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Annihilators and Power Values of Generalized Skew Derivations on Lie Ideals

Vincenzo De Filippis

Abstract. Let R be a prime ring of characteristic different from 2, let Q_r be its right Martindale quotient ring, and let C be its extended centroid. Suppose that F is a generalized skew derivation of R, L a non-central Lie ideal of R, $0 \neq a \in R$, $m \geq 0$ and $n, s \geq 1$ fixed integers. If

$$a\big(u^m F(u)u^n\big)^s = 0$$

for all $u \in L$, then either $R \subseteq M_2(C)$, the ring of 2×2 matrices over C, or $m = 0$ and there exists $b \in Q_r$ such that $F(x) = bx$, for any $x \in R$, with $ab = 0$.

1 Introduction

Let R be an associative ring with center $Z(R)$. Several papers in the literature evidence how the structure of R is closely related to the behaviour of some additive maps defined of R.

In [23] Herstein proves that if R is a prime ring, n is a fixed positive integer, and $d \neq 0$ is a derivation of R such that $d(x)^n \in Z(R)$ for all $x \in R$, then $R \subseteq M_2(K)$, the ring of 2×2 matrices over a field K. Bergen and Carini [2] show that the same conclusion holds for prime rings of characteristic different from 2 if $d(x)^n \in Z(R)$, for all x in some non-central Lie ideals of R. Later, in [28], Lee and Lin prove that if R is a semiprime ring with a non-zero derivation d, L is a Lie ideal of R, n is a fixed integer, and $a \in R$ such that $\mathrm{ad}(x)^n = 0$, for all $x \in L$, then $\mathrm{ad}(I) = 0$, for I the ideal of R generated by $[L, L]$. Moreover, if the characteristic of R is different from 2, then $\mathrm{ad}(L) = 0$. Furthermore, if R is prime, then $a = 0$.

Therefore, any investigation of derivations in prime rings from the algebraic point of view will definitely be interesting.

In [3], Chang and Lin consider the situation when $d(u)u^n = 0$ for all $u \in \rho$ and $u^n d(u) = 0$ for all $u \in \rho$, where ρ is a nonzero right ideal of R and d is a non-zero derivation of R. They show that if R is a prime ring and n is a fixed positive integer, then $d(\rho)\rho = 0$, and if $u^n d(u) = 0$ for all $u \in \rho$, then $R \cong M_2(F)$, the 2×2 matrices over a field F of two elements.

In [19], Dhara and Sharma give a generalization of the above stated results. Precisely speaking, they prove that if R is a prime ring of characteristic different from 2, d a non-zero derivation of R, $a \in R$, L a non-central Lie ideal of R, and s, t, n fixed

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integers such that $ax^s d(x)^n x^t \in Z(R)$ for all $x \in L$, then either $a = 0$ or $R \subseteq M_2(K)$ for a field K.

Following this line of investigation, we obtained a result [17] having the same flavour of the above mentioned ones, replacing the derivation d by a generalized derivation $F: R \to R$. We recall that a *generalized derivation* is an additive mapping F satisfying the rule $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$ and for a derivation d of R. In light of this definition, we prove that if $u^s F(u) u^t = 0$ for all u in a noncommutative Lie ideal L of R, where $s(\geq 0), t(\geq 0)$ are fixed integers, then $F(x) = 0$ for all $x \in R$ unless char $R = 2$ and R satisfies S_4 [17].

Continuing the above line in [18], we extended the previous cited results by considering an annihilating condition and proved the following theorem.

Theorem 1.1 *Let R be a prime ring, let Q_r be its right Martindale quotient ring, and let C be its extended centroid. Let F be a nonzero generalized derivation of R and L a noncommutative Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(u^s F(u) u^t)^n = 0$ for all $u \in L$, where $s \geq 0, t \geq 0, n \geq 1$ are fixed integers. Then $s = 0$ and there exists $b \in Q_r$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$, unless $R \subseteq M_2(C)$, the ring of 2×2 matrices over C.*

The aim of this paper is to extend the above theorem to the case where the generalized derivation is replaced by a generalized skew derivation. More precisely, let α be an automorphism of R. An additive mapping $d: R \to R$ is called a *skew derivation* of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$ and α is called an *associated automorphism* of d. An additive mapping $G: R \to R$ is said to be a *generalized skew derivation* of R if there exists a skew derivation d of R with associated automorphism α such that

$$G(xy) = G(x)y + \alpha(x)d(y)$$

for all $x, y \in R$, d is said to be an *associated skew derivation* of G, and α is called an *associated automorphism* of G. Any mapping of R with form $G(x) = ax + \alpha(x)b$ for some $a, b \in R$ and $\alpha \in \mathrm{Aut}(R)$, is called *inner generalized skew derivation*. In particular, if $a = -b$, then G is called *inner skew derivation*. If a generalized skew derivation (resp. a skew derivation) is not inner, then it is usually called *outer*. Hence the concept of generalized skew derivation unifies the notions of skew derivation and generalized derivation, which have been investigated by many researchers from various points of view (see [4–10, 27, 29]).

The main result of this article is the following theorem.

Theorem 1.2 *Let R be a prime ring of characteristic different from 2, let Q_r be its right Martindale quotient ring, and let C be its extended centroid. Suppose that F is a generalized skew derivation of R, L a non-central Lie ideal of R, $0 \neq a \in R$, $m \geq 0$, and $n, s \geq 1$ fixed integers. If*

$$a\big(u^m F(u) u^n\big)^s = 0$$

for all $u \in L$, then either $R \subseteq M_2(C)$, the ring of 2×2 matrices over C, or $m = 0$, and there exists $b \in Q_r$ such that $F(x) = bx$, for any $x \in R$, with $ab = 0$.

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2 Preliminaries

In what follows, let Q_r be the right Martindale quotient ring of R and let $C = Z(Q_r)$ be the center of Q_r. Then C is usually called the *extended centroid* of R and is a field when R is a prime ring. It should be remarked that Q_r is a centrally closed prime C-algebra. We refer the reader to [1] for the definitions and the related properties of these objects.

It is well known that automorphisms, derivations, and skew derivations of R can be extended to Q_r. In [4] Chang extends the definition of generalized skew derivation to the right Martindale quotient ring Q_r of R as follows: by a (right) generalized skew derivation we mean an additive mapping $G: Q_r \to Q_r$ such that $G(xy) = G(x)y + \alpha(x)d(y)$ for all $x, y \in Q$, where d is a skew derivation of R and α is an automorphism of R. Moreover, there exists $G(1) = a \in Q_r$ such that $G(x) = ax + d(x)$ for all $x \in R$.

We now fix some notation and collect some existing results that will be used in the sequel.

Let us denote by $\mathrm{SDer}(Q_r)$ the set of all skew-derivations of Q_r. By a *skew-derivation word* we mean an additive mapping Δ of the form $\Delta = d_1 d_1 \dots d_m$, where $d_i \in \mathrm{SDer}(Q_r)$. A *skew-differential polynomial* is a generalized polynomial with coefficients in Q_r of the form $\Phi(\Delta_j(x_i))$ involving noncommutative indeterminates x_i on which the derivation words Δ_j act as unary operations. The skew-differential polynomial $\Phi(\Delta_j(x_i))$ is said to be a *skew-differential identity* on a subset T of Q_r if it vanishes on any assignment of values from T to its indeterminates x_i.

Let R be a prime ring, SD_{int} be the C-subspace of $\mathrm{SDer}(Q_r)$ consisting of all inner skew-derivations of Q_r, and let d and δ be two non-zero skew-derivations of Q_r. The following results can be considered as the consequences of [11–14].

In particular, we have the following fact.

Fact 2.1 In [16] Chuang and Lee investigate polynomial identities with skew derivations. They prove that if $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, in [16, Theorem 1] they prove that if $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, D is an outer skew derivation of R, and α is an outer automorphism of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i, and z_i are distinct indeterminates.

Fact 2.2 By [16] we can state the following result. If d is a non-zero skew-derivation of R and

$$\Phi\big(x_1, \dots, x_n, d(x_1), \dots, d(x_n)\big)$$

is a skew-differential polynomial identity of R, then one of the following statements holds:

(i) either $d \in SD_{\mathrm{int}}$;
(ii) or R satisfies the generalized polynomial identity

$$\Phi(x_1, \dots, x_n, y_1, \dots, y_n).$$

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Fact 2.3 Let R be a prime ring and let I be a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [11]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [13, Theorem 1]).

Fact 2.4 Let R be a domain and $\alpha \in \mathrm{Aut}(R)$ be an automorphism of R that is outer. In [24] Kharchenko proves that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

We also would like to recall a reduced version of Theorem 1.1 that will be useful in the sequel.

Theorem 2.5 *Let R be a prime ring, L a noncommutative Lie ideal of R, and $F(x) = bx + xc$, for any $x \in R$ and fixed elements $b, c \in Q_r$. Suppose that there exists $0 \neq a \in R$ such that $a(u^s F(u) u^t)^n = 0$ for all $u \in L$, where $s \geq 0, t \geq 0, n \geq 1$ are fixed integers. Then $s = 0$, $c \in C$, and $a(b + c) = 0$, unless $R \subseteq M_2(C)$, the ring of 2×2 matrices over C.*

3 Proof of the Main Result

Let us begin with the following lemma.

Lemma 3.1 *Let R be a dense subring of the ring of linear transformations of a vector space V over a division ring D, and let R contain nonzero linear transformations of finite rank. Let I be a noncentral two-sided ideal of R, $m \geq 0$ and $n, s \geq 1$ fixed integers, $0 \neq a \in R$, and α an automorphism of R and suppose $b, c \in R$ and $F(x) = bx + \alpha(x)c$ such that*

$$a\Big(u^m F(u) u^n\Big)^s = 0$$

for all $u \in [I, I]$. If $F \neq 0$ and R does not satisfy s_4, then one of the following holds:

(i) $\dim_D V \leq 2$;

(ii) *$m = 0$ and there exists $b' \in Q_r$ such that $F(x) = b'x$, for any $x \in R$, with $ab' = 0$.*

Proof We assume $\dim_D V \geq 3$.

Since R is a primitive ring with nonzero socle, by [21, p. 79] there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in R$, hence

$$a\Big(u^m (bu + TuT^{-1}c) u^n\Big)^s = 0$$

for all $u \in [I, I]$. Assume first that v and $T^{-1}cv$ are D-dependent for all $v \in V$. By [15, Lemma 1], there exists $\lambda \in D$ such that $T^{-1}cv = v\lambda$, for all $v \in V$. In this case, for all $x \in R$,

$$\begin{aligned} F(x)v &= (bx + TxT^{-1}c)v = bxv + TxT^{-1}cv = bxv + T(xv\lambda) = bxv + T\big((xv)\lambda\big) \\ &= bxv + T(T^{-1}c)(xv) = bxv + cxv = (b + c)xv. \end{aligned}$$

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This means that $(F(x) - (b+c)x)V = (0)$, for all $x \in R$ and since V is faithful, it follows that $F(x) = (b+c)x$, for all $x \in R$, and

$$a\Big(u^m(b+c)u^{n+1}\Big)^s = 0$$

for all $u \in [I, I]$. By Theorem 2.5 and since $R \nsubseteq M_2(C)$, it follows that $m = 0$ and $a(b+c) = 0$, as required.

Thus, there exists $v_0 \in V$ such that v_0 and $T^{-1}cv_0$ are linearly D-independent. Since $\dim_D V \geq 3$, there exists $w \in V$ such that w, v_0 and $T^{-1}cv_0$ are linearly D-independent. By the density of R, there exist $r_1, r_2, r_3 \in I$ such that

$$r_1v_0 = 0, \qquad r_1w = T^{-1}v_0 + T^{-1}(c-b)v_0, \qquad r_1T^{-1}cv_0 = v_0,$$
$$r_2v_0 = T^{-1}cv_0, \qquad r_2T^{-1}cv_0 = w.$$

Thus,

$$0 = a\Big([r_1, r_2]^m(b[r_1, r_2] + T[r_1, r_2]T^{-1}c)[r_1, r_2]^n\Big)^s v_0 = av_0.$$

Of course, for any $u \in V$ such that $\{u, v_0\}$ are linearly D-dependent, $au = 0$. Now let $u \in V$ such that $\{u, v_0\}$ are linearly D-independent and $au \neq 0$. By the above argument it follows that u and $T^{-1}cu$ are linearly D-dependent. Moreover, since $\{u+v_0, v_0\}$ are linearly D-independent and $a(u+v_0) \neq 0$, then $\{u+v_0, T^{-1}c(u+v_0)\}$ are linearly D-independent. Analogously, one can see that $\{u-v_0, T^{-1}c(u-v_0)\}$ are linearly D-independent.

Therefore, there exist $\lambda_u, \lambda_{u+v_0}, \lambda_{u-v_0} \in D$ such that

$$T^{-1}cu = u\lambda_u, \quad T^{-1}c(u+v_0) = (u+v_0)\lambda_{u+v_0}, \quad T^{-1}c(u-v_0) = (u-v_0)\lambda_{u-v_0}.$$

In other words, we have

$$u\lambda_u + T^{-1}cv_0 = u\lambda_{u+v_0} + v_0\lambda_{u+v_0} \tag{3.1}$$

and

$$u\lambda_u - T^{-1}cv_0 = u\lambda_{u-v_0} - v_0\lambda_{u-v_0}. \tag{3.2}$$

By comparing (3.1) with (3.2) we get both

$$u(2\lambda_u - \lambda_{u+v_0} - \lambda_{u-v_0}) + v_0(\lambda_{u-v_0} - \lambda_{u+v_0}) = 0 \tag{3.3}$$

and

$$2T^{-1}cv_0 = u(\lambda_{u+v_0} - \lambda_{u-v_0}) + v_0(\lambda_{u+v_0} + v_0\lambda_{u-v_0}). \tag{3.4}$$

By (3.3) and the facts that $\{u, v_0\}$ are D-independent and $\operatorname{char}(R) \neq 2$, we have $\lambda_u = \lambda_{u-v_0} = \lambda_{u+v_0}$. By (3.4) it follows $2T^{-1}cv_0 = 2v_0\lambda_u$. Since $\{T^{-1}cv_0, v_0\}$ are D-independent, the conclusion $\lambda_u = 0$ follows, that is $\lambda_{u-v_0} = \lambda_{u+v_0} = 0$. Thus, $T^{-1}cu = 0$ and $T^{-1}c(u+v_0) = 0$, which implies the contradiction $T^{-1}cv_0 = 0$.

The above argument shows that $au = 0$, for any $u \in V$. Therefore $aV = (0)$ and so $a = 0$, a contradiction. ■

Now we consider the case where F is an inner generalized skew derivation of R.

Proposition 3.2 *Let R be a prime ring of characteristic different from 2, let Q_r be its right Martindale quotient ring, and let C be its extended centroid. Let I be a noncentral*

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two-sided ideal of R, let $m \geq 0$ and $n, s \geq 1$ be fixed integers, let $0 \neq a \in R$, let α be an automorphism of R, and suppose $b, c \in R$ and $F(x) = bx + \alpha(x)c$ satisfy

$$a\big(u^m F(u) u^n\big)^s = 0$$

for all $u \in [I, I]$. Then one of the following holds:

(i) *$R \subseteq M_2(C)$, the ring of 2×2 matrices over C;*
(ii) *α is the identity mapping on R, $m = 0$, $c \in C$ and $a(b + c) = 0$;*
(iii) *$m = 0$, $c = 0$, and $ab = 0$.*

Proof In all that follows we can assume that α is not the identity mapping on R; otherwise, the conclusion follows from Theorem 2.5.

Suppose first that α is X-inner. Thus, there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$. So

$$a\big(u^m (bu + quq^{-1}c) u^n\big)^s = 0$$

for all $u \in [I, I]$. Since I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [11]), it follows that

$$a\big(u^m (bu + quq^{-1}c) u^n\big)^s = 0$$

for all $u \in [Q_r, Q_r]$. If $q^{-1}c \in C = Z(Q_r)$, then $F(x) = (b + c)x$, for all $x \in R$, and the conclusion follows from Theorem 2.5. Thus we may assume that $q^{-1}c \notin C$, and

$$a\big([x_1, x_2]^m (b[x_1, x_2] + q[x_1, x_2]q^{-1}c)[x_1, x_2]^n\big)^s \tag{3.5}$$

is a non-trivial generalized polynomial identity for Q_r. By Martindale's theorem [30], Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over D, where D is a finite dimensional division ring over C. By Lemma 3.1 we know that either $\dim_C RC = 4$ or $\dim_D V \leq 2$. In this last case it follows that either $Q_r \cong D$ or $Q_r \cong M_2(D)$, the ring of 2×2 matrices over D. More generally, we assume that $Q_r \cong M_k(D)$, for $k \leq 2$.

If C is finite, then D is a field by Wedderburn's Theorem. On the other hand, if C is infinite, let $\overline{C}$ be the algebraic closure of C, then by the van der Monde determinant argument, we see that $Q_r \otimes_C \overline{C}$ satisfies the same generalized polynomial identity (3.5). Moreover,

$$Q_r \bigotimes_C \overline{C} \cong M_k(D) \bigotimes_C \overline{C} \cong M_k\big(D \bigotimes_C \overline{C}\big) \cong M_t(\overline{C}),$$

for some $t \geq 1$.

Considering Lemma 3.1 and the fact Q_r is not commutative, we assert that $t = 2$. Hence R is an order in a 4-dimensional central simple algebra, as required.

Hence, we can assume that α is X-outer. By [12, Theorem 1], Q_r satisfies

$$a\big([x_1, x_2]^m (b[x_1, x_2] + \alpha([x_1, x_2])c)[x_1, x_2]^n\big)^s.$$

Moreover, by [12, Main Theorem] Q_r is a GPI-ring. Thus, Q_r is a primitive ring having nonzero socle and its associated division ring D is a finite-dimensional over C. By

Lemma 3.1 we also have $\dim_D V \le 2$. From now on we may assume that Q_r satisfies

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]+\alpha([x_1,x_2])c\big)[x_1,x_2]^n\Big)^2.$$

Since any $\alpha(x_i)$-word degree is 2 and either $\mathrm{char}(R)=0$ or $\mathrm{char}(R)\ge 3$, then, by [13, Theorem 3], Q_r satisfies the identity

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]+[y_1,y_2]c\big)[x_1,x_2]^n\Big)^2.$$

In particular Q_r satisfies both

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]+[x_1,x_2]c\big)[x_1,x_2]^n\Big)^2 \tag{3.6}$$

and

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]\big)[x_1,x_2]^n\Big)^2. \tag{3.7}$$

Applying Theorem 2.5 to (3.6) yields that either $R\subseteq M_2(C)$ or $m=0$, $c\in C$, and $a(b+c)=0$. In this last case and by (3.7), we have that

$$a\Big(b[x_1,x_2][x_1,x_2]^n\Big)^2$$

is a generalized polynomial identity for Q_r. Hence, again by Theorem 2.5, it follows that $ab=0$, so that $ac=0$ holds. Since $c\in C$ and $a\ne 0$, then $c=0$, and we are done. ∎

The reminder of this paper will be devoted to the proof of the main theorem. We remark that in [4] Chang shows that any (right) generalized skew derivation of R can be uniquely extended to the right Martindale quotient ring Q_r of R as follows: a (right) generalized skew derivation is an additive mapping $F\colon Q_r\to Q_r$ such that $F(xy)=F(x)y+\alpha(x)d(y)$ for all $x,y\in Q_r$, where d is a skew derivation of R and α is an automorphism of R. Notice that there exists $F(1)=b\in Q_r$ such that $F(x)=bx+d(x)$ for all $x\in R$.

Proof First, we notice that if $d=0$, then $F(x)=bx$, for all $x\in R$ and we are done by Theorem 2.5. On the other hand, if $\alpha=\mathrm{id}_R$, that is the associated automorphism α is the identity mapping on R, then d is a ordinary derivation of R, so that F is a generalized derivation of R. In this case the conclusion follows from Theorem 1.1. Hence in the sequel we assume both $d\ne 0$ and $\alpha\ne\mathrm{id}_R$.

Since $\mathrm{char}(R)\ne 2$ then there exists an ideal I of R such that $0\ne[I,R]\subseteq L$ (see [22, pp. 4-5], [20, Lemma 2, Proposition 1], [26, Theorem 4]). By the assumption, we have

$$a\big(u^mF(u)u^n\big)^s=0$$

for all $u\in[I,I]$ and also for all $u\in[Q_r,Q_r]$ (see [16, Theorem 2]). Since Q_r satisfies

$$a\Big([x_1,x_2]^m\Big(b[x_1,x_2]+d([x_1,x_2])\Big)[x_1,x_2]^n\Big)^s,$$

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Q_r satisfies

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]+d(x_1)x_2+\alpha(x_1)d(x_2) - d(x_2)x_1-\alpha(x_2)d(x_1)\big)[x_1,x_2]^n\Big)^s. \tag{3.8}$$

If d is an inner skew derivation of R, then the result follows from Proposition 3.2.

Assume that d is X-outer. By [16, Theorem 1] and (3.8) it follows that Q_r satisfies the generalized polynomial identity

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]+y_1x_2+\alpha(x_1)y_2-y_2x_1-\alpha(x_2)y_1\big)[x_1,x_2]^n\Big)^s.$$

Therefore, Q_r satisfies both

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]\big)[x_1,x_2]^n\Big)^s \tag{3.9}$$

and

$$a\Big([x_1,x_2]^m\big(b[x_1,x_2]+y_1x_2-\alpha(x_2)y_1\big)[x_1,x_2]^n\Big)^s. \tag{3.10}$$

By applying Theorem 2.5 to (3.9) and since $a\neq 0$, one of the following holds: either $R\subseteq M_2(C)$ or $m=0$ and $ab=0$. In the latter case the relation (3.10) reduces to

$$a\Big(\big(b[x_1,x_2]+y_1x_2-\alpha(x_2)y_1\big)[x_1,x_2]^n\Big)^s. \tag{3.11}$$

Let us first consider the case when α is an inner automorphism of R. Then there exists an invertible element $q\in Q_r$ such that $\alpha(x)=qxq^{-1}$. Since $1\neq\alpha\in \mathrm{Aut}(R)$, we can assume $q\notin C$. Hence Q_r satisfies the generalized polynomial identity

$$a\Big(\big(b[x_1,x_2]+y_1x_2-qx_2q^{-1}y_1\big)[x_1,x_2]^n\Big)^s. \tag{3.12}$$

Replacing y_1 by qx_1 in (3.12), it follows that Q_r satisfies

$$a\Big(\big(b[x_1,x_2]+q[x_1,x_2]\big)[x_1,x_2]^n\Big)^s,$$

and by Theorem 2.5 again, it follows that $0=a(b+q)=aq$; that is, $a=0$, a contradiction.

Finally, we assume that α is X-outer. By [12, Main Theorem] Q_r is a GPI-ring. Thus, Q_r is a primitive ring having nonzero socle and its associated division ring D is a finite-dimensional over C.

Let us assume that $\dim_D V\geq 3$.

Since R is a primitive ring with nonzero socle, by [21, p. 79] there exists a semilinear automorphism $T\in \mathrm{End}(V)$ such that $\alpha(x)=TxT^{-1}$ for all $x\in R$. By (3.11) we know that

$$a\Big(\big(b[x_1,x_2]+y_1x_2-Tx_2T^{-1}y_1\big)[x_1,x_2]^n\Big)^s$$

is satisfied by Q_r. We notice that, if for any $v\in V$ there exists $\lambda_v\in D$ such that $T^{-1}v=v\lambda_v$. By a standard argument, it follows that there exists a unique $\lambda\in D$ such

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that $T^{-1}v = v\lambda$, for all $v \in V$ (see for example [15, Lemma 1]). In this case, we conclude that

$$\alpha(x)v = (TxT^{-1})v = Txv\lambda$$

and

$$\big(\alpha(x) - x\big)v = T(xv\lambda) - xv = T(T^{-1}xv) - xv = 0.$$

Since V is faithful, we know that α is the identity mapping, which is a contradiction. Thus, there exists $v \in V$ such that v and $T^{-1}v$ are D-independent. Since $\dim_D V \geq 3$, there exists $w \in V$ such that w, v and $T^{-1}v$ are linearly D-independent. By the density of R, there exist $r_1, r_2, r_3, r_4 \in Q_r$ such that

$$r_1 v = 0, \qquad r_1 w = 2T^{-1}v - T^{-1}bv, \qquad r_1 T^{-1}v = v,$$
$$r_2 v = T^{-1}v, \qquad r_2 T^{-1}v = w, \qquad r_4 v = 0,$$
$$r_4 T^{-1}v = v - bv.$$

Therefore,

$$0 = a\Big(\big(b[r_1, r_2] + r_4 r_2 - Tr_2 T^{-1} r_4\big)[r_1, r_2]^n\Big)^s v = av.$$

By using the same argument as in Lemma 3.1, we get the contradiction $a = 0$.

Hence, $\dim_D V \leq 2$ and by (3.11) we have that Q_r satisfies

$$a\Big(\big(b[x_1, x_2] + y_1 x_2 - \alpha(x_2) y_1\big)[x_1, x_2]^n\Big)^2.$$

Since $\alpha(x_2)$-word degree is 2 and either $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) \geq 3$, by [13, Theorem 3], Q_r satisfies the identity

$$a\Big(\big(b[x_1, x_2] + y_1 x_2 - z_2 y_1\big)[x_1, x_2]^n\Big)^2. \tag{3.13}$$

In particular, if we take $z_2 = 0$, then

$$a\Big(\big(b[x_1, x_2] + y_1 x_2\big)[x_1, x_2]^n\Big)^2$$

is a generalized polynomial identity for Q_r. If $\dim_D V = 2$, then there exist $v, w \in V$ such that w, v are linearly D-independent. By the density of R, there exist $r_1, r_2, r_3 \in Q_r$ such that

$$r_1 v = 0, r_1 w = v, r_2 v = w, r_3 w = v - bv.$$

Thus,

$$0 = a\Big(\big(b[r_1, r_2] + r_3 r_2\big)[r_1, r_2]^n\Big)^2 v = av.$$

As above, the contradiction $a = 0$ follows. Therefore, $\dim_D V = 1$; that is, Q_r is a domain. Since $ab = 0$ and $a \neq 0$, we have $b = 0$. By (3.13), Q_r satisfies

$$\big(y_1 x_2 - z_2 y_1\big)[x_1, x_2].$$

In particular, for $y_1 = x_1$ and $z_2 = x_2$, one has that $[x_1, x_2]^2$ is a polynomial identity for Q_r. Hence, Q_r must be commutative, which is a contradiction again. ∎

4 A Potential Topic for Further Research

In the current presentation we deal with generalized skew derivations acting on Lie ideals in prime rings. In our final result, we definitely describe the form of a generalized skew derivation F and the structure of a prime ring R, in the case

$$a\Big(u^m F(u)u^n\Big)^s = 0, \quad \forall u \in L$$

where L is a non-central Lie ideal of R, $0 \neq a \in R$ and m, n, s are suitable fixed integers. Nevertheless, there are several interesting open questions related to our work. In this final section we will propose a potential topic for future further research.

In a recent paper [25], Koşan and Lee propose the following new definition. Let $d: R \to Q_r$ be an additive mapping and $b \in Q_r$. An additive map $F: R \to Q_r$ is called a *left b-generalized derivation*, with associated mapping d, if $F(xy) = F(x)y + bxd(y)$, for all $x, y \in R$. In the same paper it is proved that if R is prime ring, then d is a derivation of R. For simplicity of notation, this mapping F will be called *X-generalized derivation* with associated pair (b, d). Clearly, any generalized derivation with associated derivation d is an X-generalized derivation with associated pair $(1, d)$. Similarly the mapping $x \mapsto ax + b[x, c]$, for $a, b, c \in Q_r$, is an X-generalized derivation with associated pair $(b, ad(c))$, where $ad(c)(x) = [x, c]$ denotes the inner derivation of R induced by the element c. More generally, the mapping $x \mapsto ax + qxc$, for $a, q, c \in Q_r$, is an X-generalized derivation with associated pair $(q, \mathrm{ad}(c))$. This mapping is called *inner X-generalized derivation*. Moreover, if $\alpha \in \mathrm{Aut}(R)$, with $\alpha(x) = qxq^{-1}$ for q an invertible element of Q_r, and F is the inner generalized skew derivation with associated automorphism α, then F is an X-generalized derivation with associated pair $(q, \mathrm{ad}(q^{-1}b))$, for a suitable element $b \in Q_r$.

There arises the question of whether there exists a unified definition of X-generalized derivation and generalized skew derivation. In view of this idea, we now give a new definition that is a common generalization of the previous two definitions.

Definition 4.1 Let R be an associative algebra, let $b \in Q_r$, $d: R \to R$ be a linear mapping, and let α be an automorphism of R. A linear mapping $F: R \to R$ is called an *X-generalized skew derivation* of R, with associated term (b, α, d) if

$$F(xy) = F(x)y + b\alpha(x)d(y)$$

for all $x, y \in R$.

I would like to give some examples of additive mappings that are X-generalized skew derivations.

Example 4.2 Let R be an associative algebra and α be an automorphism of R. The mapping

$$F: R \longrightarrow R, \quad x \longmapsto ax + b\alpha(x)c$$

is an X-generalized skew derivation of R with associated term (b, α, d), where a, b and c are fixed elements in R and $d(x) = \alpha(x)c - cx$, for all $x \in R$. Indeed, for all

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$x, y \in R$,

$$F(xy) = axy + b\alpha(xy)c = axy - b\alpha(x)cy + b\alpha(x)cy + b\alpha(x)\alpha(y)c$$
$$= (ax + b\alpha(x)c)y + b\alpha(x)(\alpha(y)c - cy) = F(x)y + b\alpha(x)d(y),$$

where $d(y) := \alpha(y)c - cy$ is an inner skew derivation of R induced by the element $c \in R$, with associated automorphism α. Such X-generalized skew derivations are called *inner X-generalized skew derivations*.

Example 4.3 Let R be an associative algebra, let F be an X-generalized derivation of R with associated pair (b, d), where d is a derivation and b is an invertible element of R. Then the mapping

$$G: R \longrightarrow R, \quad x \longmapsto aF(x)$$

is an X-generalized skew derivation of R. Indeed, for all $x, y \in R$,

$$G(xy) = aF(xy) = aF(x)y + abxd(y) = aF(x)y + abxb^{-1}bd(y)$$
$$= G(x)y + a\alpha(x)bd(y),$$

where $\alpha(x) := bx^{-1}$. Thus, G is an X-generalized skew derivation, with associated term (a, α, bd). It should be remarked that the mapping $x \mapsto bd(x)$ is a skew derivation of R with associated automorphism α.

Example 4.4 Let R be an associative algebra, let I_R be the identical mapping of R, let α be an automorphism of R, and $a, b \in R$. Then $F(x) = b(\alpha - aI_R)(x)$ is an X-generalized skew derivation of R with associated term $(b, \alpha, \alpha - I_R)$. Indeed, for all $x, y \in R$ we get

$$F(xy) = b(\alpha - I_R)(xy) = b\alpha(x)\alpha(y) - baxy$$
$$= b\alpha(x)\alpha(y) - b\alpha(x)y + b\alpha(x)y - baxy$$
$$= b(\alpha - I_R)(x)y + b\alpha(x)(\alpha - I_R)(y) = F(x)y + b\alpha(x)d(y),$$

where $d(x) = \alpha(x) - x$ for all $x \in R$. Note that d is a skew derivation of R in this case.

Example 4.5 Let R be an associative algebra, let α be an automorphism of R, let d be a skew derivation associated with α and $b \in R$. Then $F(x) = b(\alpha - d)(x)$ is an X-generalized skew derivation of R with associated term $(b, \alpha, \alpha - I_R - d)$. Indeed, for all $x, y \in R$, we have

$$F(xy) = b(\alpha - d)(xy) = b\alpha(x)\alpha(y) - bd(x)y - b\alpha(x)d(y)$$
$$= b\alpha(x)\alpha(y) - b\alpha(x)y + b\alpha(x)y - bd(x)y - b\alpha(x)d(y)$$
$$= F(x)y + b\alpha(x)\big(\alpha(y) - y - d(y)\big),$$

where the skew derivation $\delta(x) = \alpha(x) - x - d(x)$ is the additive mapping associated with F.

Example 4.6 Let R be an associative algebra, α an automorphism of R, d a skew derivation associated with α, F a generalized skew derivation associated with α and

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d, and $b \in R$. Then $G(x) = b(\alpha - F)(x)$ is an X-generalized skew derivation of R with associated term $(b, \alpha, \alpha - I_R - d)$. Indeed, for all $x, y \in R$, we obtain

$$\begin{aligned} G(xy) = b(\alpha - F)(xy) &= b\alpha(x)\alpha(y) - bF(x)y - b\alpha(x)d(y) \\ &= b\alpha(x)\alpha(y) - b\alpha(x)y + b\alpha(x)y - bF(x)y - b\alpha(x)d(y) \\ &= G(x)y + b\alpha(x)\big(\alpha(y) - y - d(y)\big). \end{aligned}$$

According to the above examples, we can conclude that general results about X-generalized skew derivations may give useful and powerful corollaries about derivations, generalized derivations, skew derivations and generalized skew derivations.

In view of this and taking account of Theorem 1.2, one natural question arises.

Question 4.7 Let R be a prime ring of characteristic different from 2, let Q_r be its right Martindale quotient ring, and let C be its extended centroid. Suppose that F is a X-generalized skew derivation of R, L a non-central Lie ideal of R, $0 \neq a \in R$, $m \geq 0$ and $n, s \geq 1$ fixed integers, such that

$$a\big(u^m F(u) u^n\big)^s = 0, \quad \forall u \in L.$$

(i) How do we describe the form of F?
(ii) What we can say about the structure of R?

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Department of Mathematics and Computer Science, University of Messina, 98166, Messina, Italy
e-mail: defilippis@unime.it