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ON THE OPERATORS WHICH ARE INVERTIBLE MODULO AN OPERATOR IDEAL

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We study the semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ of left and right invertible operators modulo an operator ideal $\mathcal{A}$, respectively. We show that these semigroups allow us to obtain useful characterisations of the radical $\mathcal{A}^{\text{rad}}$ of $\mathcal{A}$. For example, $\mathcal{A}^{\text{rad}}$ is the perturbation class for $\mathcal{A}_l$ and $\mathcal{A}_r$.

1. INTRODUCTION

Atkinson [**3**] studied the operators which are left invertible $\Phi_l(X,Y)$ or right invertible $\Phi_r(X,Y)$ modulo $\mathcal{K}$, with $\mathcal{K}$ the compact operators. He proved that an operator $T \in \mathcal{L}(X,Y)$ belongs to Φ_l or Φ_r if and only if the kernel and the range of T are complemented and additionally, the kernel is finite dimensional or the range is finite codimensional, respectively. Yood [**19**] obtained some perturbation results for these classes and Lebow and Schechter [**12**] proved that the *inessential* operators form the perturbation class for $\Phi_l(X)$ and $\Phi_r(X)$.

Yang [**18**] extended some results of [**3, 19**] to operators invertible modulo $\mathcal{W}$, with $\mathcal{W}$ the weakly compact operators. His aim was to study a generalised Fredholm theory in which the reflexive spaces played the role of the finite dimensional spaces. Moreover, Astala and Tylli [**4**] compared the left-invertible operators modulo $\mathcal{W}$ with the Tauberian operators and other classes of operators defined in terms of weak compactness.

In this paper we study the classes $\mathcal{A}_l$ and $\mathcal{A}_r$ of operators which are left and right invertible respectively, modulo an operator ideal $\mathcal{A}$. We show that these classes are open semigroups in the sense of [**2**], and that there is a close connection between $\mathcal{A}_l$, $\mathcal{A}_r$ and the radical $\mathcal{A}^{\text{rad}}$ (in the sense of [**15**]) of the operator ideal $\mathcal{A}$. In fact, if $\mathcal{A}$, $\mathcal{B}$ are operator ideals, then the equalities $\mathcal{A}^{\text{rad}} = \mathcal{B}^{\text{rad}}$, $\mathcal{A}_l = \mathcal{B}_l$ and $\mathcal{A}_r = \mathcal{B}_r$ are equivalent. We obtain characterisations of $\mathcal{A}^{\text{rad}}$ simpler than the original definition in [**15**] and we show that $\mathcal{A}^{\text{rad}}$ is the perturbation class for $\mathcal{A}_l$, $\mathcal{A}_l$ and $\mathcal{A}_l \cap \mathcal{A}_r$.

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Denoting by $\mathcal{W}$ the weakly compact operators, we compare $\mathcal{W}_l$ with Φ_l and $\mathcal{W}_r$ with Φ_r. They coincide in $\mathcal{L}(X,Y)$ when one of the spaces X, Y has the Dunford-Pettis property. Moreover, we give several examples of operator ideals $\mathcal{A}$ which are maximal in the sense that $\mathcal{B} \subset \mathcal{A}$ for each operator ideal $\mathcal{B}$ such that $\mathrm{Sp}\,(\mathcal{A}) = \mathrm{Sp}\,(\mathcal{B})$; in particular, $\mathcal{A} = \mathcal{A}^{\mathrm{rad}}$. Observe that it is not known whether $\mathcal{K}^{\mathrm{rad}}$ is maximal in this sense ([**15**, 4.3.7]).

In this paper paper, X, Y and Z are Banach spaces and X^* is the *dual space* of X. Also $\mathcal{L}(X,Y)$ is the set of all (continuous linear) operators from X into Y. For $T \in \mathcal{L}(X,Y)$, $N(T)$ and $R(T)$ are the *kernel* and the *range* of T, and $T^* \in \mathcal{L}(Y^*, X^*)$ is the *conjugate* operator of T.

Given a closed subspace M of X, we denote by J_M the inclusion of M into X, and by Q_M the quotient map from X onto X/M. A subspace M of X is *complemented* if it is the range of a continuous linear projection on X. Of course, complemented subspaces are closed.

We denote by $\mathcal{L}$ the class of all operators between Banach spaces; that is, the union of all the sets $\mathcal{L}(X,Y)$, and by $\mathcal{F}$ and $\mathcal{G}$ the subclasses of all *finite rank* and all *bijective* operators, respectively. Given a subclass $\mathcal{A} \subset \mathcal{L}$, the *components* of $\mathcal{A}$ are the subsets

$$\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y).$$

We write $\mathcal{A}(X)$ in the case $X = Y$.

DEFINITION 1.1: ([**15**]) A subclass $\mathcal{A} \subset \mathcal{L}$ is an *operator ideal* if it satisfies the following conditions:

(α_1) $\mathcal{F} \subset \mathcal{A}$.

(α_2) $\mathcal{A}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$.

(α_3) If $B \in \mathcal{L}(W,X)$, $T \in \mathcal{A}(X,Y)$ and $A \in \mathcal{L}(Y,Z)$, then $ATB \in \mathcal{A}(W,Z)$.

Every operator ideal $\mathcal{A}$ has associated a *space ideal* $\mathrm{Sp}\,(\mathcal{A})$, given by

$$\mathrm{Sp}\,(\mathcal{A}) := \{X \ : \ \text{the identity } I_X \in \mathcal{A}\}.$$

The *dual operator ideal* $\mathcal{A}^d$ of $\mathcal{A}$ is defined by $\mathcal{A}^d = \{T \in \mathcal{L} : T^* \in \mathcal{A}\}$. Clearly $\mathrm{Sp}\left(\mathcal{A}^d\right) = \left\{X : X^* \in \mathrm{Sp}\,(\mathcal{A})\right\}$.

EXAMPLES OF OPERATOR IDEALS. ([**15**]) Let $T \in \mathcal{L}(X,Y)$.

$\mathcal{K}$ *compact* operators: $T \in \mathcal{K}$ if it takes bounded sets into relatively compact sets.

$\mathcal{SS}$ *strictly singular* operators: $T \in \mathcal{SS}$ if there is no infinite dimensional closed subspace M of X so that the restriction $T \mid M$ is an isomorphism.

$\mathcal{W}$ *weakly compact* operators: $T \in \mathcal{W}$ if it takes bounded sets into relatively weakly compact sets.

$\mathcal{CC}$ *completely continuous* operators: $T \in \mathcal{CC}$ if it takes weakly Cauchy sequences into convergent sequences.

$\mathcal{UC}$ unconditionally converging operators: $T \in \mathcal{UC}$ if it takes weakly unconditionally Cauchy series into unconditionally convergent series.

The space ideals of $\mathcal{K}$, $\mathcal{W}$, $\mathcal{CC}$ and $\mathcal{UC}$ are the finite dimensional spaces, the reflexive spaces, the spaces with the Schur property and the spaces that contain no subspaces isomorphic to c_0, respectively.

The following concept of semigroup was introduced in [**2**].

DEFINITION 1.2: A subclass $\mathcal{S} \subset \mathcal{L}$ is an *operator semigroup* (a *semigroup,* for short) if it satisfies the following conditions:

(σ_1) $\mathcal{G} \subset \mathcal{S}$.

(σ_2) $S \in \mathcal{S}(W,Y)$ and $T \in \mathcal{S}(X,Z)$ if and only if $S \oplus T \in \mathcal{S}(W \oplus X, Y \oplus Z)$.

(σ_3) If $T \in \mathcal{S}(X,Y)$ and $S \in \mathcal{S}(Y,Z)$, then $ST \in \mathcal{S}(X,Z)$.

EXAMPLES OF SEMIGROUPS. [**2**] Every operator ideal $\mathcal{A}$ has associated two semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$, given by

$$\mathcal{A}_+ := \{T \in \mathcal{L} : S \in \mathcal{L},\, TS \in \mathcal{A} \Rightarrow S \in \mathcal{A}\};$$

that is, $T \in \mathcal{A}_+(X,Y)$ if and only if for every Z and every $S \in \mathcal{L}(Z,X)$, we have $TS \in \mathcal{A}$ implies $S \in \mathcal{A}$. Analogously,

$$\mathcal{A}_- := \{T \in \mathcal{L} : S \in \mathcal{L},\, ST \in \mathcal{A} \Rightarrow S \in \mathcal{A}\}.$$

The semigroups $\mathcal{K}_+$, $\mathcal{K}_-$, $\mathcal{W}_+$ and $\mathcal{W}_-$ coincide with the upper semi-Fredholm, lower semi-Fredholm, Tauberian ([**10**]) and cotauberian operators ([**17**]), respectively. The semigroups $\mathcal{UC}_+$ and $\mathcal{UC}_-$ were studied in [**8**]. We refer to [**2**] for further results.

DEFINITION 1.3: Let X and Y be Banach spaces, and let $\mathcal{S}$ be a semigroup such that $\mathcal{S}(X,Y)$ is non-empty.

The component $P\mathcal{S}(X,Y)$ of the *perturbation class* $P\mathcal{S}$ of $\mathcal{S}$ is defined by

$$P\mathcal{S}(X,Y) := \Big\{K \in \mathcal{L}(X,Y) : T + K \in \mathcal{S}(X,Y) \text{ for every } T \in \mathcal{S}(X,Y)\Big\}.$$

PROPOSITION 1.4. ([**12, 2**]) *Let $\mathcal{S}$ be a semigroup. Then*

(a) *$P\mathcal{S}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$ and $P\mathcal{S}(X)$ is a two-sided ideal in $\mathcal{L}(X)$.*

(b) *If $\mathcal{S}(X,Y)$ is an open subset of $\mathcal{L}(X,Y)$, then $P\mathcal{S}(X,Y)$ is closed.*

An even more general concept of perturbation class was introduced by Lebow and Schechter [**12**].

2. MAIN RESULTS

In this section $\mathcal{A}$ is an operator ideal. We introduce and study the semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ of operators which are left or right invertible modulo $\mathcal{A}$.

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DEFINITION 2.1: Let $T \in \mathcal{L}(X,Y)$. We define the classes $\mathcal{A}_l$ and $\mathcal{A}_r$ as follows:

$T \in \mathcal{A}_l$ if there exists $A \in \mathcal{L}(Y,X)$ such that $I_X - AT \in \mathcal{A}(X)$.

$T \in \mathcal{A}_r$ if there exists $B \in \mathcal{L}(Y,X)$ such that $I_Y - TB \in \mathcal{A}(Y)$.

Let us denote by $\mathcal{RR}$ the *relatively regular* operators ([**3**]); that is, the operators with complemented kernel and range. The relatively regular operators provide basic examples of operators in $\mathcal{A}_l$ and $\mathcal{A}_r$.

EXAMPLES 2.2.

(a) Let $T \in \mathcal{L}(X,Y)$. Then

$$T \in \mathcal{K}_l \Longleftrightarrow T \in \mathcal{RR} \text{ and } \dim N(T) < \infty;$$

$$T \in \mathcal{K}_r \Longleftrightarrow T \in \mathcal{RR} \text{ and } \dim Y/R(T) < \infty.$$

(b) Let $T \in \mathcal{RR}(X,Y)$. Then

$$N(T) \in \mathrm{Sp}\,(\mathcal{A}) \Longleftrightarrow T \in \mathcal{A}_l \quad \text{and} \quad Y/R(T) \in \mathrm{Sp}\,(\mathcal{A}) \Longleftrightarrow T \in \mathcal{A}_r,$$

but Example (c) below shows that neither $\mathcal{A}_l$ nor $\mathcal{A}_r$ is contained in $\mathcal{RR}$ in general.

(c) If $X \in \mathrm{Sp}\,(\mathcal{A})$, then

$$\mathcal{A}_l(X,Y) = \mathcal{L}(X,Y) \text{ and } \mathcal{A}_r(Y,X) = \mathcal{L}(Y,X), \text{ for every } Y.$$

The following result can be derived easily from the definitions.

PROPOSITION 2.3. *The classes $\mathcal{A}_l$ and $\mathcal{A}_r$ are semigroups in the sense of Definition 1.2. Moreover, $\mathcal{A}_l \subset \mathcal{A}_+$ and $\mathcal{A}_r \subset \mathcal{A}_-$.*

REMARK 2.4. The class $\mathcal{A}_l \cap \mathcal{A}_r$ is also a semigroup. We have $\mathcal{K}_l \cap \mathcal{K}_r = \mathcal{K}_+ \cap \mathcal{K}_-$, the Fredholm operators, but an example in [**9**] shows that $\mathcal{W}_l \cap \mathcal{W}_r \neq \mathcal{W}_+ \cap \mathcal{W}_-$.

Indeed, for every $T \in \mathcal{L}(X,Y)$ we consider the operator $T^{co} \in \mathcal{L}(X^{**}/X, Y^{**}/Y)$, given by $T^{co}(z+X) = T^{**}z + Y$. Since $T \in \mathcal{W}_+$ if and only if T^{co} is injective, and $T \in \mathcal{W}_-$ if and only if T^{co} has dense range, $T \in \mathcal{W}_+ \cap \mathcal{W}_-$ whenever T^{co} is bijective.

On the other hand, for $X = \ell_2(J)$, where J is the quasireflexive James' space, X^{**}/X is isomorphic to ℓ_2 and $T \in \mathcal{L}(X)$ belongs to $\mathcal{W}_l \cap \mathcal{W}_r$ if and only if T^{co} is regular with respect to the lattice structure in ℓ_2. Since there are examples of operators $T \in \mathcal{L}(X)$ such that T^{co} is bijective but not regular, $\mathcal{W}_l \cap \mathcal{W}_r \neq \mathcal{W}_+ \cap \mathcal{W}_-$.

We refer to [**9**] for further details.

The following result gives basic properties of $\mathcal{A}_l$ and $\mathcal{A}_r$.

PROPOSITION 2.5. *Let $S \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$.*

(a) *$ST \in \mathcal{A}_l \Rightarrow T \in \mathcal{A}_l$ and $ST \in \mathcal{A}_r \Rightarrow S \in \mathcal{A}_r$.*

(b) *For every $K \in \mathcal{A}(X,Y)$,*

$$T \in \mathcal{A}_l \Rightarrow T + K \in \mathcal{A}_l \quad \text{and} \quad T \in \mathcal{A}_r \Rightarrow T + K \in \mathcal{A}_r.$$

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(c) *$T \in \mathcal{A}^d{}_l \Rightarrow T^* \in \mathcal{A}_r$ and $T \in \mathcal{A}^d{}_r \Rightarrow T^* \in \mathcal{A}_l$. However, the converse implications are not true.*

(d) *$\mathcal{A}_l(X,Y) \neq \emptyset$ if and only if $\mathcal{A}_r(Y,X) \neq \emptyset$.*

(e) *$\mathcal{A}_l(X,Y)$ and $\mathcal{A}_r(X,Y)$ are open subsets of $\mathcal{L}(X,Y)$.*

PROOF: We indicate the proof of some of the results. The remaining ones are direct consequences of the definitions.

(c) Let $J : c_0 \to \ell_\infty$ be the natural inclusion. Then $J \notin \mathcal{K}_l$ because c_0 is not complemented in ℓ_∞. However, $N(J^*)$ is complemented by the lifting property of ℓ_1; hence $J^* \in \Phi_r = \mathcal{K}_r$. Note that $\mathcal{K} = \mathcal{K}^d$ by Schauder's theorem.

(e) Let $T \in \mathcal{A}_l(X,Y)$ and $A \in \mathcal{L}(Y,X)$ such that $AT = I_X + K$ with $K \in \mathcal{A}(X)$. If $S \in \mathcal{L}(X,Y)$ and $\|A\| \cdot \|S\| < 1$, then $I_X + AS$ is invertible, hence $A(T+S) = I_X + AS + K \in \mathcal{A}_l$ by part (b); thus $T + S \in \mathcal{A}_l$ by part (a). ∎

The concept of radical of an operator ideal is due to Pietsch [**15**, Section 4.3].

DEFINITION 2.6: ([**15**]) The *radical* $\mathcal{A}^{\text{rad}}$ of $\mathcal{A}$ is defined as follows:

$$\mathcal{A}^{\text{rad}}(X,Y) := \left\{ K \in \mathcal{L}(X,Y) : \begin{array}{l} \text{for every } S \in \mathcal{L}(Y,X), \text{ there exists } U \in \mathcal{L}(X), \\ \text{so that } I_X - U(I_X - SK) \in \mathcal{A} \end{array} \right\}.$$

$\mathcal{A}^{\text{rad}}$ is an operator ideal that contains $\mathcal{A}$. Moreover, $\mathrm{Sp}\left(\mathcal{A}^{\text{rad}}\right) = \mathrm{Sp}(\mathcal{A})$ and $\mathcal{A}^{\text{rad}}(X,Y)$ is always closed in $\mathcal{L}(X,Y)$ ([**15**]).

LEMMA 2.7. *In the definition of $\mathcal{A}^{\text{rad}}$ we can write $I_X - (I_X - SK)U \in \mathcal{A}$ instead of $I_X - U(I_X - SK) \in \mathcal{A}$.*

PROOF: If $K \in \mathcal{A}^{\text{rad}}$ and $L_1 := I_X - U(I_X - SK) \in \mathcal{A}$, then $I_X - U \in \mathcal{A}^{\text{rad}}$. Thus there exists $W \in \mathcal{L}(X)$ so that $I_X - WU \in \mathcal{A}$, and

$$(I_X - SK)U = WU(I_X - SK)U - L_2 = W(I_X - L_1)U - L_2 = I_X - L_3,$$

with $L_2, L_3 \in \mathcal{A}$; hence $I_X - (I_X - SK)U \in \mathcal{A}$.

The converse implication is similar. ∎

The best known example of radical is $\mathcal{I} := \mathcal{F}^{\text{rad}}$, the *inessential operators*. These operators were introduced by Yood [**19**] in the study of the perturbation of Fredholm operators. Kleinecke [**11**] introduced the name *inessential operators* and proved that $\mathcal{I}(X)$ is the biggest ideal of operators in $\mathcal{L}(X)$ with Riesz spectrum.

REMARK 2.8. ([**15**, 4.3.8])

(a) $\mathcal{I}$ is contained in $\mathcal{A}^{\text{rad}}$ for every $\mathcal{A}$.
(b) $\mathcal{A} \subset \mathcal{B}$ implies $\mathcal{A}^{\text{rad}} \subset \mathcal{B}^{\text{rad}}$ for every pair of operator ideals $\mathcal{A}$ and $\mathcal{B}$.
(c) $\left(\mathcal{A}^{\text{rad}}\right)^{\text{rad}} = \mathcal{A}^{\text{rad}}$.

REMARK 2.9. In general, $\mathcal{A}^{\text{rad}}$ is much bigger than $\mathcal{A}$. For example [**7**, Theorem 1], $\mathcal{L}(X,Y) = \mathcal{I}(X,Y)$ and $\mathcal{L}(Y,X) = \mathcal{I}(Y,X)$ in the following cases:

(i) X contains no copies of ℓ_∞ and $Y = \ell_\infty$.
(ii) X contains no complemented copies of c_0 and $Y = C[0,1]$;
(iii) X contains no complemented copies of ℓ_p and $Y = L_p[0,1]$, $1 \leqslant p < \infty$.

Pietsch conjectured that the following question has a positive answer.

QUESTION 2.10. ([**15**, 4.3.7]) *Is $\mathcal{A}^{\text{rad}}$ the biggest operator ideal $\mathcal{B}$ such that* $\text{Sp}\,(\mathcal{B}) = \text{Sp}\,(\mathcal{A})$ *?*

As far as we know, the question is open even for $\mathcal{F}^{\text{rad}} = \mathcal{I}$.

The semigroups associated to an operator ideal allow us to give a simpler description of the radical.

PROPOSITION 2.11. *For every $T \in \mathcal{L}(X,Y)$, the following assertions are equivalent:*

(a) $T \in \mathcal{A}^{\text{rad}}(X,Y)$.
(b) *For every* $S \in \mathcal{L}(Y,X)$, $I_X - ST \in \mathcal{A}_l(X)$.
(b′) *For every* $S \in \mathcal{L}(Y,X)$, $I_Y - TS \in \mathcal{A}_l(Y)$.
(c) *For every* $S \in \mathcal{L}(Y,X)$, $I_X - ST \in \mathcal{A}_r(X)$.
(c′) *For every* $S \in \mathcal{L}(Y,X)$, $I_Y - TS \in \mathcal{A}_r(Y)$.

PROOF: (a) $\Leftrightarrow$ (b). Assume $T \in \mathcal{A}^{\text{rad}}(X,Y)$ and $S \in \mathcal{L}(Y,X)$. Then there exists $U \in \mathcal{L}(X)$ such that $K_1 := I_X - U(I_X - ST) \in \mathcal{A}$. Thus $U(I_X - ST) = I_X - K_1 \in \mathcal{A}_l \cap \mathcal{A}_r$, hence $I_X - ST \in \mathcal{A}_l$ (Proposition 2.5.a).

Conversely, if $I_X - ST \in \mathcal{A}_l$ for every $S \in \mathcal{L}(Y,X)$, then any left inverse modulo $\mathcal{A}$ of $I_X - ST$ can be taken as U in Definition 2.6.

The proof of (a) $\Leftrightarrow$ (c) is similar to that of (a) $\Leftrightarrow$ (b), using Lemma 2.7.

(b) $\Leftrightarrow$ (b′) and (c) $\Leftrightarrow$ (c′). If $U \in \mathcal{L}(X)$ is a left (respectively, right) inverse of $I_X - ST$ modulo $\mathcal{A}$, then $I_Y + TUS$ is a left (respectively, right) inverse of $I_Y - TS$ modulo $\mathcal{A}$. ∎

COROLLARY 2.12. $\mathcal{L}(X,Y) = \mathcal{A}^{\text{rad}}(X,Y) \Longleftrightarrow \mathcal{L}(Y,X) = \mathcal{A}^{\text{rad}}(Y,X)$.

Let us see that the perturbation classes of the semigroups $\mathcal{A}_l$, $\mathcal{A}_r$ and $\mathcal{A}_l \cap \mathcal{A}_r$ coincide with the radical $\mathcal{A}^{\text{rad}}$.

THEOREM 2.13. *Let $\mathcal{S}$ be one of the semigroups $\mathcal{A}_l$, $\mathcal{A}_r$ or $\mathcal{A}_l \cap \mathcal{A}_s$. If $\mathcal{S}(X,Y)$ is non-empty, then*

$$P\mathcal{S}(X,Y) = \mathcal{A}^{\text{rad}}(X,Y).$$

PROOF: We give the proof for $\mathcal{S} = \mathcal{A}_l$. The proof of the other cases is analogous. Let $K \in \mathcal{A}^{\text{rad}}(X,Y)$. For every $T \in \mathcal{A}_l(X,Y)$, we can select $A \in \mathcal{L}(Y,X)$ so that

$$D_1 := I_X - AT \in \mathcal{A}(X).$$

For the operator $-A \in \mathcal{L}(Y,X)$, the definition of $\mathcal{A}^{\text{rad}}$ gives us another operator $U \in \mathcal{L}(X)$ so that $D_2 := I_X - U(I_X + AK) \in \mathcal{A}$. Since $\mathcal{A} \subset P\mathcal{A}_l$ (Proposition 2.5.b),

$$UA(T+K) = U(I_X - D_1 + AK) = I_X - D_2 - UD_1 \in \mathcal{A}_l;$$

hence $T + K \in \mathcal{A}_l$. Thus $\mathcal{A}^{\text{rad}}(X,Y) \subset P(\mathcal{A}_l)(X,Y)$.

For the converse inclusion, we show first that

$$K \in P\mathcal{A}_l(X,Y), A \in \mathcal{L}(Y) \Rightarrow AK \in P\mathcal{A}_l(X,Y). \tag{1}$$

If $U \in \mathcal{A}_l(X,Y)$ and A is invertible, then $U + AK = A(A^{-1}U + K) \in \mathcal{A}_l$; hence $AK \in P\mathcal{A}_l$. For the general case it is enough to observe that every $A \in \mathcal{L}(Y)$ can be written as the sum of two invertible operators.

Now, let $K \in \mathcal{L}(X,Y)$, $K \notin \mathcal{A}^{\text{rad}}$. By Proposition 2.11, there exists $A \in \mathcal{L}(Y,X)$ such that $I_X - AK \notin \mathcal{A}_l(X)$.

Let $U \in \mathcal{A}_l(X,Y)$. Then $U(I_X - AK) = U - (UA)K \notin \mathcal{A}_l(X,Y)$. Therefore $(UA)K \notin P\mathcal{A}_l$, and (1) implies $K \notin P\mathcal{A}_l(X)$. ∎

The semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ determine and are determined by $\mathcal{A}^{\text{rad}}$.

PROPOSITION 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals. Then*

$$\mathcal{A}^{\text{rad}}(X) = \mathcal{B}^{\text{rad}}(X) \Longleftrightarrow \mathcal{A}_l(X) = \mathcal{B}_l(X) \Longleftrightarrow \mathcal{A}_r(X) = \mathcal{B}_r(X).$$

PROOF: Assume that $\mathcal{A}^{\text{rad}}(X) = \mathcal{B}^{\text{rad}}(X)$. Since $\mathcal{A} \subset \mathcal{A}^{\text{rad}} = \left(\mathcal{A}^{\text{rad}}\right)^{\text{rad}}$ (Remark 2.8), $\mathcal{A}_l \subset {\mathcal{A}^{\text{rad}}}_l$ and $\mathcal{A}_r \subset {\mathcal{A}^{\text{rad}}}_r$. To prove the other two equalities it is enough to show that ${\mathcal{A}^{\text{rad}}}_l(X) \subset \mathcal{A}_l(X)$ and ${\mathcal{A}^{\text{rad}}}_r(X) \subset \mathcal{A}_r(X)$.

If $T \in {\mathcal{A}^{\text{rad}}}_l(X)$, then we can find $A \in \mathcal{L}(X)$ such that $I_X - AT \in \mathcal{A}^{\text{rad}}$. By Theorem 2.13, $AT \in \mathcal{A}_l$; hence $T \in \mathcal{A}_l$.

The proof for $\mathcal{A}_r$ is similar.

If $\mathcal{A}_l(X) = \mathcal{B}_l(X)$ or $\mathcal{A}_r(X) = \mathcal{B}_r(X)$, then it follows from Proposition 2.11 that $\mathcal{A}^{\text{rad}}(X) = \mathcal{B}^{\text{rad}}(X)$. ∎

COROLLARY 2.15. $\mathcal{A}^{\text{rad}} = \mathcal{B}^{\text{rad}} \Longleftrightarrow \mathcal{A}_l = \mathcal{B}_l \Longleftrightarrow \mathcal{A}_r = \mathcal{B}_r$.

PROOF: It is enough to note that the proof of Proposition 2.14 shows that $\mathcal{A}(X) = \mathcal{A}^{\text{rad}}(X)$ implies $\mathcal{A}_l(X,Y) = \mathcal{B}_l(X,Y)$ and $\mathcal{A}_r(Y,X) = \mathcal{B}_r(Y,X)$ for every Y, and $\mathcal{A}_l(X) = \mathcal{B}_l(X)$ or $\mathcal{A}_r(X) = \mathcal{B}_r(X)$ for every X implies $\mathcal{A}^{\text{rad}}(X,Y) = \mathcal{B}^{\text{rad}}(X,Y)$ for every X, Y. ∎

3. Examples

Here we give information on the semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ in some cases. First we recall some useful characterisations of the inessential operators.

Theorem 3.1. [14, 1] *For $K \in \mathcal{L}(X, Y)$ the following assertions are equivalent:*

(a) *K is inessential.*
(b) *For every $S \in \mathcal{L}(Y, X)$ the kernel of $I_X - SK$ is finite dimensional.*
(b′) *For every $S \in \mathcal{L}(Y, X)$ the kernel of $I_Y - KS$ is finite dimensional.*
(c) *For every $S \in \mathcal{L}(Y, X)$ the cokernel $X/\overline{R(I_X - SK)}$ is finite dimensional.*
(c′) *For every $S \in \mathcal{L}(Y, X)$ the cokernel $Y/\overline{R(I_Y - KS)}$ is finite dimensional.*

Question 3.2. *Is it possible to find characterisations of $\mathcal{A}^{\mathrm{rad}}$ similar to Theorem 3.1 for other semigroups $\mathcal{A}$?*

Recall that a Banach space X is said to have the *Dunford-Pettis property* if $\mathcal{W}(X, Y) \subset \mathcal{CC}(X, Y)$ for every Banach space Y. The spaces $C(K)$ and $L_1(\mu)$ have the Dunford-Pettis property. We refer to [6] for additional information.

Proposition 3.3. *Assume that X or Y has the Dunford-Pettis property. Then*

$$\mathcal{W}_l(X, Y) = \Phi_l(X, Y) \quad \text{and} \quad \mathcal{W}_r(X, Y) = \Phi_r(X, Y).$$

Proof: Recall that $\mathcal{K}_l = \Phi_l$ and $\mathcal{K}_r = \Phi_r$. If X has the Dunford-Pettis property and $T \in \mathcal{W}(X, Y)$, then $ST \in (\mathcal{W} \cap \mathcal{CC})(X)$ for every $S \in \mathcal{L}(Y, X)$. Thus $(ST)^2 \in \mathcal{K}$. In particular, $\dim N(I - ST) < \infty$. By Theorem 3.1, T is inessential. Hence,

$$\mathcal{K}(X, Y) \subset \mathcal{W}(X, Y) \subset \mathcal{I}(X, Y).$$

Since $\mathcal{K}_l = \mathcal{I}_l$, it follows from Corollary 2.15 that $\mathcal{W}_l(X, Y) = \mathcal{K}_l(X, Y)$.

The proof of the other cases is analogous. ∎

Corollary 3.4. *If X has the Dunford-Pettis property and Y is a non-complemented closed subspace of X, then the inclusion $J_Y : Y \to X$ does not belong to $\mathcal{W}_l$.*

Proof: Clearly, $J_Y \notin \Phi_l(Y, X) = \mathcal{W}_l(Y, X)$. ∎

Remark 3.5. The Hardy space H^1 is a closed non-complemented subspace of L^1. Since L^1 has the Dunford-Pettis property, it follows from the previous corollary that the inclusion of H^1 into L^1 does not belong to $\mathcal{W}_l$. This result was previously established in [4, Lemma 8] using techniques of harmonic analysis.

Apart from the inessential operators, we can show other examples of operator ideals which coincide with its radical.

DEFINITION 3.6: We say that an operator ideal $\mathcal{A}$ is *radical* if $\mathcal{A} = \mathcal{A}^{\text{rad}}$.

DEFINITION 3.7: Let Z_0 be a fixed infinite dimensional Banach space. An operator $T \in \mathcal{L}(X, Y)$ belongs to the class Z_0-$\mathcal{NC}$ if there is no closed subspace M of X isomorphic to Z_0 so that the restriction $T|_M$ is an isomorphism, and $T(M)$ is complemented in Y.

For some spaces, like $Z_0 = \ell_1 \oplus \ell_2$, the identity I_{Z_0} is not in Z_0-$\mathcal{NC}$, but we can write it as the sum of two projections that are both in Z_0-$\mathcal{NC}$. Thus Z_0-$\mathcal{NC}(Z_0)$ is not a subspace of $\mathcal{L}(Z_0)$. However, we shall give examples of spaces so that Z_0-$\mathcal{NC}$ is an operator ideal.

PROPOSITION 3.8. *Let Z_0 be a fixed infinite dimensional Banach space. Then*

(a) *Z_0-$\mathcal{NC}$ satisfies properties (α_1) and (α_3) in Definition 1.1. Moreover,*

$$I_X \in Z_0\text{-}\mathcal{NC} \iff X \text{ contains no complemented copy of } Z_0 .$$

(b) *Suppose that Z_0-$\mathcal{NC}$ is an operator ideal. Then every operator ideal $\mathcal{A}$ that satisfies $\mathrm{Sp}(\mathcal{A}) = \mathrm{Sp}(Z_0\text{-}\mathcal{NC})$ is contained in Z_0-$\mathcal{NC}$. In particular, Z_0-$\mathcal{NC}$ is a radical operator ideal.*

PROOF: (a) Since Z_0 is infinite dimensional, the finite rank operators are contained in Z_0-$\mathcal{NC}$. So it satisfies property (α_1).

To prove property (α_3), observe that for every operator $T \in \mathcal{L}(X, Y)$, if the restriction $T|_M$ on a closed subspace M of X is an isomorphism and $T(M)$ is complemented in Y, then M is complemented in X. Indeed, if $Y = N \oplus T(M)$, then $X = T^{-1}(N) \oplus M$.

We take $B \in \mathcal{L}(W, X)$, $T \in \mathcal{L}(X, Y)$ and $A \in \mathcal{L}(Y, Z)$, and we suppose that $ATB \notin Z_0$-$\mathcal{NC}(W, Z)$. Then there exists a closed subspace M of W isomorphic to Z_0 so that $ATB|_M$ is an isomorphism and $(ATB)(M)$ is complemented in Z. Let N be a closed subspace of Z such that $Z = N \oplus (ATB)(M)$. Then $Y = A^{-1}(N) \oplus (TB)(M)$. Moreover, $B(M)$ is isomorphic to Z_0 and $T|_{B(M)}$ is an isomorphism; hence $T \notin Z_0$-$\mathcal{NC}$. Thus Z_0-$\mathcal{NC}$ satisfies property (α_3).

The second assertion of part (a) is immediate.

(b) Note that $Z_0 \notin \mathrm{Sp}(\mathcal{A}) = \mathrm{Sp}(Z_0\text{-}\mathcal{NC})$, by part (a). Suppose that $T \in \mathcal{L}(X, Y)$, but $T \notin Z_0$-$\mathcal{NC}$. Then there exists a subspace M of X isomorphic to Z_0 such that $T|_M$ is an isomorphism and both M and $T(M)$ are complemented. Thus we can find $A \in \mathcal{L}(Y, Z_0)$ and $B \in \mathcal{L}(Z_0, X)$ so that $ATB = I_{Z_0}$; hence $T \notin \mathcal{A}$. ∎

Let us see that the class of all operators preserving no complemented copies of one of the spaces ℓ_p, $1 \leqslant p \leqslant \infty$ or c_0 is a radical operator ideal.

THEOREM 3.9. *Suppose that Z_0 is one of the spaces ℓ_p, $1 \leqslant p \leqslant \infty$, or c_0. Then Z_0-$\mathcal{NC}$ is a radical operator ideal.*

PROOF: By Proposition 3.8, we only have to show that Z_0-$\mathcal{NC}(X, Y)$ is a subspace of $\mathcal{L}(X, Y)$. In order to do that, let $S \in Z_0$-$\mathcal{NC}(X, Y)$ and $T \in \mathcal{L}(X, Y)$, and assume that M is a subspace of X isomorphic to Z_0 so that $(S+T)|_M$ is an isomorphism and $(S+T)(M)$ is complemented in Y. We have to show that $T \notin Z_0\text{-}\mathcal{NC}(X,Y)$.

First we assume that Z_0 is ℓ_p with $1 \leqslant p < \infty$, or c_0. Let P be a projection on Y such that $R(P) = (S+T)(M)$. Then $PS|_M + PT|_M = (S+T)|_M$, and by Proposition 3.8(a), $PS|_M \in Z_0\text{-}\mathcal{NC}$.

We claim that $PS|_M \in \mathcal{SS}$; that is, it is strictly singular. Indeed, otherwise we could find a closed, infinite dimensional subspace M_1 of M such that $PS|_{M_1}$ is an isomorphism. By [**13**, Proposition 2.a.2], $PS(M_1)$ contains a subspace N_2 which is isomorphic to Z_0 and complemented in $R(P)$. Since we can write $N_2 = PS(M_2)$, where M_2 is a closed subspace of M_1, $PS|_{M_2} \notin Z_0\text{-}\mathcal{NC}(X,Y)$, a contradiction.

Note that $(S+T)|_M \in \mathcal{K}_l(M,Y)$, because it is an isomorphism with complemented range (see Example 2.2). Moreover, $\mathcal{SS} \subset \mathcal{I}$ and $\mathcal{I}$ is the perturbation class of $\mathcal{K}_l$ [**15**]. Thus $PT|_M \in \mathcal{K}_l$.

So, there exists a finite codimensional closed subspace N of M (which is isomorphic to M) such that $PT|_N$ is an isomorphism and $PT(M) = PT(N)$. Thus $T \notin Z_0\text{-}\mathcal{NC}$.

In the case $Z_0 = \ell_\infty$, we can repeat a similar argument, using three facts. First, that every subspace of a Banach space isomorphic to ℓ_∞ is complemented; second, that for every non-weakly compact operator $R : \ell_\infty \to Z$, there exists a subspace M of ℓ_∞ isomorphic to ℓ_∞ so that $R \mid M$ is an isomorphism [**13**, Proposition 2.f.4]; and third, that every weakly compact $R : \ell_\infty \to Z$ is inessential [**7**, Theorem 1]. ∎

Remark 3.10. (a) Theorem 3.9 gives a positive answer to Question 2.10 for $Z_0\text{-}\mathcal{NC}$, when Z_0 is one of the spaces ℓ_p, $1 \leqslant p \leqslant \infty$ or c_0.

(b) It follows from the results of Bessaga and Pelczynski in [**5**] (see [**16**, Lemma 9.2 in Chapter C.II.]) that the conjugate T^* of $T \in \mathcal{L}(X,Y)$ is unconditionally converging, that is, $T \in \mathcal{UC}^d$, if and only if X has no closed subspace N isomorphic to ℓ_1 such that TJ_N is an isomorphism and $T(N)$ is complemented in Y.

Thus, it follows from Theorem 3.9 that $\mathcal{UC}^d$ is the radical operator ideal $\ell_1\text{-}\mathcal{NC}$.

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