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# A TIME-LIKE SURFACE IN MINKOWSKI 3-SPACE WHICH CONTAINS PSEUDOCIRCLES

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Simple characterizations of a pseudosphere or a plane in Minkowski 3-space by the existence of pseudocircles are given.

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## 1. Introduction

One of the simple characterizations of spheres in Euclidean 3-space $\mathbb{E}^3$ is as follows:

A circle in $\mathbb{E}^3$ of an arbitrary given radius can be pressed entirely on an arbitrary position of a surface. (*)

The condition (*) is quite natural because an observer is an inhabitant of an ambient space. However, (*) requires a very large quantity of information because of its condition "an arbitrary position". In [2], Ogiue and Takagi give a simple condition for a compact surface to be a sphere.

**Theorem** ([2], Ogiue–Takagi). *Let $S$ be a surface in Euclidean 3-space $\mathbb{E}^3$. Suppose that, through each point $p \in S$, there exist two circles of $\mathbb{E}^3$ such that*

(1) *they are contained in $S$ in a neighbourhood of $p$,*

(2) *they are tangent to each other at $p$.*

*Then $S$ is locally a plane or a sphere.*

In this paper we consider characterizations for a pseudosphere $S_1^2(r, a)$ in Minkowski 3-space $\mathbb{M}^3$ (for definition, see Section 2) analogous to the Ogiue–Takagi's result. The normal vector field on a pseudosphere in $\mathbb{M}^3$ is space-like, so that we stick to a surface $S$ in $\mathbb{M}^3$ such that the normal vector field on $S$ is space-like. Such a surface is called *a time-like surface*. There is another important class of surfaces in Minkowski 3-space $\mathbb{M}^3$ which is called *space-like surfaces*. In this case the induced metric on the surface is positively definite, so that we can give characterizations of a hyperbolic surface $H_1^2(r, a)$ by exactly the same arguments as those in [2]. Thus we do not consider this case in this paper.



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The notion of pseudocircles in Minkowski 3-space is given analogus to that of circles in Euclidean 3-space. Let $\gamma(s)$ be a time-like or a space-like curve (cf., Section 2). We can define the principal normal $\mathbf{N}(s)$, the binormal $\mathbf{B}(s)$, the curvature $k(s)$ and the torsion $\tau(s)$, so that the Frenet–Serret type formula for such a curve holds (cf., Section 2). We say that $\gamma(s)$ is a *pseudocircle* if the torsion is equal to zero and the curvature is positive constant along the curve.

Firstly, we have a simple characterization for a pseudosphere in $\mathbb{M}^3$ which is peculiar to pseudospheres in Minkowski 3-space.

**Theorem A.** *Let $S$ be a time-like surface in Minkowski 3-space $\mathbb{M}^3$. Suppose that, through each point $p \in S$, there exist two pseudocircles $\gamma_1, \gamma_2$ of $\mathbb{M}^3$ such that*

(1) *$\gamma_1, \gamma_2$ are contained in $S$ in a neighbourhood of $p$,*
(2) *$\gamma_1, \gamma_2$ are tangent to each other at $p$,*
(3) *$\delta(\gamma_1(s))\delta(\gamma_2(s)) = -1$, where $\delta(\gamma_1(s)) = \langle \mathbf{N}(s), \mathbf{N}(s) \rangle$.*

*Then $S$ is locally a pseudosphere.*

If we remove the condition (3) in the assumption in Theorem A, we have the following theorem analogous to the Ogiue–Takagi's result.

**Theorem B.** *Let $S$ be a time-like surface in Minkowski 3-space $\mathbb{M}^3$. Suppose that, through each point $p \in S$, there exist two pseudocircles $\gamma_1, \gamma_2$ of $\mathbb{M}^3$ such that*

(1) *$\gamma_1, \gamma_2$ are contained in $S$ in a neighbourhood of $p$,*
(2) *$\gamma_1, \gamma_2$ are tangent to each other at $p$.*

*Then $S$ is locally a time-like plane or a pseudosphere.*

We remark that an intersection of a pseudosphere with an arbitrary plane is a pseudocircle or a pair of light-like lines. We have given characterizations of a pseudosphere by the existence of light-like lines in **[1]**. The method of the proofs of the theorems we used here is analogous to that of Theorem 1 in **[2]**.

All surfaces and maps considered here are of class $C^\infty$ unless stated otherwise.

## 2. Basic notions

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be the usual oriented 3-dimensional vector space and differential manifold, which is oriented by $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ and given the Euclidean differentiable structure. *Minkowski 3-space* is defined by $\mathbb{M}^3 = \{\mathbb{R}^3, I_{(2,1)}\}$, where $I_{(2,1)} = dx_1^2 + dx_2^2 + dx_3^2$. Thus the metric tensor is given by $\langle \mathbf{X}, \mathbf{Y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$, where $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3)$. A vector $\mathbf{X}$ in $\mathbb{M}^3$ is called *light-like* if $\langle \mathbf{X}, \mathbf{X} \rangle = 0$, *space-like* if $\langle \mathbf{X}, \mathbf{X} \rangle > 0$ and *time-like* if $\langle \mathbf{X}, \mathbf{X} \rangle < 0$. A curve $\gamma$ is called *light-like* if its tangent vactor field $\dot{\gamma}$ is always light-like. We also say that a curve $\gamma$ is *non-light-like* if its tangent vector field $\dot{\gamma}$ is always space-like or time-like. The *pseudosphere* is defined to be

$$S_1^2(r, a) = \{(x_1, x_2, x_3) | (x_1 - a_1)^2 + (x_2 - a_2)^2 - (x_3 - a_3)^2 = r^2\},$$

where $a = (a_1, a_2, a_3)$ is the centre and $r > 0$ is the radius of $S_1^2(r, a)$.

The Levi–Civita connection of $\mathbb{M}^3$ is denoted by $\tilde{\nabla}$. Let $S$ be a surface in $\mathbb{M}^3$. We say that $S$ is *time-like* if the normal vector field to $S$ is space-like. On the time-like surface $S$ we have the Levi–Civita connection corresponding to the induced Lorenzian metric on $S$. We denote it by $\nabla$. The *normal connection* on $S$ is denoted by $\nabla^\perp$. Let $\sigma$ be a second fundamental form of $S$ in $\mathbb{M}^3$ which is given by

$$\sigma(\mathbf{X}, \mathbf{Y}) = \tilde{\nabla}_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{X}}\mathbf{Y},$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector fields tangent to $S$. Since $S$ is a time-like surface, there exists (at least locally) a unit normal vector field $\xi$ on $S$. We have the following formula:

$$\langle \sigma(\mathbf{X}, \mathbf{Y}), \xi \rangle = \langle -\tilde{\nabla}_{\mathbf{X}}\xi, \mathbf{Y} \rangle,$$

so that $-\tilde{\nabla}_{\mathbf{X}}\xi$ is the *shape operator* in this case. We denote that $A\mathbf{X} = -\tilde{\nabla}_{\mathbf{X}}\xi$. It is well-known that the shape operator is self-adjoint with respect to $\langle , \rangle$ (i.e., $\langle A\mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{X}, A\mathbf{Y} \rangle$). The *covariant derivative* $\nabla'_{\mathbf{X}}\sigma$ of $\sigma$ is defined by

$$\nabla'_{\mathbf{X}}\sigma(\mathbf{Y}, \mathbf{Z}) = \nabla^\perp_{\mathbf{X}}\sigma(\mathbf{Y}, \mathbf{Z}) - \sigma(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}) - (\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}).$$

We say that a point $p \in S$ is *umbilical* if there is a normal vector $\mathbf{Z} \in T_pS^\perp$ such that $\sigma(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle \mathbf{Z}$ for all $\mathbf{X}, \mathbf{Y} \in T_pS$. A time-like surface $S$ is *totally umbilical* provided every point of $S$ is umbilical. It is well-known that $S$ is totally umbilical if and only if its shape operator is scalar (cf., §4, Lemma 21 in [3]). The following result gives a characterization of a pseudosphere or a time-like plane in $\mathbb{M}^3$ (cf., §4, Propositions 13, 36 in [3]).

**Proposition 2.1.** *If $S$ is totally umbilical, then $S$ is an open set in a pseudosphere or a time-like plane.*

In order to define the notion of pseudocircles, we introduce the notion of *Lorenzian exterior products*. For any vectors $\mathbf{X} = (x_1, x_2, x_3), \mathbf{Y} = (y_1, y_2, y_3) \in \mathbb{M}^3$, the *Lorenzian exterior product* (*with respect to* $\langle , \rangle$) is defined to be

$$\mathbf{X} \wedge \mathbf{Y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_2 y_1 - x_1 y_2).$$

Let $\gamma$ be a non-light-like curve. We may assume that $\gamma$ is parameterized by arclength $s$. Thus we have $\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle = \varepsilon(\gamma(s)) = \pm 1$. We call $\varepsilon(\gamma(s))$ the *causal character* of $\gamma$. We can also define the notion of the *curvature* $k(s)$ of the curve $\gamma$ by $k(s) = \sqrt{|\langle \ddot{\gamma}(s), \ddot{\gamma}(s) \rangle|}$. We define the *principal normal vector* $\mathbf{N}(s)$ by $\ddot{\gamma}(s) = k(s)\mathbf{N}(s)$ except the point where $k(s) = 0$. We call $\delta(\gamma(s)) = \langle \mathbf{N}(s), \mathbf{N}(s) \rangle$ the *second causal character* of $\gamma(s)$. We also define the *binormal vector* $\mathbf{B}(s)$ by $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$, where $\mathbf{T}(s) = \dot{\gamma}(s)$. We can show that $\dot{\mathbf{B}}(s) = \tau(s)\mathbf{N}(s)$ for some function $\tau(s)$. We call $\tau(s)$ the *torsion* of $\gamma$ at $s$. By exactly the same arguments as in the case for a curve in Euclidean 3-space, we have the following Frenet–Serret formulas (cf., [**4**]):

$$\begin{cases} \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{T}(s) = k(s)\mathbf{N}(s) \\ \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{N}(s) = -\varepsilon(\gamma(s))\delta(\gamma(s))k(s)\mathbf{T}(s) + \varepsilon(\gamma(s))\tau(s)\mathbf{B}(s) \\ \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{B}(s) = \tau(s)\mathbf{N}(s). \end{cases} \tag{2.1}$$

We remark that if the curvature $k$ never vanishes then the second causal character $\delta(s)$ is constant along the curve $\gamma$. When the $\gamma$ is a plane curve, then we have $\tau(s) \equiv 0$.

We now define the notion of pseudocircle analogous to the notion of circles in Euclidean space $\mathbb{E}^3$. Let $\gamma$ be a non-light-like curve in $\mathbb{M}^3$. We say that $\gamma$ is a *pseudocircle* if the torsion $\tau$ always vanishes and the curvature $k$ is a positive constant along the curve. By the Frenet–Serret formulas, the curve $\gamma$ is a pseudocircle if and only if there exists a positive real number $k$ such that

$$\begin{cases} \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{T}(s) = k\mathbf{N}(s) \\ \tilde{\nabla}_{\mathbf{T}(s)}\mathbf{N}(s) = -\varepsilon(\gamma(s))\delta(\gamma(s))k\mathbf{T}(s). \end{cases} \tag{2.2}$$

We call $\gamma$ a *space-like pseudocircle* if $\varepsilon(\gamma) = 1$ and a *time-like pseudocircle* if $\varepsilon(\gamma) = -1$. We also call $\gamma$ a *positive pseudocircle* if $\delta(\gamma) = 1$ and $\gamma$ a *negative pseudocircle* if $\delta(\gamma) = -1$. On a time-like plane, the causal characters of the tangent vector and the normal vector are different, so that we only have three cases. Those are the positive space-like pseudocircles, the positive time-like pseudocircles or the negative space-like pseudocircles. We have the following two lemmas which give characterizations of a pseudocircle in $\mathbb{M}^3$. The proofs are given by straightforward computations.

**Lemma 2.2.** *Let $\gamma = \gamma(s)$ be a non-light-like curve in $\mathbb{M}^3$ parameterized by arclength $s$. Then $\gamma = \gamma(s)$ is a pseudocircle of curvature $k$ if and only if*

$$\tilde{\nabla}_{\dot{\gamma}}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} + \varepsilon(\gamma)\delta(\gamma)k^2\dot{\gamma} = 0 \tag{2.3}$$

**Lemma 2.3.** *Let $\gamma = \gamma(s)$ be a non-light-like curve in $\mathbb{M}^3$ parameterized by arclength $s$. Then $\gamma = \gamma(s)$ is a pseudocircle of curvature $k$ if and only if it satisfies*

$$\begin{cases} \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\dot{\gamma} + \varepsilon(\gamma)\delta(\gamma)k^2\dot{\gamma} + \tilde{\nabla}_{\dot{\gamma}}\sigma(\dot{\gamma}, \dot{\gamma}) - \nabla^{\perp}_{\dot{\gamma}}\sigma(\dot{\gamma}, \dot{\gamma}) = 0 \\ \nabla'_{\dot{\gamma}}\sigma(\dot{\gamma}, \dot{\gamma}) + 3\sigma(\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}) = 0. \end{cases} \tag{2.4}$$

We can show that the intersection of a pseudosphere $S^2_1(r, a)$ with a non-degenerate plane is a pseudocircle or a pair of light-like lines.

## 3. Proof of results

In this section we give proofs of the theorems stated in the introduction. The method of the proofs almost follows along the line of the proof of Theorem 1 in [2]. However, there are some differences from the original proof caused by the index of the metric tensor. There are common parts of the proofs for Theorems A and B, so we firstly start to prove Theorem A.

**Proof of Theorem A.** Let $k_i$ be the curvature of $\gamma_i$ $(i = 1, 2)$. Let $X_p$ be the common unit tangent vector of $\gamma_1$ and $\gamma_2$ at $p$. Then $\mathbf{X} : p \mapsto X_p$ defines a vector field on $S$, which may not be continuous. Let $\mathbf{Y}$ be a unit vector field on $S$ which satisfies $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$ and $\langle \mathbf{X} \wedge \mathbf{Y}, \xi \rangle > 0$, where $\xi$ is the unit normal vector field on $S$. Let $\mathbf{X}_1, \mathbf{X}_2$ be smooth unit vector fields on a neighbourhood of $\gamma_1 \cup \gamma_2$ such that $\mathbf{X}_i$ is equal to the unit tangent vector of $\gamma_i$ on $\gamma_i$ and $\mathbf{X}_i(p) = X_p$ $(i = 1, 2)$. It follows that $\langle \nabla_{\mathbf{X}_i} \mathbf{X}_i, \mathbf{X}_i \rangle = 0$. Thus there exist real numbers $c_i$ $(i = 1, 2)$ such that $\nabla_{\mathbf{X}_i} \mathbf{X}_i(p) = c_i \mathbf{Y}(p)$. Suppose that $c_1 = c_2$. Since $\tilde{\nabla}_{\mathbf{X}_i} \mathbf{X}_i = \nabla_{\mathbf{X}_i} \mathbf{X}_i + \sigma(\mathbf{X}_i, \mathbf{X}_i)$ $(i = 1, 2)$, and $\mathbf{X}_i(p) = \mathbf{X}_p$, we have $\tilde{\nabla}_{\mathbf{X}_1} \mathbf{X}_1(p) = \tilde{\nabla}_{\mathbf{X}_2} \mathbf{X}_2(p)$. The last equation means that $k_1 = k(\gamma_1(p)) = k(\gamma_2(p)) = k_2$, so that $\gamma_1 = \gamma_2$ on a neighbourhood of $p$. This contradicts to the assumptions (i.e., $c_1 \neq c_2$).

It follows from Lemma 2.3 that

$$\begin{cases} \nabla'_{\mathbf{X}} \sigma(\mathbf{X}, \mathbf{X}) + 3\sigma(\mathbf{X}, c_1 \mathbf{Y}) = 0 \\ \nabla'_{\mathbf{X}} \sigma(\mathbf{X}, \mathbf{X}) + 3\sigma(\mathbf{X}, c_2 \mathbf{Y}) = 0 \end{cases} \tag{3.1}$$

at each point. Therefore we have $\sigma(\mathbf{X}, \mathbf{Y}) = 0$. It follows that $\langle A\mathbf{X}, \mathbf{Y} \rangle = \langle \sigma(\mathbf{X}, \mathbf{Y}), \xi \rangle = 0$. Since $\dim S = 2$, we see that $A\mathbf{X}$ is parallel to $\mathbf{X}$. Thus $\mathbf{X}$ is (and hence $\mathbf{Y}$ is also) a principal vector at each point. Let $\lambda$ and $\mu$ be principal curvature and $\eta_1$ and $\eta_2$ the corresponding principal vectors so that $A\eta_1 = \lambda\eta_1$ and $A\eta_2 = \mu\eta_2$. Put $S_0 = \{p \in S | \lambda(p) \neq \mu(p)\}$. If $S_0 = \emptyset$, then $S$ is totally umbilical. Therefore we suppose that $S_0 \neq \emptyset$. Then $\eta_1$ and $\eta_2$ are smooth vector fields in some neighbourhood of each point of $S_0$. Since $\langle \eta_1, \eta_2 \rangle = 0$, we may assume that $\eta_1$ is space-like and $\eta_2$ is time-like. Since $\langle \eta_1, \eta_1 \rangle = 1$, we have

$$\nabla_{\eta_1} \eta_2 = \alpha\eta_1 \text{ and } \nabla_{\eta_2} \eta_2 = \beta\eta_1, \tag{3.2}$$

by a straightforward computation.

Put $S_{0,i} = \{p \in S | \mathbf{X}(p) = \eta_1(p)\}$ $(i = 1, 2)$. Then $S_0 = S_{0,1} \cup S_{0,2}$, and it is easily seen that $S_0 \subset \overline{S_{0,1}} \cup \mathrm{Int} S_{0,2}$ and $S_0 \subset \overline{S_{0,2}} \cup \mathrm{Int} S_{0,1}$ and hence that $S_{0,1}$ or $S_{0,2}$ has interior points, or $S_{0,1}$ or $S_{0,2}$ is dense in $S_0$. We now distinguish the following two cases.

*Case* (1) $S_{0,1}$ has interior points or $S_{0,1}$ is dense in $S_0$.

*Case* (2) $S_{0,2}$ has interior points or $S_{0,2}$ is dense in $S_0$.

The situation is different between two cases. In case (1) both of two pseudocircles through $p$ are space-like. On the other hand, both of the pseudocircles are positive time-like in case (2), so we only consider case (1).

In this case $X(p)$ is space-like. Since $\sigma(\mathbf{X}, \mathbf{Y}) = 0$, we obtain

$$\begin{aligned}\nabla'_{\eta_1}\sigma(\eta_1,\eta_1) &= \nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1) - 2\sigma(\eta_1,\nabla_{\eta_1}\eta_1)\\ &= \nabla^{\perp}_{\eta_1}(\lambda\xi) - 2\alpha\sigma(\eta_1,\eta_2)\\ &= \nabla^{\perp}_{\eta_1}(\lambda\xi)\\ &= (\nabla^{\perp}_{\eta_1}\lambda)\xi + \lambda\nabla^{\perp}_{\eta_1}\xi\\ &= (\nabla^{\perp}_{\eta_1}\lambda)\xi.\end{aligned}$$

It follows from (3.1) and $\sigma(\mathbf{X},\mathbf{Y})=0$ that

$$\nabla'_{\mathbf{X}}\sigma(\mathbf{X},\mathbf{X}) = 0. \tag{3.3}$$

Therefore we have $\nabla_{\mathbf{X}}\lambda = 0$ on $S_{0,1}$. If $p$ is an interior point of $S_{0,1}$, then

$$\nabla_{\eta_1}\lambda = 0 \tag{3.4}$$

holds in some neighbourhood of $p$. If $S_{0,1}$ is dense in $S_0$, then, by continuity, (3.4) holds on $S_0$.

We choose an orthonormal frame field $e_1, e_2$ in a sufficiently small tubular neighbourhood of $\gamma_1$ in such a way that $e_1 = \dot{\gamma}_1$ along $\gamma_1$, and put $\nabla_{e_1}e_1 = ae_2$ and $\nabla_{e_2}e_1 = be_2$ so that $\nabla_{e_1}e_2 = ae_1$ and $\nabla_{e_2}e_2 = be_1$. Let $(h_{ij})$ be the matrix of the shape operator with respect to $e_1$ and $e_2$ (i.e., $-\tilde{\nabla}_{e_i}\xi = h_{i1}e_1 + h_{i2}e_2 \quad (i=1,2)$).

On the other hand, we have

$$\tilde{\nabla}_{e_1}\sigma(e_1,e_1) - \nabla^{\perp}_{e_1}\sigma(e_1,e_1) = -(h_{11}^2 e_1 + h_{11}h_{12}e_2)$$

and

$$\nabla_{e_1}\nabla_{e_1}e_1 = a^2 e_1 + (\nabla_{e_1}a)e_2.$$

By Lemma 2.3, we have

$$\{a^2 e_1 + (\nabla_{e_1}a)e_2\} + \delta(\gamma_1)k^2 e_1 - \{h_{11}^2 e_1 + h_{11}h_{12}e_2\} = 0,$$

so that, along $\gamma_1$,

$$\delta(\gamma_1)k^2 = -a^2 + h_{11}^2 \tag{3.5}$$

and

$$\nabla_{e_1}a = h_{11}h_{12}. \tag{3.6}$$

It also follows from the second equation of Lemma 2.3 that

$$\nabla^{\perp}_{e_1}\sigma(e_1,e_1) + \sigma(e_1,\nabla_{e_1}e_1) = 0.$$

Since

$$\nabla^{\perp}_{e_1}\sigma(e_1, e_1) = \nabla^{\perp}_{e_1}\{\langle -\nabla_{e_1}\xi, e_1\rangle\xi\} = (\nabla^{\perp}_{e_1}h_{11})\xi$$

and

$$\sigma(e_1, \nabla_{e_1}e_1) = \sigma(e_1, ae_2) = -ah_{12}\xi,$$

we have

$$\nabla_{e_1}h_{11} - ah_{12} = 0. \tag{3.7}$$

We remark that $\eta_1, e_1$ are space-like and $\eta_2, e_2$ are time-like. We define $\theta$ such that

$$\begin{cases} e_1 = \eta_1\cosh\theta + \eta_2\sinh\theta \\ e_2 = \eta_1\sinh\theta + \eta_2\cosh\theta. \end{cases} \tag{3.8}$$

So we have

$$\begin{cases} Ae_1 = \lambda\eta_1\cosh\theta + \mu\eta_2\sinh\theta \\ Ae_2 = \lambda\eta_1\sinh\theta + \mu\eta_2\cosh\theta, \end{cases}$$

from which we get

$$\begin{cases} h_{11} = \lambda\cosh^2\theta - \mu\sinh^2\theta \\ h_{12} = -(\lambda-\mu)\cosh\theta\sinh\theta \\ h_{22} = -\lambda\sinh^2\theta + \mu\cosh^2\theta. \end{cases} \tag{3.9}$$

By differentiating (3.8) with respect to $e_1$, we have

$$\nabla_{e_1}e_1 = \eta_1\{\alpha\cosh\theta + \beta\sinh\theta + \nabla_{e_1}\theta\}\sinh\theta + \eta_2\{\alpha\cosh\theta + \beta\sinh\theta + \nabla_{e_1}\theta\}\cosh\theta.$$

Since $\nabla_{e_1}e_1 = ae_2 = a\eta_1\sinh\theta + a\eta_2\cosh\theta$, we obtain

$$a = \alpha\cosh\theta + \beta\sinh\theta + \nabla_{e_1}\theta. \tag{3.10}$$

We also have $(\nabla_{\eta_1}A)(\eta_2) = \nabla_{\eta_1}(A\eta_2) - A\nabla_{\eta_1}\eta_2 = (\nabla_{\eta_1}\mu)\eta_2 + \alpha(\mu-\lambda)\eta_1$ and $(\nabla_{\eta_2}A)(\eta_1) = (\nabla_{\eta_2}\lambda)\eta_1 + \beta(\lambda-\mu)\eta_2$. By the equation of Coddazzi $(\nabla_{\eta_1}A)(\eta_2) - (\nabla_{\eta_2}A)(\eta_1) = 0$, we have

$$\begin{cases} -\alpha(\lambda-\mu) = \nabla_{\eta_2}\lambda \\ \beta(\lambda-\mu) = \nabla_{\eta_1}\mu. \end{cases} \tag{3.11}$$

Therefore by (3.4), (3.7), (3.9), (3.10) and (3.11) we obtain

$$\{3(\lambda-\mu)\cosh\theta\cdot\nabla_{e_1}\theta-\sinh^2\theta\cdot\nabla_{\eta_2}\mu\}\sinh\theta=0. \tag{3.12}$$

We remark that $\theta=0$ at $p$. Then the point $p$ has one of the following properties:
(A) There exists no sequence $\{p_n\in\gamma_1|\theta(p_n)\neq 0\}$ with $\lim p_n=p$.
(B) There exists a sequence $\{p_n\in\gamma_1|\theta(p_n)\neq 0\}$ with $\lim p_n=p$.

If $p$ is a point which has the property (A), then it is clear that the integral curve of $\eta_1$ through $p$ coincides with $\gamma_1$ on the connected component of $\{q\in\gamma_1|\theta(q)=0\}$ containing $p$.

If $p$ is a point with the property (B), it follows from (3.12) that

$$3(\lambda-\mu)\cosh\theta\cdot\nabla_{e_1}\theta-\sinh^2\theta\cdot\nabla_{\eta_2}\mu=0. \tag{3.13}$$

Since $\lambda\neq\mu$, taking the limit of (3.13), we obtain

$$\nabla_{e_1}\theta=0. \tag{3.14}$$

Applying $\nabla_{e_1}$ to (3.12) and substituting the relation (3.13) into the equation on the set $\{p_n\in\gamma_1|\theta(p_n)\neq 0\}$, we obtain

$$\begin{aligned}&\frac{\sinh\theta\cdot\nabla_{\eta_2}\mu}{3(\lambda-\mu)}\{3(\lambda-\mu)\cosh\theta\cdot\nabla_{e_1}\theta-\sinh^2\theta\cdot\nabla_{\eta_2}\mu\}\\&\quad+\big\{3(\nabla_{e_1}\lambda-\nabla_{e_1}\mu)\cosh\theta\cdot\nabla_{e_1}\theta+3(\lambda-\mu)\sinh\theta\cdot(\nabla_{e_1}\theta)^2\\&\qquad+3(\lambda-\mu)\cosh\theta\cdot\nabla_{e_1}\nabla_{e_1}\theta-2\sinh\theta\cosh\theta\cdot\nabla_{e_1}\nabla_{\eta_2}\mu-\sinh^2\theta\cdot\nabla_{e_1}\nabla_{\eta_2}\mu\big\}=0.\end{aligned}$$

Taking the limit of the above equation we have

$$\nabla_{e_1}\nabla_{e_1}\theta=0, \tag{3.15}$$

since $\sinh\theta=0$ and $\cosh\theta=1$. It is clear that the (3.14) and (3.15) hold even if $p$ has the property (A).

It follows from (3.10) and (3.14) that $a(p)=\alpha(p)$. This, together with (3.5) and (3.9), yields

$$\delta(\gamma_1)k^2=-(\alpha(p))^2+(\lambda(p))^2, \tag{3.16}$$

which means that the curvature of $\gamma_1$ is $\sqrt{\delta\{-(\alpha(p))^2+(\lambda(p))^2\}}$.

For $\gamma_2$, by exactly the same arguments as those of $\gamma_1$, we obtain

$$\delta(\gamma_2)k_2^2=-(\alpha(p))^2+(\lambda(p))^2, \tag{3.17}$$

where $k_2$ is the curvature of $\gamma_2$. Until here we have not used the condition that $\delta(\gamma_1)\delta(\gamma_2)=-1$. It follows from (3.16), (3.15) and the assumption $\delta(\gamma_1)\delta(\gamma_2)=-1$ that

$$(-(\alpha(p))^2 + (\lambda(p))^2)^2 < 0$$

This is a contradiction, so that $S_0 = \emptyset$ and $S$ is totally umbilical. Since the time-like plane does not satisfy the condition $\delta(\gamma_1)\delta(\gamma_2) = -1$, $S$ is an open set of a pseudosphere.

**Proof of Theorem B.** Since we already proved Theorem A, we may consider the case that $\delta(\gamma_1)\delta(\gamma_2) = 1$. We use the same notation as in the proof of Theorem A. So our purpose is to show that $S_0 = \emptyset$. Therefore we suppose that $S_0 = \emptyset$, so that we also distinguish the following two cases:

*Case* (1) $S_{0,1}$ has interior points or $S_{0,1}$ is dense in $S_0$.

*Case* (2) $S_{0,2}$ has interior points or $S_{0,2}$ is dense in $S_0$.

Both of the pseudocircles through $p$ are space-like in case (1) and are positive time-like in case (2). For a positive time-like curve, we have $\varepsilon\delta = -1$, so that the equations (2.3) and (2.4) have the same form as equations for a negative space-like curve. It follows that the almost all arguments for positive time-like curves are the same as those for negative space-like curves. Hence, we only consider case (1).

We may use the equations (3.1)–(3.17) for the proof of Theorem B. Since $\delta(\gamma_1)\delta(\gamma_2) = 1$, $\gamma_1$ and $\gamma_2$ are space-like pseudocircles with the common same curvature $\sqrt{-(\alpha(p))^2 + (\lambda(p))^2}$.

Applying $\nabla_{e_1}$ to (3.10), we obtain $\nabla_{\eta_1} a = \nabla_{\eta_1}\alpha$ at $p$ because of (3.14) and (3.15).

On the other hand, from (3.6) we get $\nabla_{\eta_1} a = \nabla_{e_1} a = h_{11}h_{12} = 0$ at $p$. Therefore we have $\nabla_{\eta_1}\alpha = 0$ at $p$. Since $p$ is arbitrary, we get $\nabla_{\eta_1}\alpha = 0$ on $S_{0,1}$. If $p$ is an interior point of $S_{0,1}$, then

$$\nabla_{\eta_1}\alpha = 0 \tag{3.18}$$

holds in some neighbourhood of $p$. If $S_{0,1}$ is dense in $S_0$, then, by continuity, (3.18) holds on $S_0$. It follows from (3.18) that

$$\nabla_{\eta_1}\nabla_{\eta_1}\eta_1 = \nabla_{\eta_1}(\alpha\eta_2) = \alpha^2\eta_1.$$

Moreover, since $\eta_1$ is a principal vector, we get

$$\begin{aligned}
&\langle\tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1) - \nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1), V\rangle = \langle\tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1), V\rangle - \langle\nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1), V\rangle \\
&= \langle\tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1), V\rangle = \langle\tilde{\nabla}_{\eta_1}\xi, V\rangle = \langle(\nabla_{\eta_1}\lambda)\xi + \lambda\tilde{\nabla}_{\eta_1}\xi, V\rangle = \lambda\langle\tilde{\nabla}_{\eta_1}\xi, V\rangle \\
&= \lambda\langle -A\eta_1, V\rangle = -\lambda^2\langle\eta_1, V\rangle
\end{aligned}$$

for any tangent vector field $V$ on $S$. Therefore we obtain

$$\nabla_{\eta_1}\nabla_{\eta_1}\eta_1 + (-\alpha^2 + \lambda^2)\eta_1 + \tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1) - \nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1) = 0. \tag{3.19}$$

Furthermore, since $\nabla_{\eta_1}\eta_1 = \alpha\eta_2$, it follows from $\sigma(\mathbf{X},\mathbf{Y}) = 0$ and (3.3) that $(\nabla'_{\eta_1}\sigma)(\eta_1,\eta_1) + 3\sigma(\eta_1, \nabla_{\eta_1}\eta_1) = 0$ on $S_{0,1}$. If $p$ is an interior point of $S_{0,1}$,

$$(\nabla'_{\eta_1}\sigma)(\eta_1, \eta_1) + 3\sigma(\eta_1, \nabla_{\eta_1}\eta_1) = 0 \tag{3.20}$$

holds in some neighbourhood of $p$. If $S_{0,1}$ is dense in $S_0$, then, by continuity, (3.20) holds on $S_0$. By (3.19), (3.20) and Lemma 2.3, we see that the integral curve of $\eta_1$ through $p$ is a space-like pseudocircle of curvature $\sqrt{\delta(\eta_1)\{-(\alpha(p))^2 + (\lambda(p))^2\}}$, where $\delta(\eta_1) = \langle\eta_1, \eta_1\rangle$. By (3.4) and (3.18), the above curvature is constant along the curve. It is clear that $\delta(\eta_1) = \delta(\gamma_1)$. Since we can apply the same arguments to $\gamma_2$, letting $\theta_i\,(i = 1, 2)$ be the function defined the same as the function of (3.8), we consider the following cases:

$(A)_i$: There exists no sequence $\{p_n \in \gamma_i | \theta_i(p_n) \neq 0\}$ with $p = \lim p_n$.

$(B)_i$: There exists a sequence $\{p_n \in \gamma_i | \theta_i(p_n) \neq 0\}$ with $p = \lim p_n$.

It is clear that $(A)_1$ and $(A)_2$ does not occur. If $(B)_1$ and $(A)_2$, then $\gamma_1$ and $\gamma_2$ have the same curvature and the integral curve of $\eta_1$ through $p$ coincide with $\gamma_2$. Then we have

$$\tilde{\nabla}_{\eta_1}\eta_1 = \nabla_{\eta_1}\eta_1 + \sigma(\eta_1, \eta_1) = \alpha\eta_2 + \lambda\xi.$$
$$\tilde{\nabla}_{e_1}e_1 = \nabla_{e_1}e_1 + \sigma(e_1, e_1) = ae_2 + h_{11}\xi.$$

Since $d = a$, $\lambda = h_{11}$ and $\eta_2 = e_2$ at $p$, we obtain $\tilde{\nabla}_{\eta_1}\eta_1 = \tilde{\nabla}_{e_1}e_1$. This equation means that $\gamma_1$ and $\gamma_2$ coincide in a neighbourhood of $p$. This contradicts the assumption. By the same arguments as the above case, the case $(A)_1$ and $(B)_2$ contradicts the assumption. If $(B)_1$ and $(B)_2$, then $\gamma_1, \gamma_2$ and the integral curve of $\eta_1$ through $p$ have the same curvature and hence, we can show that normal vectors for these vectors are the same. It follows that $\gamma_1$ and $\gamma_2$ coincide in a neighbourhood of $p$. This contradicts the assumption, so that $S_0 = \emptyset$. This completes the proof.

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