iv), it follows that $\mathbf{S}\neg\mathbf{U}(\mathbf{UU}(\mathcal{X})) \subseteq \mathbf{U}\neg\mathbf{S}(\mathbf{UU}(\mathcal{X})) \subseteq \mathbf{U}\neg\mathbf{SUS}(\mathcal{X})$ since $\mathbf{US}(\mathcal{X}) \subseteq \mathbf{UU}(\mathcal{X})$ by Theorem 2.3(iii), and $\mathbf{U}\neg\mathbf{SUS}(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{SU}\neg(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{SSS}(\mathcal{X}) \subseteq \mathbf{USSSS}(\mathcal{X}) = \mathbf{USS}(\mathcal{X})$, showing that all classes are equal. Also, as a consequence of Theorem 3.4, $\mathbf{SUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUU}(\mathcal{X})$, which completes the proof. □

Corollary 3.7. *For all subclasses* $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{SUUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USSS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USU}\neg(\mathcal{X}) = \mathbf{S}\neg\mathbf{US}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUS}\neg(\mathcal{X}) = \mathbf{U}\neg\mathbf{SU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SS}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$.

Proof. Using Theorem 3.1 for the subclass $\neg\mathcal{X}$ of $\mathcal{A}$, $\mathbf{UUU}\neg(\neg\mathcal{X}) = \mathbf{UU}\neg(\mathcal{X})$, which implies that $\mathbf{UUU}(\mathcal{X}) = \mathbf{UU}\neg(\mathcal{X})$ and dually $\mathbf{SSS}(\mathcal{X}) = \mathbf{SS}\neg(\mathcal{X})$. Similarly, using $\neg\mathcal{X}$ for Corollary 3.6, $\mathbf{SUU}(\neg\mathcal{X}) = \mathbf{S}\neg\mathbf{USS}(\neg\mathcal{X}) = \mathbf{S}\neg\mathbf{USU}(\neg\mathcal{X}) = \mathbf{S}\neg\mathbf{US}\neg(\neg\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\neg\mathcal{X}) = \mathbf{U}\neg\mathbf{SUU}(\neg\mathcal{X}) = \mathbf{U}\neg\mathbf{SUS}(\neg\mathcal{X}) = \mathbf{U}\neg\mathbf{SU}\neg(\neg\mathcal{X}) = \mathbf{U}\neg\mathbf{SSS}(\neg\mathcal{X}) = \mathbf{USS}(\neg\mathcal{X})$. Therefore, combining with Theorem 2.2(iii), $\mathbf{SUUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USSS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USU}\neg(\mathcal{X}) = \mathbf{S}\neg\mathbf{US}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUS}\neg(\mathcal{X}) = \mathbf{U}\neg\mathbf{SU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SS}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$. □

As a summary for the later discussion on $RT_{\mathcal{A}}$ and $RT^*_{\mathcal{A}}$, we present Theorem 3.1 and Corollaries 3.6 and 3.7 in terms of class operators.

Theorem 3.8. *The following hold for any universal class of finite associative rings:*

(i) $\mathbf{UU}\neg\mathbf{S} = \mathbf{US}\neg\mathbf{U} = \mathbf{UUU}\neg = \mathbf{UU}$;

(ii) $\mathbf{SS}\neg\mathbf{U} = \mathbf{SU}\neg\mathbf{S} = \mathbf{SSS}\neg = \mathbf{SS}$;

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(iii) $\mathbf{SUU} = \mathbf{S}\neg\mathbf{USS} = \mathbf{S}\neg\mathbf{USU} = \mathbf{S}\neg\mathbf{US}\neg = \mathbf{S}\neg\mathbf{UUU} = \mathbf{U}\neg\mathbf{SUU} = \mathbf{U}\neg\mathbf{SUS} = \mathbf{U}\neg\mathbf{SU}\neg = \mathbf{USS}$;

(iv) $\mathbf{SUUU} = \mathbf{S}\neg\mathbf{USSS} = \mathbf{S}\neg\mathbf{USU}\neg = \mathbf{S}\neg\mathbf{US} = \mathbf{S}\neg\mathbf{UUU} = \mathbf{U}\neg\mathbf{SUUU} = \mathbf{U}\neg\mathbf{SUS}\neg = \mathbf{U}\neg\mathbf{SU} = \mathbf{USSS}$.

The presence of simple rings in radical and semisimple classes ensures the presence of radical–semisimple subclasses. For a subclass $\mathcal{X} \subseteq \mathcal{A}$, let $\mathcal{X}_s$ be any collection of simple rings contained in $\mathcal{X}$. Then $\mathcal{X}_s$ is hereditary and homomorphically closed and generates coinciding radical and semisimple classes.

Proposition 3.9. *For any subclass $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{US}(\mathcal{X}_s) = \mathbf{SU}(\mathcal{X}_s)$.*

Proof. Since $\mathcal{X}_s$ is homomorphically closed, $\mathcal{X}_s \subseteq \mathbf{UU}(\mathcal{X}_s)$ and $\mathbf{SU}(\mathcal{X}_s) \subseteq \mathbf{SUUU}(\mathcal{X}_s)$. From Theorem 2.3(iii), $\mathbf{SUUU}(\mathcal{X}_s) \subseteq \mathbf{SU}(\mathcal{X}_s)$ and so $\mathbf{SU}(\mathcal{X}_s) = \mathbf{SUUU}(\mathcal{X}_s)$.

The dual argument uses the hereditariness of $\mathcal{X}_s$ and Corollary 3.7. The class $\mathbf{US}(\mathcal{X}_s) = \mathbf{USSS}(\mathcal{X}_s) = \mathbf{SUUU}(\mathcal{X}_s)$ and so $\mathbf{US}(\mathcal{X}_s) = \mathbf{SU}(\mathcal{X}_s)$. □

As is the case in the universal class of finite groups [6], more of the duality for **U** and **S** and its consequences is preserved for finite associative rings.

Theorem 3.10. *The following conditions on a subclass $\mathcal{X} \subseteq \mathcal{A}$ are equivalent:*

(i) *$\mathcal{X}$ is a hereditary radical class;*
(ii) *$\mathcal{X}$ is a homomorphically closed semisimple class;*
(iii) *$\mathcal{X}$ is a radical–semisimple class.*

Proof. If a subclass $\mathcal{X}$ of $\mathcal{A}$ is a hereditary radical class (i), then $\mathcal{X} = \mathbf{US}(\mathcal{X}) = \mathbf{S}\neg(\mathbf{US}(\mathcal{X}))$. By Corollary 3.7, $\mathbf{S}\neg\mathbf{US}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$, so $\mathcal{X}$ is radical–semisimple, showing (ii) and (iii). The dual argument shows that (ii) implies (i) and (iii). That (iii) implies (i) and (ii) follows from the properties of radical and semisimple classes. □

Weaker conditions than homomorphically closed for semisimple classes and hereditary for radical classes suffice to ensure that a class is radical–semisimple.

Theorem 3.11. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the following are equivalent:*

(i) $\mathcal{X} = \mathbf{US}(\mathcal{X}) = \mathbf{SU}(\mathcal{X})$;
(ii) $\mathcal{X} = \mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X})$;
(iii) *$\mathcal{X}$ is a radical class with condition (2^*);*
(iv) *$\mathcal{X}$ is a semisimple class with condition (1^*).*

Proof. For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, given (i) it follows that $\mathcal{X} = \mathbf{US}(\mathcal{X}) = \mathbf{US}(\mathbf{SU}(\mathcal{X})) = \mathbf{USSS}(\mathcal{X})$ by Theorem 2.3(iv) and, dually, $\mathcal{X} = \mathbf{SU}(\mathbf{US}(\mathcal{X})) = \mathbf{SUUU}(\mathcal{X})$, which is (ii). That (i) implies (iii) and (iv) follows from $\mathcal{X}$ being hereditary and homomorphically closed, respectively, and similarly for (ii) implying (iii) and (iv). That (ii) implies (i) follows from the definition of radical–semisimple. If $\mathcal{X}$ is a radical class with (2^*), which is (iii), then $\mathcal{X} = \mathbf{U}(\mathbf{S}(\mathcal{X})) = \mathbf{U}(\mathbf{SSS}(\mathcal{X}))$ by Theorem 2.4 and $\mathbf{USSS}(\mathcal{X}) =$

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$\mathbf{SUUU}(\mathcal{X})$ by Corollary 3.7, which is (ii). The dual argument shows that given (iv), $\mathcal{X}$ is a semisimple class with condition (1*); then $\mathcal{X} = \mathbf{S}(\mathbf{U}(\mathcal{X})) = \mathbf{S}(\mathbf{UUU}(\mathcal{X}))$ by Theorem 2.4 and $\mathbf{SUUU}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$ again by Corollary 3.7, which is (ii). □

For the universal class of all associative rings the 'concrete radical classes', which includes the Jacobson radical, are all hereditary [5] and it follows from Theorem 3.10 that these are all radical–semisimple classes in this setting.

Corollary 3.12. *A radical class $\mathcal{R}$ is hereditary if and only if $\mathcal{R}$ is of the form* $\mathbf{USS}(\mathcal{X})$ *for some subclass $\mathcal{X}$ of $\mathcal{A}$.*

Proof. If a radical class $\mathcal{R}$ is hereditary, then condition (2*) holds and, by Theorem 3.11, $\mathcal{R} = \mathbf{USS}(\mathbf{S}(\mathcal{R}))$ and so $\mathcal{R}$ is of the form $\mathbf{USS}(\mathcal{X})$. The converse is clear since $\mathbf{USS}(\mathcal{X})$ is a radical class for all subclasses $\mathcal{X}$ of $\mathcal{A}$ by definition, and $\mathbf{USS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USS}(\mathcal{X})$ by Corollary 3.6. □

Corollary 3.13. *If condition* (2*) *holds for a subclass $\mathcal{X} \subseteq \mathcal{A}$, then the radical class* $\mathbf{US}(\mathcal{X})$ *is a radical–semisimple class. Dually, if condition* (1*) *holds for a subclass $\mathcal{X} \subseteq \mathcal{A}$, then the semisimple class* $\mathbf{SU}(\mathcal{X})$ *is a radical–semisimple class.*

Proof. Let $\mathcal{X}$ be a subclass of $\mathcal{A}$ where condition (2*) holds. From Theorem 2.4, $\mathbf{S}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$ and so $\mathbf{US}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$, a radical–semisimple class by Corollary 3.7. The dual argument proves the second part. □

In this finite setting we can extend Theorem 2.1 to the following result and its dual.

Theorem 3.14. *The largest hereditary subclass of a radical class is a radical–semisimple class.*

Proof. Let $\mathcal{R}$ be a radical class and so $\mathcal{R} = \mathbf{US}(\mathcal{R})$. By Theorem 2.1, $\mathbf{S}\neg(\mathcal{R})$ is a radical class and, since $\mathcal{R}$ is closed under extensions, $\mathbf{S}\neg(\mathcal{R}) = \mathbf{SU}(\mathcal{R})$ by Lemma 3.2. Hence, $\mathbf{S}\neg(\mathcal{R})$ is a radical–semisimple class. □

Corollary 3.15. *The largest homomorphically closed subclass of a semisimple class is a radical–semisimple class.*

Proof. Let $\mathcal{S}$ be a semisimple class. Then the largest homomorphically closed subclass of $\mathcal{S}$ is $\mathbf{U}\neg(\mathcal{S}) = \mathbf{U}\neg\mathbf{SU}(\mathcal{S})$. By Corollary 3.7, $\mathbf{U}\neg\mathbf{SU}(\mathcal{S}) = \mathbf{USSS}(\mathcal{S}) = \mathbf{SUUU}(\mathcal{S})$, showing that $\mathbf{U}\neg(\mathcal{S})$ is radical–semisimple. □

For any subclass $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{S}(\mathcal{X})$ is a semisimple class when $\mathcal{X}$ is homomorphically closed [10] or hereditary (by Theorem 2.3(i)). We can strengthen this result for classes of finite associative rings.

Proposition 3.16. *For a subclass $\mathcal{X} \subseteq \mathcal{A}$, if $\mathcal{X} \subseteq$* $\mathbf{SU}(\mathcal{X})$*, then* $\mathbf{S}(\mathcal{X})$ *is a semisimple class.*

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Proof. If a subclass $\mathcal{X}$ of $\mathcal{A}$ is contained in $\mathbf{SU}(\mathcal{X})$, then $\mathbf{SSU}(\mathcal{X}) \subseteq \mathbf{S}(\mathcal{X})$. If $A \in \mathbf{S}(\mathcal{X})$, then A has no nonzero accessible subring in $\mathcal{X}$ and hence A has no nonzero accessible subring in $\mathbf{SS}(\mathcal{X})$. Therefore, $\mathbf{S}(\mathcal{X}) \subseteq \mathbf{SSS}(\mathcal{X}) = \mathbf{SSU}(\mathcal{X})$ by Theorem 2.3(iv). Hence, $\mathbf{S}(\mathcal{X}) = \mathbf{SSU}(\mathcal{X})$ and, by Theorem 2.3(i), $\mathbf{S}(\mathcal{X})$ is a semisimple class. □

Every accessible subring of a homomorphic image of $A \in \mathcal{A}$ is a homomorphic image of an accessible subring of A [9, Lemma 4.1]. We can dualise this result to an extent for finite associative rings as follows.

Proposition 3.17. *For all $A \in \mathcal{A}$, every simple homomorphic image of an accessible subring of A is a simple accessible subring of a homomorphic image of A.*

Proof. As the relations 'is an accessible subring of' and 'is a homomorphic image of' are reflexive, the case for the simple accessible subring of any A in $\mathcal{A}$ follows directly. Now, from Theorem 2.3(ii), $A \in \neg\mathbf{UUU}(\mathcal{X})$ if and only if A has a nonzero homomorphic image B such that every simple homomorphic image of B is in $\mathcal{X}$ if and only if A has a simple homomorphic image in $\mathcal{X}$. Dually, $A \in \neg\mathbf{SSS}(\mathcal{X})$ if and only if A has a simple accessible subring in $\mathcal{X}$. If C is a simple homomorphic image of an accessible subring of A, then $A \in \neg\mathbf{S}(\neg\mathbf{UUU}(\{C\}))$. Suppose that C is not the simple accessible subring of a homomorphic image of A. Then $A \notin \neg\mathbf{U}(\neg\mathbf{SSS}(\{C\}))$ and hence $\neg\mathbf{S}\neg\mathbf{UUU}(\{C\}) \neq \neg\mathbf{U}\neg\mathbf{SSS}(\{C\})$, which is a contradiction by Corollary 3.6. □

Whether the requirement that a homomorphic image of an accessible subring is simple can be removed, extending Proposition 3.17 to all homomorphic images of accessible subrings of $A \in \mathcal{A}$, that is, $\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}\neg\mathbf{S}(\mathcal{X})$, is not known to the authors. It is true that for all $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{S}\neg\mathbf{UU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SS}(\mathcal{X})$.

The following two propositions use properties of the universal class elements to help determine an upper bound for the order of the semigroup $RT^*_{\mathcal{A}}$. Let $\mathcal{A}'$ be a subuniversal class of $\mathcal{A}$.

Proposition 3.18. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}'$, $\neg\mathbf{UUU}(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$ if and only if every $A \in \mathcal{A}'$ has a single simple homomorphic image. Dually, $\neg\mathbf{SSS}(\mathcal{X}) = \mathbf{SS}(\mathcal{X})$ for all $\mathcal{X}$ if and only if all A have a single simple accessible subring.*

Proof. Suppose that some $A \in \mathcal{A}'$ has both B and C as simple nonisomorphic homomorphic images. Then $A \in \neg\mathbf{U}(\mathbf{UU}(\{B\}))$ but $A \notin \mathbf{UU}(\{B\})$. Conversely, if $\neg\mathbf{UUU}(\mathcal{X}) \neq \mathbf{UU}(\mathcal{X})$ for all $\mathcal{X} \subseteq \mathcal{A}'$, then as $\mathbf{UU}(\mathcal{X}) \subseteq \neg\mathbf{UUU}(\mathcal{X})$, there exists some nonzero $A \in \neg\mathbf{UUU}(\mathcal{X})$ which is not in $\mathbf{UU}(\mathcal{X})$. That is, some nonzero A has a homomorphic image in $\mathbf{UU}(\mathcal{X})$ and so A has a simple homomorphic image in $\mathcal{X}$. However, $A \notin \mathbf{UU}(\mathcal{X})$ and so A must have at least one other simple homomorphic image that is not in $\mathcal{X}$, showing that A must have more than one simple homomorphic image.

The dual argument proves the second part. □

Proposition 3.19. *If $\mathcal{X}$ is a subclass of $\mathcal{A}$ such that for all $A \in \mathcal{X}$, every nonzero proper ideal B of A is simple, then $\mathbf{US}(\mathcal{X}) = \mathbf{USU}\neg(\mathcal{X})$ and $\mathbf{SU}(\mathcal{X}) = \mathbf{SUS}\neg(\mathcal{X})$.*

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Proof. For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, suppose that $A \in \mathcal{X}$ is such that every nonzero proper ideal B of A is simple. Then every nonzero proper homomorphic image of A is also simple. If $A \in \mathbf{US}(\mathcal{X})$, then all simple homomorphic images of A are in $\mathbf{US}(\mathcal{X})$ and hence in $\mathbf{U}\neg(\mathcal{X})$. If $A \notin \mathcal{X}$, then A has a nonzero accessible subring B in $\mathcal{X}$. Since B is simple, $B \in \mathbf{U}\neg(\mathcal{X})$. Therefore, $A \in \mathbf{US}(\mathbf{U}\neg(\mathcal{X}))$.

If $A \in \mathcal{X}$, then $A \in \mathbf{U}\neg(\mathcal{X})$ and, again, $A \in \mathbf{USU}\neg(\mathcal{X})$, showing that $\mathbf{US}(\mathcal{X}) \subseteq \mathbf{USU}\neg(\mathcal{X})$. Since $\mathbf{USU}\neg(\mathcal{X})$ is always contained in $\mathbf{US}(\mathcal{X})$, the two classes are equal. The dual argument shows that $\mathbf{SUS}\neg(\mathcal{X}) = \mathbf{SU}(\mathcal{X})$. □

The maximum order of $RT_{\mathcal{A}}$ and $RT^*_{\mathcal{A}}$ for the universal class of finite associate rings can now be determined with a complete listing of possible radical and semisimple class operators.

Theorem 3.20. *The operator semigroup $RT_{\mathcal{A}}$ has at most 12 elements.*

Proof. Starting with the classes $\mathbf{U}(\mathcal{X})$ and $\mathbf{S}(\mathcal{X})$ and applying operators $\mathbf{U}$ and $\mathbf{S}$ to each in turn generates $\mathbf{UU}(\mathcal{X})$, $\mathbf{US}(\mathcal{X})$, $\mathbf{SU}(\mathcal{X})$ and $\mathbf{SS}(\mathcal{X})$. Repeating generates classes $\mathbf{UUU}(\mathcal{X})$, $\mathbf{SUU}(\mathcal{X})$, $\mathbf{USU}(\mathcal{X})$, $\mathbf{SUS}(\mathcal{X})$ and $\mathbf{SSS}(\mathcal{X})$, as the three other possibilities are not distinct under Theorem 2.2(iii), Theorem 2.3(iv) and Corollary 3.6. Using Theorem 2.3(i) and (iv) with Corollary 3.7, a further application of $\mathbf{U}$ and $\mathbf{S}$ to each of these five classes gives the last distinct class $\mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X})$. In summary, $RT_{\mathcal{A}} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{SUU}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{SUUU}\}$ and the order of $RT_{\mathcal{A}}$ is at most 12. □

This is the semigroup mentioned in the background for the universal class $\mathcal{A}_1$. The operators $\mathbf{UU}, \mathbf{US}, \mathbf{UUU}, \mathbf{USS}(= \mathbf{SUU}), \mathbf{USU}$ and $\mathbf{USSS}(= \mathbf{SUUU})$ will always generate radical classes, and the operators $\mathbf{SU}, \mathbf{SS}, \mathbf{SUU}, \mathbf{SUS}, \mathbf{SSS}$ and $\mathbf{SUUU}$ generate semisimple classes for any subclass $\mathcal{X}$. Using a similar proof strategy we can determine an upper bound for the order of the semigroup generated by $\mathbf{U}, \mathbf{S}$ and $\neg$.

Theorem 3.21. *The operator semigroup $RT^*_{\mathcal{A}}$ has at most 46 elements.*

Proof. Beginning with the three classes $\mathbf{U}(\mathcal{X}), \mathbf{S}(\mathcal{X})$ and $\neg(\mathcal{X})$ and applying operators $\mathbf{U}, \mathbf{S}$ and $\neg$, we obtain $\mathbf{UU}(\mathcal{X})$, $\mathbf{SU}(\mathcal{X})$, $\neg\mathbf{U}(\mathcal{X})$, $\mathbf{US}(\mathcal{X})$, $\mathbf{SS}(\mathcal{X})$, $\neg\mathbf{S}(\mathcal{X})$, $\mathbf{U}\neg(\mathcal{X})$, $\mathbf{S}\neg(\mathcal{X})$ and $\neg\neg(\mathcal{X}) = \mathcal{X}$.

Repeating this process and using Theorems 2.3(iv) and 3.1 and Corollaries 3.6 and 3.7 as appropriate, additions are $\mathbf{UUU}(\mathcal{X})$, $\mathbf{SUU}(\mathcal{X})$, $\neg\mathbf{UU}(\mathcal{X})$, $\mathbf{USU}(\mathcal{X})$, $\mathbf{SSS}(\mathcal{X})$, $\neg\mathbf{SU}(\mathcal{X})$, $\mathbf{S}\neg\mathbf{U}(\mathcal{X})$, $\mathbf{SUS}(\mathcal{X})$, $\neg\mathbf{US}(\mathcal{X})$, $\neg\mathbf{SS}(\mathcal{X})$, $\mathbf{U}\neg\mathbf{S}(\mathcal{X})$, $\mathbf{SU}\neg(\mathcal{X})$, $\neg\mathbf{U}\neg(\mathcal{X})$, $\mathbf{US}\neg(\mathcal{X})$ and $\neg\mathbf{S}\neg(\mathcal{X})$. In the next round $\mathbf{SUUU}(\mathcal{X})$, $\neg\mathbf{UUU}(\mathcal{X})$, $\neg\mathbf{SUU}(\mathcal{X})$, $\neg\mathbf{USU}(\mathcal{X})$, $\neg\mathbf{SSS}(\mathcal{X})$, $\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$, $\neg\mathbf{SUS}(\mathcal{X})$, $\neg\mathbf{U}\neg\mathbf{S}(\mathcal{X})$, $\mathbf{USU}\neg(\mathcal{X})$, $\neg\mathbf{SU}\neg(\mathcal{X})$, $\mathbf{S}\neg\mathbf{U}\neg(\mathcal{X})$, $\mathbf{SUS}\neg(\mathcal{X})$, $\neg\mathbf{US}\neg(\mathcal{X})$ and $\mathbf{U}\neg\mathbf{S}\neg(\mathcal{X})$ are generated. Lastly, one further application gives distinct classes $\neg\mathbf{SUUU}(\mathcal{X})$, $\neg\mathbf{USU}\neg(\mathcal{X})$, $\neg\mathbf{S}\neg\mathbf{U}\neg(\mathcal{X})$, $\neg\mathbf{SUS}\neg(\mathcal{X})$ and $\neg\mathbf{U}\neg\mathbf{S}\neg(\mathcal{X})$.

This gives $RT^*_{\mathcal{A}} = \{\mathbf{U}, \mathbf{S}, \neg, \mathbf{UU}, \mathbf{SU}, \neg\mathbf{U}, \mathbf{US}, \mathbf{SS}, \neg\mathbf{S}, \mathbf{U}\neg, \mathbf{S}\neg, \neg\neg, \mathbf{UUU}, \mathbf{SUU}, \neg\mathbf{UU}, \mathbf{USU}, \mathbf{SSS}, \neg\mathbf{SU}, \mathbf{S}\neg\mathbf{U}, \mathbf{SUS}, \neg\mathbf{US}, \neg\mathbf{SS}, \mathbf{U}\neg\mathbf{S}, \mathbf{SU}\neg, \neg\mathbf{U}\neg, \mathbf{US}\neg, \neg\mathbf{S}\neg, \mathbf{SUUU}, \neg\mathbf{UUU}, \neg\mathbf{SUU}, \neg\mathbf{USU}, \neg\mathbf{SSS}, \neg\mathbf{S}\neg\mathbf{U}, \neg\mathbf{SUS}, \neg\mathbf{U}\neg\mathbf{S}, \mathbf{USU}\neg, \neg\mathbf{SU}\neg, \mathbf{S}\neg\mathbf{U}\neg, \mathbf{SUS}\neg, \neg\mathbf{US}\neg, \mathbf{U}\neg\mathbf{S}\neg, \neg\mathbf{SUUU}, \neg\mathbf{USU}\neg, \neg\mathbf{S}\neg\mathbf{U}\neg, \neg\mathbf{SUS}\neg, \neg\mathbf{U}\neg\mathbf{S}\neg\}$, at most 46 elements. □

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In addition to the six radical operators listed above for $RT_{\mathcal{A}}$, we gain an extra two operators which always generate radical classes, namely **USU¬** and **US¬**. Dually, the two extra operators which always generate semisimple classes are **SUS¬** and **SU¬**.

For the universal class $\mathcal{A}_1 = \{0, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2^0\}$, every homomorphic image of each accessible subring of $A \in \mathcal{A}_1$ is an accessible subring of a homomorphic image of A and so for all subclasses $\mathcal{X} \subseteq \mathcal{A}_1$, $\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}\neg\mathbf{S}(\mathcal{X})$. The equalities in Propositions 3.18 and 3.19 hold and $RT^*_{\mathcal{A}_1} = \{$**U**, **S**, ¬, **UU**, **SU**, ¬**U**, **US**, **SS**, ¬**S**, **U**¬, **S**¬, ¬¬, **UUU**, **SUU**, **USU**, **SSS**, ¬**SU**, **S**¬**U**, **SUS**, ¬**US**, ¬**U**¬, ¬**S**¬, **SUUU**, ¬**SUU**, ¬**USU**, ¬**S**¬**U**, ¬**SUS**, **S**¬**U**¬, ¬**SUUU**, ¬**S**¬**U**¬$\}$ with $|RT^*_{\mathcal{A}_1}| = 30$. Note that $RT^*_{\mathcal{A}_1}$ has the same six radical operators and corresponding semisimple operators as $RT_{\mathcal{A}_1}$.

In this universal class, $\{0, \mathbb{Z}_2^0\}$ is nil and $\mathbf{USSS}(\{0, \mathbb{Z}_2^0\}) = \mathbf{SUUU}(\{0, \mathbb{Z}_2^0\}) = \{0, \mathbb{Z}_2^0\}$. This is a smaller radical–semisimple class than the universal class itself, which is expected in the universal class of all associative rings [13]. What is emerging are radical and semisimple classes which account for the properties of the radical ideal in terms of the substructure of the class elements rather than a property that the elements of the ideal might have [18].

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R. G. MCDOUGALL,
Faculty of Science, Health, Education and Engineering,
University of the Sunshine Coast, Maroochydore,
Queensland 4588, Australia
e-mail: rmcdouga@usc.edu.au

L. K. THORNTON,
Faculty of Science, Health, Education and Engineering,
University of the Sunshine Coast, Maroochydore,
Queensland 4588, Australia
e-mail: lauren.thornton@research.usc.edu.au

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