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Fundamental Group of Simple C^*-algebras with Unique Trace III

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Abstract. We introduce the fundamental group $\mathcal{F}(A)$ of a simple σ-unital C^*-algebra A with unique (up to scalar multiple) densely defined lower semicontinuous trace. This is a generalization of *Fundamental Group of Simple C^*-algebras with Unique Trace I and II* by Nawata and Watatani. Our definition in this paper makes sense for stably projectionless C^*-algebras. We show that there exist separable stably projectionless C^*-algebras such that their fundamental groups are equal to $\mathbb{R}_+^\times$ by using the classification theorem of Razak and Tsang. This is a contrast to the unital case in Nawata and Watatani. This study is motivated by the work of Kishimoto and Kumjian.

1 Introduction

Let M be a factor of type II_1 with a normalized trace τ. Murray and von Neumann introduced the fundamental group $\mathcal{F}(M)$ of M in [28]. They showed that if M is hyperfinite, then $\mathcal{F}(M) = \mathbb{R}_+^\times$. Since then there have been many papers on the computation of fundamental groups. Voiculescu [41] showed that $\mathcal{F}(L(\mathbb{F}_\infty))$ of the group factor of the free group $\mathbb{F}_\infty$ contains the positive rationals, and Radulescu proved that $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+^\times$ in [36]. Connes [7] showed that if G is an ICC group with property (T), then $\mathcal{F}(L(G))$ is a countable group. Popa showed that any countable subgroup of $\mathbb{R}_+^\times$ can be realized as the fundamental group of some factor of type II_1 in [34]. Furthermore, Popa and Vaes [35] exhibited a large family $\mathcal{S}$ of subgroups of $\mathbb{R}_+^\times$, containing $\mathbb{R}_+^\times$ itself, all of its countable subgroups, as well as uncountable subgroups with any Hausdorff dimension in $(0,1)$, such that for each $G \in \mathcal{S}$ there exist many free ergodic measure preserving actions of $\mathbb{F}_\infty$ for which the associated II_1 factor M has the fundamental group equal to G. In our previous paper [29], we introduced the fundamental group $\mathcal{F}(A)$ of a simple unital C^*-algebra A with a normalized trace τ based on the computation of Picard groups by Kodaka [22–24]. The fundamental group $\mathcal{F}(A)$ is defined as the set of the numbers $\tau \otimes Tr(p)$ for some projection $p \in M_n(A)$ such that $pM_n(A)p$ is isomorphic to A. We computed the fundamental groups of several C^*-algebras and showed that any countable subgroup of $\mathbb{R}_+^\times$ can be realized as the fundamental group of a separable simple unital C^*-algebra with a unique trace in [29,30]. Note that the fundamental groups of separable simple unital C^*-algebras are countable.

In this paper we introduce the fundamental group of a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace. We

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do not assume that C^*-algebras are unital. In particular, our definition in this paper makes sense for stably projectionless C^*-algebras. Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. The fundamental group $\mathcal{F}(A)$ of A is defined as the set of the numbers $d_\tau(h_1)/d_\tau(h_2)$ for some nonzero positive elements $h_1, h_2 \in A \otimes \mathbb{K}$ such that $\overline{h_1(A \otimes \mathbb{K})h_1}$ is isomorphic to $\overline{h_2(A \otimes \mathbb{K})h_2}$ and $d_\tau(h_2) < \infty$, where d_τ is the dimension function defined by τ. Then the fundamental group $\mathcal{F}(A)$ of A is a multiplicative subgroup of $\mathbb{R}_+^\times$. We show that if A is unital, then our definition in this paper coincides with the previous definition in [29, 30]. Hence if $A \otimes \mathbb{K}$ is separable and has a nonzero projection, then $\mathcal{F}(A)$ is a countable multiplicative subgroup of $\mathbb{R}_+^\times$. By contrast, we show that there exist separable, simple, stably projectionless C^*-algebras such that their fundamental groups are equal to $\mathbb{R}_+^\times$ by using the classification theorems of Razak [37] and Tsang [40]. This study is motivated by the work of Kishimoto and Kumjian in [20]. (See Example 4.21.)

2 Hilbert C^*-modules and Induced Traces

We say a C^*-algebra A is *σ-unital* if A has a countable approximate unit. In particular, if A is σ-unital, then there exists a positive element $h \in A$ such that $\{h^{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is an approximate unit. Such a positive element h is called *strict positive* in A. Let $\mathcal{X}$ be a right Hilbert A-module and let $\mathcal{H}(A)$ denote the set of isomorphic classes $[\mathcal{X}]$ of countably generated right Hilbert A-modules. (See [26, 27] for the basic facts on Hilbert modules.) We denote by $L_A(\mathcal{X})$ the algebra of the adjointable operators on $\mathcal{X}$. For $\xi, \eta \in \mathcal{X}$, a "rank one operator" $\Theta_{\xi,\eta}$ is defined by $\Theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle_A$ for $\zeta \in \mathcal{X}$. We denote by $K_A(\mathcal{X})$ the closure of the linear span of "rank one operators" $\Theta_{\xi,\eta}$ and by $\mathbb{K}$ the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space. Let $\mathcal{X}_A$ be a right Hilbert A-module A with the obvious right A-action and $\langle a, b\rangle_A = a^*b$ for $a, b \in A$. Then $K_A(\mathcal{X}_A)$ is isomorphic to A. Hence if A is unital, then $K_A(\mathcal{X}_A) = L_A(\mathcal{X}_A)$. A multiplier algebra, denoted by $M(A)$, of a C^*-algebra A is the largest unital C^*-algebra that contains A as an essential ideal. It is unique up to isomorphism over A and isomorphic to $L_A(\mathcal{X}_A)$. Let H_A denote the standard Hilbert module $\{(x_n)_{n\in\mathbb{N}}; x_n \in A, \sum x_n^* x_n \text{ converges in } A\}$ with an A-valued inner product $\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\rangle = \sum x_n^* y_n$. Then there exists a natural isomorphism ψ of $A \otimes \mathbb{K}$ to $K_A(H_A)$ and ψ can be uniquely extended to an isomorphism $\tilde{\psi}$ of $M(A \otimes \mathbb{K})$ to $L_A(H_A)$. For simplicity of notation, we use the same letter x for $\tilde{\psi}(x)$, where $x \in M(A \otimes \mathbb{K})$.

A finite subset $\{\xi_i\}_{i=1}^n$ of $\mathcal{X}$ is called a *finite basis* if $\eta = \sum_{i=1}^n \xi_i\langle \xi_i, \eta\rangle_A$ for any $\eta \in \mathcal{X}$. More generally, we call a sequence $\{\xi_i\}_{i\in\mathbb{N}} \subseteq \mathcal{X}$ a *countable basis* of $\mathcal{X}$ if $\eta = \sum_{i=1}^\infty \xi_i\langle \xi_i, \eta\rangle_A$ in norm for any $\eta \in \mathcal{X}$; see [17, 18, 42]. It is also called a standard normalized tight frame, as in [12, 13]. A countable basis $\{\xi_i\}_{i\in\mathbb{N}}$ always converges unconditionally, that is, for any $\eta \in \mathcal{X}$, the net associating $\sum_{i\in F} \xi_i\langle \xi_i, \eta\rangle_A$ with each finite subset $F \subseteq \mathbb{N}$ is norm converging to η. It is a consequence of the following estimate: for every $\xi \in \mathcal{X}$, $a, b \in K_A(\mathcal{X})$, with $0 \le a \le b \le 1$, $\|\xi - b\xi\|^2 \le \|\xi\|\|\xi - a\xi\|$.

Proposition 2.1 *Let A be a simple C^*-algebra and $\mathcal{X}$ a right Hilbert A-module. As-*

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sume that $K_A(\mathcal{X})$ is σ-unital. Then $\mathcal{X}$ has a countable basis.

Proof Consider a right ideal $\{\Theta_{\xi_0,\zeta} : \zeta \in \mathcal{X}\}$ in $K_A(\mathcal{X})$ for some $\xi_0 \in \mathcal{X}$. Then a similar argument as in [4, Lemma 2.3] proves the proposition. ■

Remark 2.2 In general, we need not assume that A is simple. If $K_A(\mathcal{X})$ is σ-unital, then $\mathcal{X}$ has a countable basis. This is an immediate consequence of Kasparov's stabilization trick [19].

Let B be a C^*algebra. An *A-B-equivalence bimodule* is an A-B-bimodule $\mathcal{F}$ that is simultaneously a full left Hilbert A-module under a left A-valued inner product ${}_A\langle \cdot, \cdot \rangle$ and a full right Hilbert B-module under a right B-valued inner product $\langle \cdot, \cdot \rangle_B$, satisfying ${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$ for any $\xi, \eta, \zeta \in \mathcal{F}$. We say that A is *Morita equivalent* to B if there exists an A-B-equivalence bimodule. It is easy to see that $K_B(\mathcal{F})$ is isomorphic to A. A dual module $\mathcal{F}^*$ of an A-B-equivalence bimodule $\mathcal{F}$ is a set $\{\xi^*; \xi \in \mathcal{F}\}$ with the operations such that $\xi^* + \eta^* = (\xi + \eta)^*$, $\lambda \xi^* = (\overline{\lambda} \xi)^*$, $b\xi^* a = (a^* \xi b^*)^*$, ${}_B\langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle_B$, and $\langle \xi^*, \eta^* \rangle_A = {}_A\langle \eta, \xi \rangle$. The bimodule $\mathcal{F}^*$ is a B-A-equivalence bimodule. We refer the reader to [38, 39] for the basic facts on equivalence bimodules and Morita equivalence.

We review basic facts on the Picard groups of C^*-algebras introduced by Brown, Green, and Rieffel in [5]. For A-A-equivalence bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$, we say that $\mathcal{E}_1$ is isomorphic to $\mathcal{E}_2$ as an equivalence bimodule if there exists a $\mathbb{C}$-liner one-to-one map Φ of $\mathcal{E}_1$ onto $\mathcal{E}_2$ with the properties such that $\Phi(a\xi b) = a\Phi(\xi)b$, ${}_A\langle \Phi(\xi), \Phi(\eta) \rangle = {}_A\langle \xi, \eta \rangle$, and $\langle \Phi(\xi), \Phi(\eta) \rangle_A = \langle \xi, \eta \rangle_A$ for $a, b \in A$, $\xi, \eta \in \mathcal{E}_1$. The set of isomorphic classes $[\mathcal{E}]$ of the A-A-equivalence bimodules $\mathcal{E}$ forms a group under the product defined by $[\mathcal{E}_1][\mathcal{E}_2] = [\mathcal{E}_1 \otimes_A \mathcal{E}_2]$. We call it the *Picard group* of A and denote it by $\mathrm{Pic}(A)$. The identity of $\mathrm{Pic}(A)$ is given by the A-A-bimodule $\mathcal{E} := A$ with ${}_A\langle a_1, a_2 \rangle = a_1 a_2^*$ and $\langle a_1, a_2 \rangle_A = a_1^* a_2$ for $a_1, a_2 \in A$. The inverse element of $[\mathcal{E}]$ in the Picard group of A is the dual module $[\mathcal{E}^*]$. Let α be an automorphism of A, and let $\mathcal{E}_\alpha^A = A$ with the obvious left A-action and the obvious A-valued inner product. We define the right A-action on $\mathcal{E}_\alpha^A$ by $\xi \cdot a = \xi \alpha(a)$ for any $\xi \in \mathcal{E}_\alpha^A$ and $a \in A$, and the right A-valued inner product by $\langle \xi, \eta \rangle_A = \alpha^{-1}(\xi^* \eta)$ for any $\xi, \eta \in \mathcal{E}_\alpha^A$. Then $\mathcal{E}_\alpha^A$ is an A-A-equivalence bimodule. For $\alpha, \beta \in \mathrm{Aut}(A)$, $\mathcal{E}_\alpha^A$ is isomorphic to $\mathcal{E}_\beta^A$ if and only if there exists a unitary $u \in A$ such that $\alpha = ad\ u \circ \beta$. Moreover, $\mathcal{E}_\alpha^A \otimes \mathcal{E}_\beta^A$ is isomorphic to $\mathcal{E}_{\alpha\circ\beta}^A$. Hence we obtain an homomorphism ρ_A of $\mathrm{Out}(A)$ to $\mathrm{Pic}(A)$. An A-B-equivalence bimodule $\mathcal{F}$ induces an isomorphism Ψ of $\mathrm{Pic}(A)$ to $\mathrm{Pic}(B)$ by $\Psi([\mathcal{E}]) = [\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]$ for $[\mathcal{E}] \in \mathrm{Pic}(A)$. Therefore if A is Morita equivalent to B, then $\mathrm{Pic}(A)$ is isomorphic to $\mathrm{Pic}(B)$.

If A is unital, then any A-B-equivalence bimodule $\mathcal{F}$ is a finitely generated projective B-module as a right module with a finite basis $\{\xi_i\}_{i=1}^n$. Put $p = (\langle \xi_i, \xi_j \rangle_A)_{ij} \in M_n(B)$. Then p is a projection and $\mathcal{F}$ is isomorphic to pB^n as a right Hilbert B-module with an isomorphism of A to $pM_n(B)p$. In the case A is σ-unital, an A-B-equivalence bimodule $\mathcal{F}$ has a countable basis $\{\xi_i\}_{i\in\mathbb{N}}$ as a right Hilbert B-module by Proposition 2.1. By [16, Lemma 2.1] and $\sum_{i=1}^n \Theta_{\xi_i,\xi_i} \le 1_{L_B(H_B)}$ for any $n \in \mathbb{N}$, we see that $\{\sum_{m=1}^N \xi_m b_m\}_{N\in\mathbb{N}}$ converges in $\mathcal{F}$. Define p by $p(b_n)_n = (\sum_{m=1}^\infty \langle \xi_n, \xi_m \rangle_B b_m)_n$ for $(b_n)_n \in H_B$. Then p is a projection in $L_B(H_B)$. Therefore $\mathcal{F}$ is isomorphic to pH_B as a right Hilbert module with an isomorphism of A to $p(B \otimes \mathbb{K})p$.

Proposition 2.3 *Let A and B be simple σ-unital C^*-algebras and $\mathcal{F}$ an A-B-equivalence bimodule. Then there exists a positive element $h \in A \otimes \mathbb{K}$ such that $\mathcal{F}$ is isomorphic to $\overline{hH_B}$ as a right Hilbert B-module with an isomorphism of A to $\overline{h(B \otimes \mathbb{K})h}$.*

Proof By the discussion above, there exists a projection $p \in M(B \otimes \mathbb{K})$ such that an A-B-equivalence bimodule $\mathcal{F}$ is isomorphic to pH_B as a right Hilbert B-module with an isomorphism of A to $p(B \otimes \mathbb{K})p$. Since $p(B \otimes \mathbb{K})p$ is σ-unital, there exists a strict positive element h in $p(B \otimes \mathbb{K})p$. A standard argument shows that $h^{\frac{1}{n}}\xi \in \overline{hH_B}$ for any $n \in \mathbb{N}$ and any $\xi \in H_B$. Since $\{h^{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is an approximate unit for $K_B(pH_B)$, we see that $pH_B = \overline{hH_B}$. Therefore $\mathcal{F}$ is isomorphic to $\overline{hH_B}$ as a right Hilbert B-module with an isomorphism of A to $\overline{h(B \otimes \mathbb{K})h}$ as a C^*-algebra. ■

Recall that a *trace* on A is a linear map τ on the positive elements of A with values in $[0, \infty]$ that vanishes at 0 and satisfies the trace identity $\tau(a^*a) = \tau(aa^*)$. If A is simple, then $\tau(a^*a) = 0$ implies $a = 0$. Define $\mathcal{M}_\tau^+ = \{a \geq 0 : \tau(a) < \infty\}$ and $\mathcal{M}_\tau = \operatorname{span}\mathcal{M}_\tau^+$. Then $\mathcal{M}_\tau$ is an ideal in A. Every trace τ on A extends a positive linear map on $\mathcal{M}_\tau$. A *normalized trace* is a state on A that is a trace. We say τ is *densely defined* if $\mathcal{M}_\tau$ is a dense ideal in A. In particular, each densely defined trace on A extends a positive linear map on the Pedersen ideal $\operatorname{Ped}(A)$, which is the minimal dense ideal in A. (See [32].) Note that if A is unital, then every densely defined trace is bounded. We review some results about inducing traces from a simple σ-unital C^*-algebra A through a right Hilbert A-module $\mathcal{X}$. See, for example, [6, 9, 14, 25, 31, 33] for induced traces in several settings. We state the relevant properties in a way that is convenient for our purposes, and we include a self-contained proof.

Proposition 2.4 *Let A and $\mathcal{X}$ be as above and let τ be a densely defined lower semicontinuous trace. For $x \in K_A(\mathcal{X})_+$ (resp. $L_A(\mathcal{X})_+$), define*

$$Tr_\tau^{\mathcal{X}}(x) := \sum_{i=1}^{\infty} \tau\big(\langle \xi_i, x\xi_i\rangle_A\big),$$

where $\{\xi_i\}_{i=1}^{\infty}$ is a countable basis of $\mathcal{X}$. Then $Tr_\tau^{\mathcal{X}}$ does not depend on the choice of basis and is a densely defined (resp. strictly densely defined), lower semicontinuous trace on $K_A(\mathcal{X})$ (resp. $L_A(\mathcal{X})$).

Proof Let $\{\xi_i\}_{i\in\mathbb{N}}$ and $\{\zeta_k\}_{k\in\mathbb{N}}$ be countable bases of $\mathcal{X}$. For any positive element $x \in K_A(\mathcal{X})_+$,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\tau\big(\langle \xi_i, x\xi_i\rangle_A\big) &= \lim_{n\to\infty}\sum_{i=1}^{n}\tau\Big(\Big\langle \sum_{k=1}^{\infty}\zeta_k\langle \zeta_k, x^{\frac{1}{2}}\xi_i\rangle_A, x^{\frac{1}{2}}\xi_i\Big\rangle_A\Big)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\tau\Big(\sum_{k=1}^{\infty}\langle x^{\frac{1}{2}}\xi_i, \zeta_k\rangle_A\langle \zeta_k, x^{\frac{1}{2}}\xi_i\rangle_A\Big).\end{aligned}$$

By the lower semicontinuity of τ and $\sum_{i=1}^{n}\Theta_{\xi_i,\xi_i}\leq 1_{L_A(\mathcal{X})}$,

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\tau\Big(\sum_{k=1}^{\infty}\langle x^{\frac12}\xi_i,\zeta_k\rangle_A\langle\zeta_k,x^{\frac12}\xi_i\rangle_A\Big)\\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\tau\big(\langle x^{\frac12}\xi_i,\zeta_k\rangle_A\langle\zeta_k,x^{\frac12}\xi_i\rangle_A\big)\\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\tau\big(\langle x^{\frac12}\zeta_k,\xi_i\rangle_A\langle\xi_i,x^{\frac12}\zeta_k\rangle_A\big)\\
&=\lim_{n\to\infty}\sum_{k=1}^{\infty}\tau\Big(\Big\langle x^{\frac12}\zeta_k,\sum_{i=1}^{n}\xi_i\langle\xi_i,x^{\frac12}\zeta_k\rangle_A\Big\rangle_A\Big)\leq\lim_{m\to\infty}\sum_{k=1}^{m}\tau\big(\langle\zeta_k,x\zeta_k\rangle_A\big).
\end{aligned}$$

Therefore $Tr_\tau^{\mathcal{X}}$ does not depend on the choice of basis. A similar argument implies that $Tr_\tau^{\mathcal{X}}(x^*x)=Tr_\tau^{\mathcal{X}}(xx^*)$ for $x\in L_A(\mathcal{X})$.

We shall show that $Tr_\tau^{\mathcal{X}}$ is densely defined on $K_A(\mathcal{X})$. Since $K_A(\mathcal{X})$ is simple, it is enough to show that there exists a nonzero element $x\in K_A(\mathcal{X})$ such that $Tr_\tau^{\mathcal{X}}(x)<\infty$. There exists a nonzero positive element $a\in A$ such that $\tau(a)<\infty$ because τ is densely defined on A. For any $\eta\in\mathcal{X}$, we have $\langle\eta a^{\frac12},\eta a^{\frac12}\rangle_A\leq\|\langle\eta,\eta\rangle_A\|a$, so $\tau(\langle\eta a^{\frac12},\eta a^{\frac12}\rangle_A))<\infty$. By the simplicity of A, there exists an element η_0 in $\mathcal{X}$ such that $\eta_0 a^{\frac12}\neq 0$. Define $\zeta:=\eta_0 a^{\frac12}$. Then we have

$$\begin{aligned}
Tr_\tau^{\mathcal{X}}(\Theta_{\zeta,\zeta})&=\sum_{n=1}^{\infty}\tau\big(\langle\xi_n,\zeta\rangle_A\langle\zeta,\xi_n\rangle_A\big)=\lim_{N\to\infty}\tau\Big(\Big\langle\zeta,\sum_{n=1}^{N}\xi_n\langle\xi_n,\zeta\rangle_A\Big\rangle_A\Big)\\
&=\tau\big(\langle\zeta,\zeta\rangle_A\big)<\infty
\end{aligned}$$

by the lower semicontinuity of τ. Therefore $Tr_\tau^{\mathcal{X}}$ is densely defined on $K_A(\mathcal{X})$. It is easy to see that $Tr_\tau^{\mathcal{X}}$ is lower semicontinuous. ∎

Remark 2.5 Since a right Hilbert A-module $\mathcal{X}$ is a $K_A(\mathcal{X})$-A-equivalence bimodule, a similar computation in the proof above implies $Tr_{Tr_\tau^{\mathcal{X}}}^{\mathcal{X}^*}=\tau$. Therefore there exists a bijective correspondence between densely defined lower semicontinuous traces on A and those on $K_A(\mathcal{X})$.

To simplify notation, we use the same letter τ for the induced trace $Tr_\tau^{\mathcal{X}_A}$ on $M(A)$. We denote by $\hat{\tau}$ the induced trace $Tr_\tau^{H_A}$ on $M(A\otimes\mathbb{K})$.

3 Multiplicative Maps of the Picard Groups to $\mathbb{R}_+^\times$

Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ_A. Define a map $\widehat{T}_{\tau_A}$ of $\mathcal{H}(A)$ to $[0,\infty]$ by

$$\widehat{T}_{\tau_A}([\mathcal{X}]):=Tr_{\tau_A}^{\mathcal{X}}(1_{L_A(\mathcal{X})}).$$

We see that $\widehat{T}_{\tau_A}([\mathcal{X}]) = \sum_{i=1}^{\infty} \tau_A(\langle \xi_i, \xi_i \rangle_A)$, where $\{\xi_i\}_{i=1}^{\infty}$ is a countable basis of $\mathcal{X}$ and does not depend on the choice of basis (see Proposition 2.4). It is easily seen that $\widehat{T}_{\tau_A}$ is well-defined. We shall compute $\widehat{T}_{\tau_A}([\overline{hH_A}])$, where h is a positive element in $A \otimes \mathbb{K}$. Let $d_{\tau_A}(h) = \lim_{n\to\infty} \widehat{\tau}_A(h^{\frac{1}{n}})$ for $h \in (A \otimes \mathbb{K})_+$. Then d_{τ_A} is a dimension function. (See, for example, [1, 2, 8].)

Proposition 3.1 *Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ_A and h a positive element in $A \otimes \mathbb{K}$. Then $\widehat{T}_{\tau_A}([\overline{hH_A}]) = d_{\tau_A}(h)$.*

Proof We may assume that $\|h\| \leq 1$. Then $\{h^{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is an increasing approximate unit for $K_A(\overline{hH_A})$ and $\lim_{n\to\infty} h^{\frac{1}{n}}\xi = \xi$ for any $\xi \in \overline{hH_A}$. Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a basis of $\overline{hH_A}$ and $\{\eta_j\}_{j\in\mathbb{N}}$ a basis of H_A. By the lower semicontinuity of τ_A and $\langle \xi_i, h^{\frac{1}{n}}\xi_i\rangle_A \leq \langle \xi_i, h^{\frac{1}{n+1}}\xi_i\rangle_A$, we have

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{\infty}\tau_A(\langle \xi_i, h^{\frac{1}{n}}\xi_i\rangle_A) &= \sum_{i=1}^{\infty}\lim_{n\to\infty}\tau_A(\langle \xi_i, h^{\frac{1}{n}}\xi_i\rangle_A)\\ &= \sum_{i=1}^{\infty}\tau_A(\langle \xi_i, \xi_i\rangle_A) = \widehat{T}_{\tau_A}([\overline{hH_A}]).\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\tau_A\big(\langle \xi_i, h^{\frac{1}{n}}\xi_i\rangle_A\big) &= \sum_{i=1}^{\infty}\tau_A\Big(\Big\langle \xi_i, h^{\frac{1}{2n}}\sum_{j=1}^{\infty}\eta_j\langle \eta_j, h^{\frac{1}{2n}}\xi_i\rangle_A\Big\rangle_A\Big)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\tau_A\big(\langle \xi_i, h^{\frac{1}{2n}}\eta_j\rangle_A\langle h^{\frac{1}{2n}}\eta_j, \xi_i\rangle_A\big)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\tau_A\big(\langle h^{\frac{1}{2n}}\eta_j, \xi_i\rangle_A\langle \xi_i, h^{\frac{1}{2n}}\eta_j\rangle_A\big)\\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\tau_A\Big(\big\langle h^{\frac{1}{2n}}\eta_j, \xi_i\langle \xi_i, h^{\frac{1}{2n}}\eta_j\rangle_A\big\rangle_A\Big)\\ &= \sum_{j=1}^{\infty}\tau_A\big(\langle \eta_j, h^{\frac{1}{n}}\eta_j\rangle_A\big) = \widehat{\tau}_A(h^{\frac{1}{n}})\end{aligned}$$

because $h^{\frac{1}{2n}}\eta_j \in \overline{hH_A}$. Therefore $\widehat{T}_{\tau_A}([\overline{hH_A}]) = d_{\tau_A}(h)$. ∎

Remark 3.2 Let p be a projection in $M(A \otimes \mathbb{K})$. Then it is easy to see that

$$\widehat{T}_{\tau_A}([pH_A]) = \widehat{\tau}_A(p).$$

The following proposition is a generalization of [30, Proposition 2.1].

Proposition 3.3 *Let A and B be simple σ-unital C^*-algebras with unique (up to scalar multiple) densely defined lower semicontinuous traces τ_A and τ_B respectively. Assume that $\tau_A(1_{M(A)}) = 1$, that is, τ_A is a normalized trace. Then for every right Hilbert A-module $\mathcal{X}$ and every A-B-equivalence bimodule $\mathcal{F}$,*

$$\widehat{T}_{\tau_B}([\mathcal{X} \otimes \mathcal{F}]) = \widehat{T}_{\tau_A}([\mathcal{X}])\widehat{T}_{\tau_B}([\mathcal{F}]).$$

Proof Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a countable basis of $\mathcal{X}$ and let $\{\eta_j\}_{j\in\mathbb{N}}$ be a countable basis of $\mathcal{F}$ as a right Hilbert B-module. Then $\{\xi_i \otimes \eta_j\}_{i,j\in\mathbb{N}}$ is a countable basis of $\mathcal{X} \otimes \mathcal{F}$ as a right Hilbert A-module. By $\tau_B(\langle \xi_i \otimes \eta_j, \xi_i \otimes \eta_j\rangle_B) \geq 0$, we have

$$\begin{aligned}\widehat{T}_{\tau_B}([\mathcal{X} \otimes \mathcal{F}]) &= \sum_{i,j=1}^{\infty} \tau_B\big(\langle \xi_i \otimes \eta_j, \xi_i \otimes \eta_j\rangle_B\big) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \tau_B\Big(\big\langle \eta_j, \langle \xi_i, \xi_i\rangle_A \eta_j\big\rangle_B\Big)\\ &= \sum_{i=1}^{\infty} Tr_{\tau_B}^{\mathcal{F}}\big(\langle \xi_i, \xi_i\rangle_A\big).\end{aligned}$$

Since $Tr_{\tau_B}^{\mathcal{F}}$ is a densely defined lower semicontinuous trace on A, there exists $\lambda \in \mathbb{R}_+^{\times}$ such that $Tr_{\tau_B}^{\mathcal{F}} = \lambda\tau_A$. The assumption $\tau_A(1_{M(A)}) = 1$ implies $\lambda = Tr_{\tau_B}^{\mathcal{F}}(1_{M(A)})$. Therefore ,

$$\sum_{i=1}^{\infty} Tr_{\tau_B}^{\mathcal{F}}\big(\langle \xi_i, \xi_i\rangle_A\big) = \sum_{i=1}^{\infty} Tr_{\tau_B}^{\mathcal{F}}(1_{M(A)})\tau_A\big(\langle \xi_i, \xi_i\rangle_A\big) = \widehat{T}_{\tau_A}([\mathcal{X}])\widehat{T}_{\tau_B}([\mathcal{F}]). \qquad \blacksquare$$

We shall consider the multiplicative map of the Picard group to $\mathbb{R}_+^{\times}$.

Proposition 3.4 *Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ_A. Assume that $\mathcal{X}$ is a nonzero right Hilbert A-module such that $\widehat{T}_{\tau_A}([\mathcal{X}]) < \infty$. Define a map $T_{\mathcal{X}}$ of $\mathrm{Pic}(K_A(\mathcal{X}))$ to $\mathbb{R}_+^{\times}$ by*

$$T_{\mathcal{X}}([\mathcal{E}]) := \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])}\widehat{T}_{\tau_A}([\mathcal{E} \otimes \mathcal{X}])$$

for $[\mathcal{E}] \in \mathrm{Pic}(K_A(\mathcal{X}))$. Then $T_{\mathcal{X}}$ is well defined and independent on the choice of trace. Moreover, $T_{\mathcal{X}}$ is a multiplicative map.

Proof If $K_A(\mathcal{X})$-$K_A(\mathcal{X})$ equivalence bimodule $\mathcal{E}'$ is isomorphic to $\mathcal{E}$, then $\mathcal{E}' \otimes \mathcal{X}$ is isomorphic to $\mathcal{E} \otimes \mathcal{X}$. Hence $T_{\mathcal{X}}([\mathcal{E}']) = T_{\mathcal{X}}([\mathcal{E}])$. A similar computation in the proof of Proposition 3.3 shows that

$$T_{\mathcal{X}}([\mathcal{E}]) = \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])}\widehat{T}_{Tr_{\tau_A}^{\mathcal{X}}}([\mathcal{E}]) = \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])} Tr_{Tr_{\tau_A}^{\mathcal{X}}}^{\mathcal{E}}(1_{L_A(\mathcal{E})}).$$

Since $\mathcal{E}$ is a $K_A(\mathcal{X})$-$K_A(\mathcal{X})$-equivalence bimodule, $K_A(\mathcal{E})$ is isomorphic to $K_A(\mathcal{X})$. The uniqueness of the trace on $K_A(\mathcal{X})$ implies

$$Tr_{Tr_{\tau_A}^{\mathcal{X}}}^{\mathcal{E}}(1_{L_A(\mathcal{E})}) = \lambda Tr_{\tau_A}^{\mathcal{X}}(1_{L_A(\mathcal{X})}) = \lambda\widehat{T}_{\tau_A}([\mathcal{X}]) < \infty$$

for some $\lambda \in \mathbb{R}_+^\times$. Therefore $T_\mathcal{X}$ is well defined. Define $\tau' := \frac{Tr_{\tau_A}^{\mathcal{X}}}{\widehat{T}_{\tau_A}([\mathcal{X}])}$. Then τ' is a normalized trace on $K_A(\mathcal{X})$. By Proposition 3.3,

$$\begin{aligned} T_\mathcal{X}([\mathcal{E}][\mathcal{E}']) &= \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])}\widehat{T}_{\tau_A}([\mathcal{E}\otimes\mathcal{E}'\otimes\mathcal{X}]) = \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])}\widehat{T}_{Tr_{\tau_A}^{\mathcal{X}}}([\mathcal{E}\otimes\mathcal{E}']) \\ &= \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])}\widehat{T}_{\tau'}([\mathcal{E}])\widehat{T}_{Tr_{\tau_A}^{\mathcal{X}}}([\mathcal{E}']) = T_\mathcal{X}([\mathcal{E}])T_\mathcal{X}([\mathcal{E}']). \end{aligned}$$

■

4 Fundamental Groups

Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ, and let h_0 be a nonzero positive element in $A\otimes\mathbb{K}$ with $d_\tau(h_0) < \infty$. Put

$$\mathcal{F}_{h_0}(A) := \left\{ d_\tau(h)/d_\tau(h_0) \in \mathbb{R}_+^\times \;\middle|\; \begin{array}{l} h \text{ is a positive element in } A\otimes\mathbb{K} \text{ such} \\ \text{that } \overline{h(A\otimes\mathbb{K})h} \cong \overline{h_0(A\otimes\mathbb{K})h_0} \end{array} \right\}.$$

Lemma 4.1 *Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ and h_0 a nonzero positive element in $A\otimes\mathbb{K}$ such that $d_\tau(h_0) < \infty$. Then $\mathcal{F}_{h_0}(A)$ is a multiplicative subgroup of $\mathbb{R}_+^\times$.*

Proof Put $\mathcal{X} = \overline{h_0H_A}$. It is enough to show that $\mathcal{F}_{h_0}(A) = Im(T_\mathcal{X})$. Let $\mathcal{E}$ be a $K_A(\mathcal{X})$-$K_A(\mathcal{X})$-equivalence bimodule. Then there exists a positive element $h \in A\otimes\mathbb{K}$ such that $\mathcal{E}\otimes\mathcal{X}$ is isomorphic to $\overline{hH_A}$ as a right Hilbert A-module with an isomorphism of $K_A(\mathcal{X})$ to $\overline{h(A\otimes\mathbb{K})h}$ by Proposition 2.3. Since $K_A(\mathcal{X})$ is isomorphic to $\overline{h_0(A\otimes\mathbb{K})h_0}$, and we have $T_\mathcal{X}([\mathcal{E}]) = d_\tau(h)/d_\tau(h_0)$ by Proposition 3.1, $Im(T_\mathcal{X}) \subset \mathcal{F}_{h_0}(A)$. Conversely let h be a positive element in $A\otimes\mathbb{K}$ such that $\overline{h(A\otimes\mathbb{K})h}$ is isomorphic to $\overline{h_0(A\otimes\mathbb{K})h_0}$. Since A is simple and $\overline{h(A\otimes\mathbb{K})h}$ is isomorphic to $K_A(\mathcal{X})$, $\mathcal{E} := \overline{hH_A}\otimes\mathcal{X}^*$ is a $K_A(\mathcal{X})$-$K_A(\mathcal{X})$-equivalence bimodule. By Proposition 3.1, $T_\mathcal{X}([\mathcal{E}]) = \frac{1}{\widehat{T}_{\tau_A}([\mathcal{X}])}\widehat{T}_{\tau_A}([\overline{hH_A}]) = d_\tau(h)/d_\tau(h_0)$. Therefore $\mathcal{F}_{h_0}(A) \subset Im(T_\mathcal{X})$. ■

Lemma 4.2 *Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. Assume that h_0 and h_1 are nonzero positive elements in $A\otimes\mathbb{K}$ such that $d_\tau(h_0), d_\tau(h_1) < \infty$. Then $\mathcal{F}_{h_0}(A) = \mathcal{F}_{h_1}(A)$.*

Proof Let $\mathcal{F} := \overline{h_0H_A}\otimes(\overline{h_1H_A})^*$. Then $\mathcal{F}$ is a $\overline{h_0(A\otimes\mathbb{K})h_0}$- $\overline{h_1(A\otimes\mathbb{K})h_1}$-equivalence bimodule by the simplicity of A, and $\mathcal{F}$ induces an isomorphism Ψ of $\operatorname{Pic}(\overline{h_0(A\otimes\mathbb{K})h_0})$ to $\operatorname{Pic}(\overline{h_1(A\otimes\mathbb{K})h_1})$ such that $\Psi([\mathcal{E}]) = [\mathcal{F}^*\otimes\mathcal{E}\otimes\mathcal{F}]$ for $[\mathcal{E}] \in \operatorname{Pic}(\overline{h_0(A\otimes\mathbb{K})h_0})$. By Proposition 3.3, $T_{\overline{h_1H_A}}(\Psi([\mathcal{E}])) = T_{\overline{h_0H_A}}([\mathcal{E}])$. Therefore $\mathcal{F}_{h_0}(A) = \mathcal{F}_{h_1}(A)$ by the proof of Lemma 4.1. ■

Put

$$\mathcal{F}(A) := \left\{ d_\tau(h_1)/d_\tau(h_2) \in \mathbb{R}_+^\times \;\middle|\; \begin{array}{l} h_1 \text{ and } h_2 \text{ are nonzero positive elements in} \\ A\otimes\mathbb{K} \text{ such that } \overline{h_1(A\otimes\mathbb{K})h_1} \cong \overline{h_2(A\otimes\mathbb{K})h_2}, \\ d_\tau(h_2) < \infty \end{array} \right\}.$$

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Theorem 4.3 *Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. Then $\mathcal{F}(A)$ is a multiplicative subgroup of $\mathbb{R}_+^\times$.*

Proof Let h_0 be a nonzero positive element in $\mathrm{Ped}(A\otimes\mathbb{K})$. By [32, Proposition 5.6.2], $\overline{h_0(A\otimes\mathbb{K})h_0}$ is contained in $\mathrm{Ped}(A\otimes\mathbb{K})$. Since $\widehat{\tau}$ is densely defined, $\mathrm{Ped}(A\otimes\mathbb{K})\subset\mathcal{M}_{\widehat{\tau}}$. Therefore $\widehat{\tau}$ is bounded on $\overline{h_0(A\otimes\mathbb{K})h_0}$ and hence $d_\tau(h_0)<\infty$. Lemma 4.2 implies $\cup_{d_\tau(h)<\infty}\mathcal{F}_h(A)=\mathcal{F}_{h_0}(A)$. It is clear that $\mathcal{F}(A)=\cup_{d_\tau(h)<\infty}\mathcal{F}_h(A)$. Consequently $\mathcal{F}(A)$ is a multiplicative subgroup of $\mathbb{R}_+^\times$ by Lemma 4.1. ∎

Definition 4.4 Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. We call $\mathcal{F}(A)$ the fundamental group of A, which is a multiplicative subgroup of $\mathbb{R}_+^\times$.

Remark 4.5 It is easy to see that $\mathcal{F}(A)$ is equal to the set

$$\left\{\widehat{\tau}(p)/\widehat{\tau}(q)\in\mathbb{R}_+^\times \;\middle|\; \begin{array}{l} p \text{ and } q \text{ are nonzero projections in} \\ M(A\otimes\mathbb{K}) \text{ such that} \\ p(A\otimes\mathbb{K})p\cong q(A\otimes\mathbb{K})q,\ \widehat{\tau}(q)<\infty \end{array}\right\}.$$

Remark 4.6 If a unique densely defined lower semicontinuous trace τ is a normalized trace, then $\mathcal{F}(A)$ is equal to the set

$$\{d_\tau(h)\in\mathbb{R}_+^\times \mid h \text{ is a positive element in } A\otimes\mathbb{K} \text{ such that } A\cong\overline{h(A\otimes\mathbb{K})h}\}.$$

Note that there exists a simple σ-unital C^*-algebra with a unique normalized trace τ, which has a densely defined lower semicontinuous trace that is not a scalar multiple of τ. For example, let A be an AF-algebra such that

$$K_0(A)=\mathbb{Z}\Big[\frac{1}{2}\Big]\oplus\mathbb{Z}\Big[\frac{1}{2}\Big],$$
$$K_0(A)_+=\big\{(q,r)\in K_0(A): q>0, r>0\big\}\cup\big\{(0,0)\big\},$$
$$\Sigma(A)=\big\{(q,r)\in K_0(A)_+: q>0, 0<r<1\big\}\cup\big\{(0,0)\big\}.$$

Then A is such a C^*-algebra.

The following corollary is shown by an argument similar to Lemma 4.2.

Corollary 4.7 *Let A and B be simple σ-unital C^*-algebras with unique (up to scalar multiple) densely defined lower semicontinuous traces. If A is Morita equivalent to B, then $\mathcal{F}(A)=\mathcal{F}(B)$.*

We shall show that if A is unital, then Definition 4.4 coincides with the previous definition in [29, 30].

Proposition 4.8 *Let A be a simple unital C^*-algebra with a unique normalized trace τ. Then*

$$\mathcal{F}(A)=\{\tau\otimes Tr(p)\in\mathbb{R}_+^\times \mid p \textit{ is a projection in } M_n(A) \textit{ such that } pM_n(A)p\cong A\}$$

where Tr is the usual unnormalized trace on $M_n(\mathbb{C})$.

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Proof Let $\mathcal{X}_A$ be a right Hilbert A-module A with the obvious right A-action and $\langle a, b\rangle_A = a^*b$ for $a, b \in A$. Since τ is a normalized trace, $\widehat{T}_\tau([\mathcal{X}_{\mathcal{A}}]) = 1$. By the proofs of Lemmas 4.1 and 4.2, $\mathcal{F}(A) = \mathcal{F}_{1\otimes e_{11}}(A) = \widehat{T}_\tau(\mathrm{Pic}(A))$ where e_{11} is a rank one projection in $\mathbb{K}$. A similar argument as in [29, Theorem 3.1] shows $\widehat{T}_\tau(\mathrm{Pic}(A)) = \{\tau \otimes Tr(p) \in \mathbb{R}_+^\times \mid p \text{ is a projection in } M_n(A) \text{ such that } pM_n(A)p \cong A\}$ because every A-A-equivalence bimodule has a finite basis. ■

We showed that K-theoretical obstruction enables us to compute fundamental groups easily in the case A is unital [29]. Therefore, if $A\otimes\mathbb{K}$ has a nonzero projection, we can compute fundamental groups easily by K-theoretical obstruction. We denote by τ_* the map $K_0(A) \to \mathbb{R}$ induced by a trace τ on A.

Definition 4.9 Let E be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. Then an *inner multiplier group* $IM(E)$ of E is defined by

$$IM(E) = \{t \in \mathbb{R}^\times \mid t \in E, t^{-1} \in E, \text{ and } tE = E\}.$$

Then $IM(E)$ is a multiplicative subgroup of $\mathbb{R}^\times$. We call $IM_+(E) := IM(E) \cap \mathbb{R}_+$ the *positive inner multiplier group* of E, which is a multiplicative subgroup of $\mathbb{R}_+^\times$.

Corollary 4.10 *Let A be a separable simple C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. Assume that $A \otimes \mathbb{K}$ has a nonzero projection. Then $\mathcal{F}(A)$ is countable. Moreover, $\tau_*(K_0(A))$ is a $\mathbb{Z}[\mathcal{F}(A)]$-module and $\mathcal{F}(A) \subset IM_+(\tau_*(K_0(A)))$.*

Proof Let p be a nonzero projection in $A\otimes\mathbb{K}$. Corollary 4.7 implies $\mathcal{F}(A) = \mathcal{F}(p(A\otimes\mathbb{K})p)$. Since $p(A \otimes \mathbb{K})p$ is a separable unital C^*-algebra, [29, Proposition 3.7] and Proposition 4.8 prove the corollary. ■

Example 4.11 Let $\mathbb{F}_n$ be a non-abelian free group with $n \geq 2$ generators. Then $C_r^*(\mathbb{F}_n)$ is a simple unital C^*-algebra with a unique normalized trace. Since $K_0(C_r^*(\mathbb{F}_n)) \cong \mathbb{Z}$, $\mathcal{F}(C_r^*(\mathbb{F}_n)) = \{1\}$. This implies that for positive elements $h_1, h_2 \in C_r^*(\mathbb{F}_n)$ if $\overline{h_1C_r^*(\mathbb{F}_n)h_1}$ is isomorphic to $\overline{h_2C_r^*(\mathbb{F}_n)h_2}$, then $d_\tau(h_1) = d_\tau(h_2)$.

Example 4.12 Let p be a prime number. Consider a tensor product algebra of a UHF algebra and the compact operators $A = M_{p^\infty} \otimes \mathbb{K}$. Then $\mathcal{F}(A) = \{p^n : n \in \mathbb{Z}\}$.

Remark 4.13 Any countable subgroup of $\mathbb{R}_+^\times$ can be realized as the fundamental group $\mathcal{F}(A)$ of a separable simple unital C^*-algebra A with a unique trace. (See [30].)

We show that there exist separable, simple, stably projectionless C^*-algebras such that their fundamental groups are equal to $\mathbb{R}_+^\times$. This is a contrast to the unital case. Recall the building blocks that are considered by Razak [37] and Tsang [40]. These algebras are subhomogeneous algebras obtained by generalized mapping torus construction as in [10, 11]. For a pair of natural numbers (n, m) with n dividing m $(m > n)$, let ρ_0 and ρ_1 be homomorphisms from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$, having multiplicities $\frac{m}{n} - 1$ and $\frac{m}{n}$ respectively. Define

$$A(n, m) = \left\{ f \in M_m(C([0, 1])) : f(0) = \rho_0(c), f(1) = \rho_1(c), c \in M_n(\mathbb{C})\right\}.$$

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Note that we may assume that the homomorphism ρ_0 maps $M_n(\mathbb{C})$ into diagonal block matrices in $M_m(\mathbb{C})$ with $\frac{m}{n}-1$ identical blocks and one zero block. On the other hand, the homomorphism ρ_1 yields matrices with $\frac{m}{n}$ identical blocks. The building block $A(n,m)$ has the following properties. (See, for example, [32, 37].)

Proposition 4.14 *We have the following:*

(i) *Every primitive ideal of $A(n,m)$ is the kernel of some point evaluation. Therefore the primitive ideal space of $A(n,m)$ is homeomorphic to $\mathbb{T}$.*

(ii) *The Pedersen ideal of $A(n,m)$ is $A(n,m)$. Therefore every densely defined lower semicontinuous trace on $A(n,m)$ is bounded.*

(iii) *For any bounded trace τ on $A(n,m)$, there exists a measure μ on $\mathbb{T}$ such that $\tau(f)=\int_{\mathbb{T}}(\frac{m-n}{m})^t Tr(f(t))d\mu(t)$ for any $f\in A(n,m)$.*

Fix an irrational $\theta\in[0,1]\backslash\mathbb{Q}$. For any $n\in\mathbb{N}$, define an injective homomorphism ϕ_n of $A(3^n,2\cdot 3^n)$ to $A(3^{n+1},2\cdot 3^{n+1})$ by

$$(\phi_n(f))(t)=\begin{cases} u_t\begin{pmatrix} f(t) & 0 & 0\\ 0 & f(t+\theta) & 0\\ 0 & 0 & 0\end{pmatrix}u_t^* & 0\le t\le 1-\theta\\ w_t\begin{pmatrix} f(t) & 0 & 0\\ 0 & f(t+\theta-1) & 0\\ 0 & 0 & f(t+\theta-1)\end{pmatrix}w_t^* & 1-\theta\le t\le 1,\end{cases}$$

where u_t and w_t are suitable continuous paths in $U(M_{2\cdot 3^{n+1}}(\mathbb{C}))$. We denote by $\phi_{n,m}$ a homomorphism $\phi_{m-1}\circ\cdots\circ\phi_n$ from $A(3^n,2\cdot 3^n)$ to $A(3^m,2\cdot 3^m)$. Let $\mathcal{O}=\varinjlim(A(3^n,2\cdot 3^n),\phi_{n,m})$.

Lemma 4.15 *With notation as above, $\mathcal{O}=\varinjlim(A(3^n,2\cdot 3^n),\phi_{n,m})$ is a separable simple stably projectionless C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous unbounded trace.*

Proof Let J be a proper two-sided closed ideal of $\mathcal{O}$, and let

$$J_n=\phi_{n,\infty}^{-1}\big(J\cap\phi_{n,\infty}\big(A(3^n,2\cdot 3^n)\big)\big).$$

Then J_n is a two-sided closed ideal of $A(3^n,2\cdot 3^n)$, and denote by F_n the corresponding closed set in $\mathbb{T}$. (See Proposition 4.14.) Since $\phi_{n,m}$ is injective, $J=\varinjlim(J_n,\phi_{n,m})$ and for n sufficiently large, J_n is a proper two-sided closed ideal of $A(3^n,2\cdot 3^n)$; that is, F_n is not empty. Put $\gamma([t])=[t-\theta]$ for any $[t]\in\mathbb{T}$. For any natural number k, we see that $F_n=F_{n+k}\cup\gamma^{-1}(F_{n+k})\cup\cdots\cup\gamma^{-k}(F_{n+k})$ by the construction of $\phi_{n,m}$ and $J_n=\phi_{n,n+k}^{-1}(J\cap\phi_{n,n+k}(A(3^n,2\cdot 3^n)))$. The same argument as in the last part of the proof of [3, Proposition 1.3] shows that $\mathcal{O}$ is simple, because γ is a minimal homeomorphism on $\mathbb{T}$.

Define $\tau_n(f)=\frac{1}{(1+2^\theta)^n}\int_{\mathbb{T}}(\frac{1}{2})^t Tr(f(t))d\mu(t)$, where μ is a normalized Haar measure on $\mathbb{T}$ and Tr is the usual unnormalized trace on $M_{2\cdot 3^n}(\mathbb{C})$. Then $\tau_n=\tau_{n+1}\circ\phi_n$, and hence there exists a densely defined lower semicontinuous trace τ on $\mathcal{O}$. Note that τ is an unbounded trace, since $\|\tau_n\|=\frac{2\cdot 3^n}{(1+2^\theta)^n}$.

We shall show the uniqueness of τ. Let τ' be a densely defined lower semicontinuous trace on $\mathcal{O}$. It is easy to see that $\tau'|_{A(3^n,2\cdot 3^n)}$ is a densely defined lower semicontinuous trace on $A(3^n,2\cdot 3^n)$, since $\mathrm{Ped}(A(3^n,2\cdot 3^n))\subseteq\mathrm{Ped}(\mathcal{O})\cap A(3^n,2\cdot 3^n)$.

Proposition 4.14 implies that for any $n \in \mathbb{N}$ there exists a measure ν_n on $\mathbb{T}$ such that $\tau'|_{A(3^n,2\cdot 3^n)}(f) = \frac{1}{(1+2^\theta)^n}\int_{\mathbb{T}}(\frac{1}{2})^t Tr(f(t))d\nu_n(t)$. By a compatibility condition, we have

$$\int_{\mathbb{T}}\Big(\frac{1}{2}\Big)^t Tr\big(f(t)\big)\,d\nu_n(t) = \frac{1}{1+2^\theta}\int_{\mathbb{T}}\Big(\frac{1}{2}\Big)^t\big(Tr(f(t)) + g(t)\big)\,d\nu_{n+1}(t),$$

where

$$g(t) = \begin{cases} Tr(f(t+\theta)) & 0 \le t \le 1-\theta, \\ 2Tr(f(t+\theta-1)) & 1-\theta \le t \le 1, \end{cases}$$

for any $f \in A(3^n, 2\cdot 3^n)$. Therefore for any $h \in C(\mathbb{T})$, we have

$$\int_{\mathbb{T}} h(t)d\nu_n(t) = \frac{1}{1+2^\theta}\int_{\mathbb{T}} h(t) + 2^\theta h(t+\theta)d\nu_{n+1}(t).$$

In the same way as in the last part of the proof of [20, Theorem 2.4], we see that ν_n is Haar measure on $\mathbb{T}$ by this condition. Consequently there exists a positive number λ such that $\tau' = \lambda\tau$. ■

The following lemma is an immediate consequence of the classification theorem of Razak [37] and Tsang [40, Theorem 3.1].

Lemma 4.16 *Let A be a simple separable AF algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace. Then $A \otimes \mathcal{O}$ is isomorphic to $\mathcal{O}$.*

Theorem 4.17 *There exist a separable simple stably projectionless nuclear C^*-algebra and non-nuclear C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace such that their fundamental groups are equal to $\mathbb{R}_+^\times$.*

Proof For any $\lambda \in \mathbb{R}_+^\times$, there exists a separable simple unital AF algebra A_λ with a unique trace such that $\lambda \in \mathcal{F}(A_\lambda)$ by [29, Corollary 3.16]. Lemma 4.16 implies $\lambda \in \mathcal{F}(\mathcal{O})$. Therefore $\mathcal{F}(\mathcal{O}) = \mathbb{R}_+^\times$. Let $\mathbb{F}_n$ be a non-abelian free group with $n \ge 2$ generators. Then $\mathcal{O} \otimes C_r^*(\mathbb{F}_n)$ is a separable, stably projectionless, non-nuclear C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace such that $\mathcal{F}(\mathcal{O} \otimes C_r^*(\mathbb{F}_n)) = \mathbb{R}_+^\times$. ■

Remark 4.18 Let h be a nonzero positive element in the Pedersen ideal of $\mathcal{O}$. Then $\overline{h\mathcal{O}h}$ is a separable stably projectionless C^*-algebra with a unique normalized trace such that $\mathcal{F}(\overline{h\mathcal{O}h}) = \mathbb{R}_+^\times$.

Remark 4.19 Recently, Jacelon [15] constructed a simple, nuclear, stably projectionless C^*-algebra W with a unique normalized trace, which shares some of the important properties of the Cuntz algebra $\mathcal{O}_2$. This C^*-algebra is an inductive limit of building blocks $A(n,m)$. Hence $W \otimes \mathbb{K}$ is isomorphic to $\mathcal{O}$ by the classification theorem of Razak [37]. Therefore Corollary 4.7 and Theorem 4.17 imply that $\mathcal{F}(W) = \mathbb{R}_+^\times$.

Recall that the fundamental group of a II_1-factor M is equal to the set of trace-scaling constants for automorphisms of $M \otimes B(\mathcal{H})$. We have a fact similar to the one discussed by Kodaka in [24]. We define the set of trace-scaling constants for automorphisms:

$$\mathfrak{S}(A) := \left\{ \lambda \in \mathbb{R}_+^\times \mid \widehat{\tau} \circ \alpha = \lambda \widehat{\tau} \text{ for some } \alpha \in \mathrm{Aut}(A \otimes K(\mathcal{H})) \right\}.$$

Proposition 4.20 *Let A be a simple σ-unital C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. Then $\mathcal{F}(A) = \mathfrak{S}(A)$.*

Proof There exists a nonzero projection p in $M(A \otimes \mathbb{K})$ such that $\widehat{\tau}(p) < \infty$ by a similar argument as in the proof of Theorem 4.3. Let $\lambda \in \mathfrak{S}(A)$, then there exists an automorphism of $A \otimes \mathbb{K}$ such that $\widehat{\tau} \circ \alpha(x) = \lambda \widehat{\tau}(x)$ for $x \in \mathcal{M}_{\widehat{\tau}}$. There exists an automorphism $\tilde{\alpha}$ of $M(A \otimes \mathbb{K})$ such that $\tilde{\alpha}(x) = \alpha(x)$ for $x \in A \otimes \mathbb{K}$. It is clear that $p(A \otimes \mathbb{K})p$ is isomorphic to $\tilde{\alpha}(p)(A \otimes \mathbb{K})\tilde{\alpha}(p)$. We have that $\widehat{\tau}(\tilde{\alpha}(p))/\widehat{\tau}(p) = \lambda$. Therefore $\lambda \in \mathcal{F}(A)$ by Remark 4.5.

Conversely, let $\lambda \in \mathcal{F}(A)$. There exist projections p and q in $M(A \otimes \mathbb{K})$ such that $p(A \otimes \mathbb{K})p$ is isomorphic to $q(A \otimes \mathbb{K})q$ and $\lambda = \widehat{\tau}(p)/\widehat{\tau}(q)$. We denote by ϕ an isomorphism of $p(A \otimes \mathbb{K})p$ to $q(A \otimes \mathbb{K})q$. Since p and q are full projections, there exist partial isometries w_1 and w_2 in $(A \otimes \mathbb{K}) \otimes \mathbb{K}$ such that $w_1^* w_1 = I \otimes I$, $w_1 w_1^* = p \otimes I$, $w_2^* w_2 = I \otimes I$, and $w_2 w_2^* = q \otimes I$ by Brown [4]. Let $\psi \colon A \otimes \mathbb{K} \otimes \mathbb{K} \to A \otimes \mathbb{K}$ be an isomorphism that induces the identity on the K_0-group. Define $\alpha = \psi \circ (adw_2^*) \circ \phi \circ (adw_1) \circ \psi^{-1}$. Then $\widehat{\tau} \circ \alpha = \lambda \widehat{\tau}$. Therefore $\lambda \in \mathfrak{S}(A)$. ■

Example 4.21 Let $\{\lambda_1, \dots, \lambda_n\}$ be nonzero positive numbers such that the closed additive subgroup of $\mathbb{R}$ generated by $\{\lambda_1, \dots, \lambda_n\}$ is $\mathbb{R}$, and $\mathcal{O}_n$ is the Cuntz algebra generated by n isometries $S_1, \dots, S_n$. There exists a one-parameter automorphism group $\alpha \colon \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_n)$ given by $\alpha_t(S_j) = e^{it\lambda_j} S_j$. Define $A := \mathcal{O}_n \rtimes_\alpha \mathbb{R}$. Then A is a simple stable separable C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ and $\mathfrak{S}(A) = \mathbb{R}_+^\times$ [20, 21]. Therefore $\mathcal{F}(A) = \mathbb{R}_+^\times$ by the corollary above.

Finally we state a direct relation between the fundamental group of C^*-algebras and that of von Neumann algebras.

Proposition 4.22 *Let A be a σ-unital infinite-dimensional simple C^*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ. Assume that τ is a normalized trace. Consider the GNS representation $\pi_\tau \colon A \to B(H_\tau)$ and the associated factor $\pi_\tau(A)''$ of type II_1. Then $\mathcal{F}(A) \subset \mathcal{F}(\pi_\tau(A)'')$. In particular, if $\mathcal{F}(\pi_\tau(A)'') = \{1\}$, then $\mathcal{F}(A) = \{1\}$.*

Proof Let h be a positive element in $A \otimes \mathbb{K}$ such that A is isomorphic to $\overline{h(A \otimes \mathbb{K})h}$. We denote by $\tilde{\tau}$ the restriction of $\widehat{\tau}$ on $\overline{h(A \otimes \mathbb{K})h}$. By the uniqueness of trace, $\pi_\tau(A)''$ is isomorphic to $\pi_{\tilde{\tau}}(\overline{h(A \otimes \mathbb{K})h})''$. Define $p := \int_0^{\|h\|} dE_t$, where $\{E_t : 0 \leq t \leq \|h\|\}$ is the spectral projections of $\pi_{\widehat{\tau}}(h)$. Then $d_\tau(h) = \widehat{\tau}(p)$. A standard argument shows $p\pi_{\widehat{\tau}}(A \otimes \mathbb{K})''p$ is isomorphic to $\pi_{\tilde{\tau}}(\overline{h(A \otimes \mathbb{K})h})''$. Therefore $\mathcal{F}(A) \subset \mathcal{F}(\pi_\tau(A)'')$. ■

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