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CLASSIFICATION OF ALGEBRAIC SURFACES WITH SECTIONAL GENUS LESS THAN OR EQUAL TO SIX. II: RULED SURFACES WITH dim $\phi_{K_X \otimes L}(X) = 1$

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Introduction. Let L be a very ample line bundle on a smooth, connected, projective, ruled not rational surface X. We have considered the problem of classifying biholomorphically smooth, connected, projected, ruled, non rational surfaces X with smooth hyperplane section C such that the genus $g = g(C)$ is less than or equal to six and dim $\phi_{\bar{L}}(X) = 1$ where $\phi_{\bar{L}}$ is the map associated to $\bar{L} = K_X \otimes L$. L. Roth in [**10**] had given a birational classification of such surfaces. If $g = 0$ or 1 then X has been classified, see [**8**].

If $g = 2 \neq h^{1,0}(X)$ by [**12**, Lemma (2.2.2)] it follows that X is a rational surface. Thus we can assume $g \geqq 3$.

Since X is ruled, $h^{2,0}(X) = 0$ and

$$\frac{L \cdot L}{8} + h^{1,0}(X) \leqq \frac{g + 1}{2},$$

see [**4**] and [**12**, p. 390]. Moreover by the classification of surfaces in $\mathbf{P}^2$ and $\mathbf{P}^3$, it follows that $h^0(L) \geqq 5$. Our study is essentially based on the adjunction process which has been introduced by the Italian school and which has been particularly studied by A. J. Sommese [**12**]. In the case in which dim $\phi_{\bar{L}}(X) = 1$, Sommese, in [**12**, p. 390], has proved that if $\phi_{\bar{L}} = r \circ s$ is the Remmert-Stein factorization of $\phi_{\bar{L}}$, then s is an embedding with the possible exception of the case when $g = 3$, $h^{1,0}(X) = 1$ and $L \cdot L = 7$ or 8. Our goal is to classify the pair $(\hat{X}, \hat{L})$ where $\hat{X}$ denotes a reduction of X to a minimal model.

In the case in which $h^{1,0}(X) = 1$, we have described some examples of $(\hat{X}, \hat{L})$ with $\hat{L}$ very ample. Those examples can be found in the following table. In the case in which $h^{1,0}(X) = 2$, there is only a pair $(\hat{X}, \hat{L})$ for which we cannot decide if $\hat{L}$ is very ample while for all the other pairs we know that $\hat{L}$ is not very ample. We would like to note that the class of surfaces with dimension one image under the adjunction mapping are known as conic bundles by algebraic geometers.

We shall mention that our study has a slight overlap with the

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classification that P. Ionescu [6] has given for projective surfaces of sectional genus less than or equal to four. We wish to thank Andrew J. Sommese for suggesting the problem and Alan Howard for helpful discussions about ruled surfaces.

0. Background material. Since most of our notations and background material are found in [12] we will establish here only the notations which are different and the results which cannot be found there. First of all we would like to fix the following notations. We let

$$d = L \cdot L,\ g = g(C) = g(L),\ \hat{d} = \hat{L} \cdot \hat{L},$$

$$c_1^2 = K_X \cdot K_X,\ \hat{c}_1^2 = K_{\hat{X}} \cdot K_{\hat{X}}.$$

We would like to remind the reader that X is gotten from a geometrically ruled surface $\hat{X}$ by blowing up a finite number of points with at most one in each fiber of the natural ruling and $\hat{L}$ is the relative line bundle.

(0.1) *Definition.* Let D be an effective divisor on a smooth, connected, projective surface X. D is k-connected if for every decomposition $D = D_1 + D_2$ into effective divisors $D_1 \cdot D_2 \geqq k$.

(0.2) PROPOSITION. *Let X be a smooth, connected, projective surface embedded by a very ample line bundle $\mathscr{L}$ in $\mathbf{P}^4$. Then*

$$\mathscr{L} \cdot \mathscr{L}(\mathscr{L} \cdot \mathscr{L} - 5) - 10(g(\mathscr{L}) - 1) + 12\chi(\mathscr{O}_X) = 2c_1^2.$$

See [5, p. 434].

(0.3) CASTELNUOVO'S INEQUALITY [2, p. 234 ff; $G + H$]. *If C is an irreducible curve embedded in $\mathbf{P}_{\mathbf{C}}^{\ell-1}$ and C belongs to no linear hyperplane $\mathbf{P}_{\mathbf{C}}^{\ell-2}$, then with d the degree of C and g the genus:*

$$g \leqq \left[\frac{d-2}{\ell-2}\right]\left(d - \ell + 1 - \left[\frac{d-\ell}{\ell-2}\right]\left(\frac{\ell-2}{2}\right)\right)$$

where [ ] is the least integer function.

(0.4) PROPOSITION. *Let X be any projective, smooth surface and let*

$$0 \to E \to F \to G \to 0$$

be the short exact sequence obtained by tensoring the sequence

$$0 \to [C]^{-1} \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

with a line bundle F, where C is a curve in X. Suppose that:

(a) *G is a very ample line bundle on C,*

(b) *E is very ample*

(c) $\ker(H^0(G) \to H^1(E))$ *gives an embedding of C.*

Then F is very ample.

Since the proof is standard we will omit it.

(0.5) *Ruled Surfaces*. Let X be a smooth, connected, projective, geometrically ruled surface, i.e., a fibration $\pi:X \to \overline{C}$, over a curve $\overline{C}$ whose fibres are $\mathbf{P}^1$. Then there exists a rank two vector bundle E (not unique) over $\overline{C}$ and an isomorphism $X = \mathscr{P}(E)$ where $\mathscr{P}(E)$ denotes the associated projective bundle of E. Let $\overline{g}$ be the genus of $\overline{C}$. Let σ be a minimal section of π, there is a line bundle $\mathfrak{L}$ on $\overline{C}$ and an extension of E of $\mathfrak{L}$ by $\mathcal{O}_{\overline{C}}$

(0.5.1) $\quad 0 \to \mathcal{O}_{\overline{C}} \to E \to \mathfrak{L} \to 0$

such that

$$X = \mathscr{P}(E) \quad \text{and} \quad \mathfrak{L} = \sigma^*\mathcal{O}_{\mathscr{P}(E)}(1) = \mathcal{O}_{\sigma(\overline{C})}(\zeta_E)$$

where ζ_E is the tautological line bundle.

$$e = -\zeta_E \cdot \zeta_E = -\deg \mathfrak{L}$$

is an invariant of the surface X. If E is decomposable, then $e \geqq 0$ and all the values of e are possible. If E is indecomposable, then

(0.5.2) $\quad -\overline{g} \leqq e \leqq 2\overline{g} - 2.$

See [**5**, p. 376 and p. 384] and [**9**, p. 191].

Let f be a fiber of $\pi:X \to \overline{C}$. Then every line bundle L on X is numerically equivalent to $\zeta_E^a \otimes \mathscr{L}^b$, i.e., $L \equiv \zeta_E^a \otimes \mathscr{L}^b$ for some integers a, b and $\mathscr{L} = \mathcal{O}_X(f)$, so

$$\deg \mathscr{L}|_{\sigma(\overline{C})} = 1,$$

(0.5.3) $\quad L \cdot L = -a^2e + 2ab \quad \text{and}$

$$2g(L) - 2 = -a^2e + ae + 2ab - 2b - 2a + 2a\overline{g}.$$

The canonical line bundle K_X of X is

$$K_X \equiv \zeta_E^{-2} \otimes \mathscr{L}^{(2\overline{g}-2-e)}.$$

Given a line bundle A on $\overline{C}$ we will denote its lift π^*A on X again by A. We have the following propositions:

(0.5.4) PROPOSITION. *Let X be a geometrically ruled surface over a curve $\overline{C}$, with invariant $e \geqq 0$.*

(i) *If $Y \equiv a\zeta_E + b\mathscr{L}$ is an irreducible curve, $Y \not\equiv \zeta_E, \mathscr{L}$, then $a > 0$, $b \geqq a \cdot e$.*

(ii) *A divisor $D \equiv a\zeta_E + b\mathscr{L}$ is ample if and only if $a > 0$, $b > a \cdot$* e.

See [**5**, p. 382].

(0.5.5) PROPOSITION. *Let X be a geometrically ruled surface over a curve $\overline{C}$, of genus $\overline{g}$ and invariant $e < 0$.*

(i) *If $Y \equiv a\zeta_E + b\mathscr{L}$ is an irreducible curve, $Y \not\equiv \zeta_E, \mathscr{L}$, then either*

$a = 1, b \geqq 0$ *or* $a \geqq 2, b \geqq \frac{1}{2}ae.$

(ii) *A divisor* $D \equiv a\zeta_E + b\mathscr{L}$ *is ample if and only if* $a > 0, b > \frac{1}{2}ae.$

See [**5**, p. 382].

The determination of the very ample divisors on a ruled surface with $\bar{g} \geqq 1$, is more difficult than in the case of a rational ruled surface i.e., a Hirzebruch surface. There is moreover the following result which is stated as an exercise in [**5**, p. 385] and it is not too difficult to prove.

(0.5.6) PROPOSITION. *Let X be a geometrically ruled surface with invariant e over an elliptic curve* ϵ. *Let*

$$L \equiv \zeta_E \otimes \mathscr{L}^b.$$

Then:

(i) *There is a section of L and* $|L|$ *has no base points if and only if* $b \geqq e + 2$.

(ii) *The linear system* $|\zeta_E + b\mathscr{L}|$ *is very ample if and only if* $b \geqq e + 3$.

(0.5.7) PROPOSITION. *Let X be a geometrically ruled surface over a curve* $\bar{C}$ *with* $\bar{g} = g(\bar{C})$ *and invariant e. Let* $L \equiv \zeta_E^a \otimes \mathscr{L}^b$ *be a line bundle on X with* $a > -2$. *Then*:

$$\text{(i)} \qquad h^1(L) = 0 \quad \text{for } b > \begin{cases} ae + 2\bar{g} - 2 + e & \text{if } e \geqq 0 \\ \frac{1}{2}ae + 2\bar{g} - 2 & \text{if } e < 0 \end{cases}$$

$$\text{(ii)} \qquad h^0(L) - h^1(L) = (a + 1)\left(b + 1 - \bar{g} - \frac{ae}{2}\right).$$

We will only sketch the proof of this proposition. In fact (i) follows showing that $L \otimes K_X^{-1}$ is ample and applying the Kodaira Vanishing Theorem. (ii) follows by the Riemann-Roch's theorem.

By a ruled surface we mean a surface birational to a geometrically ruled surface.

(0.6) PROPOSITION. *Let L be an ample and spanned line bundle on a smooth, connected, projective surface X. Assume* $h^0(L) \geqq 4$, $L \cdot L \geqq 5$. *Then* $K_X \otimes L$ *is spanned.*

See [**13**, Theorem (0.8)].

(0.7) PROPOSITION. *Let L be an ample and spanned line bundle. Suppose that L is* 3*-connected,* $h^0(L) \geqq 7$ *and* $L \cdot L \geqq 10$. *Then* $K_X \otimes L$ *is very ample.*

The proof is the same as in [**14**, p. 406].

1. The case of dim $\phi_{\bar{L}}(X) = 1$. As we have seen in the introduction

$$g = 3, \dots, 6,\ h^0(\hat{L}) \geqq h^0(L) \geqq 5 \quad \text{and}$$

$$\text{(1.0.1)} \quad \frac{d}{8} + h^{1,0}(0) \leqq \frac{g+1}{2}.$$

By Castelnuovo's inequality (0.3) if $g = 3, 4$,

$$h^{1,0}(X) = 1, \quad d \geqq 6$$

if $g = 5, 6$,

$$h^{1,0}(X) = 1, 2, \quad d \geqq 7.$$

By the long cohomology sequence of

$$0 \to K_X \to K_X \otimes L \to K_C \to 0$$

it follows that

$$h^0(K_X \otimes L) = g - 1 \quad \text{if } h^{1,0}(X) = 1 \quad \text{and}$$
$$h^0(K_X \otimes L) = g - 2 \quad \text{if } h^{1,0}(X) = 2.$$

By [**12**, p. 390] we have:

$$\hat{d} = 4g - 8h^{1,0}(X) + 4.$$

Thus we have to study the following cases:

$$\begin{array}{llll} g = 3, & \hat{d} = 8, & h^{1,0}(X) = 1, & h^0(\hat{L}) = 6, 7 \\ g = 4, & \hat{d} = 12, & h^{1,0}(X) = 1, & h^0(\hat{L}) = 9, 10 \\ g = 5, & \hat{d} = 16, & h^{1,0}(X) = 1, & h^0(\hat{L}) = 12, 13 \\ g = 5, & \hat{d} = 8, & h^{1,0}(X) = 2, & h^0(\hat{L}) = 5, 6 \\ g = 6, & \hat{d} = 20, & h^{1,0}(X) = 1, & h^0(\hat{L}) = 15, 16 \\ g = 6, & \hat{d} = 12, & h^{1,0}(X) = 2, & h^0(\hat{L}) = 6, 7, 8. \end{array}$$

(1.1) THEOREM. *There exists a surface $\hat{X}$ which is a $\mathbf{P}^1$ bundle over an elliptic curve ϵ with $e = -1$ and on which there is a very ample line bundle $\hat{L}$ with $g(\hat{L}) = 3$, $\hat{d} = 8$ and $h^0(\hat{L}) = 6$. Such a line bundle is gotten by $\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}$.*

To prove this theorem we need the following lemma.

(1.2) LEMMA. *Let $\hat{X}$ be as in the above theorem. Then $\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}$ satisfies the hypothesis in the above theorem.*

Proof. By Proposition (0.7) if we write

$$\zeta_E^2 \otimes \mathscr{L} \equiv K_X \otimes \zeta_E^4$$

then we need to show that:

(1) ζ_E^4 is ample

(2) ζ_E^4 is spanned

(3) $h^0(\zeta_E^4) \geqq 7$

(4) $\zeta_E^4 \cdot \zeta_E^4 \geqq 10$

(5) ζ_E^4 is 3-connected.

First of all

$$\zeta_E^4 \cdot \zeta_E^4 = 4\zeta \cdot 4\zeta = 16.$$

Hence (4) is satisfied. To see that $h^0(\zeta_E^4) \geqq 7$, we use Proposition (0.5.7). Therefore (3) is also satisfied. To show that (2) is true, consider the long cohomology sequence of

$$0 \to \zeta_E^4 \otimes [f]^{-1} \to \zeta_E^4 \to \zeta_{E|_f}^4 \to 0,$$

where f is a generic fibre of $\hat{X}$. $\zeta_{E_f}^4$ is spanned since it is a degree four line bundle on $\mathbf{P}^1$.

$$H^1(\zeta_E^4 \otimes [f]^{-1}) = 0$$

by the Kodaira Vanishing Theorem since

$$\zeta_E^4 \otimes [f]^{-1} \equiv (\zeta_E^{-2} \otimes \mathscr{L}) \otimes (\zeta_E^6 \otimes [f]^{-1} \otimes \mathscr{L}^{-1}) \equiv K_{\hat{X}} \otimes A$$

where

$$A \equiv \zeta_E^6 \otimes [f]^{-1} \otimes \mathscr{L}^{-1}$$

is ample by Proposition (0.5.5). Since f is a generic fiber and

$$H^0(\zeta_E^4) \to H^0(\zeta_{E|_f}^4) \to 0$$

it follows that ζ_E^4 is spanned. So (2) is also true. Since (1) follows immediately by Proposition (0.5.5), it remains only to show (5) i.e., that ζ_E^4 is 3-connected. To prove that ζ_E^4 is 3-connected we have to show that if $D \in |\zeta_E^4|$, then for every splitting $D = D_1 + D_2$ with D_1 and D_2 effective, the inequality $D_1 \cdot D_2 \geqq 3$ holds. Since D is ample $D_1 \cdot D_2 \geqq 1$ and

$$D \cdot D_i = D_1 \cdot D_i + D_2 \cdot D_i > 0 \quad \text{for } i = 1, 2.$$

Since $D \approx 4\zeta_E$ it follows that 4 divides $D \cdot D_1$ and also $D \cdot D_2$.

Assume $D_1 \cdot D_2 = 1$. Then by the above

$$D_i \cdot D_i \geqq 3 \quad \text{for } i = 1, 2.$$

By the Hodge Indew Theorem we have the contradiction

$$9 \leqq (D_1 \cdot D_1) \cdot (D_2 \cdot D_2) \leqq (D_1 \cdot D_2)^2 = 1.$$

Similar reasoning and the fact that $D \cdot D = 16$ take care of the case $D_1 \cdot D_2 = 2$.

At this point, in order to prove Theorem (1.1), it remains to prove that $\hat{d} = 8$, $g(\hat{L}) = 3$ and $h^0(\hat{L}) = 6$.

$$\hat{d} = (2\zeta_E + \mathscr{L}) \cdot (2\zeta_E + \mathscr{L}) = 4 + 4\zeta_E \cdot \mathscr{L} + \mathscr{L} \cdot \mathscr{L} = 8$$

$$2g(L) - 2 = 2g(\zeta_E^2 \otimes \mathscr{L}) - 2$$

$$= (2\zeta_E + \mathscr{L}) \cdot (K_{\hat{X}} + 2\zeta_E + \mathscr{L})$$

$$= (2\zeta_E + \mathscr{L}) \cdot (-2\zeta_E + \mathscr{L} + 2\zeta_E + \mathscr{L}) = 4$$

hence

$$g(\zeta_E^2 \otimes \mathscr{L}) = 3.$$

By Proposition (0.5.7) we have

$$h^0(\zeta_E^2 \otimes \mathscr{L}) = 6,$$

thus the proof of Theorem (1.1) is now complete.

Remark. The example in the above theorem is the only one with $\hat{L}$ very ample and with the given numerical invariants. In fact let

$$\hat{L} = \zeta_E^a \otimes \mathscr{L}^b.$$

By [**12**, Proposition (2.1)] and (0.5.3), $a = 2$, $b = 2 + e$. By (0.5.2), $e \geqq -1$. Using Proposition (0.5.4) and (0.5.5), we have that

$$\hat{L} = \zeta_E^2 \otimes \mathscr{L}^{e+2}$$

is ample only for $e = -1, 0, 1$. Moreover, since

$$\deg(\zeta_E^2 \otimes \mathscr{L}^{e+2}|_\epsilon) \leqq 2 \quad \text{for } e = 0, 1,$$

it follows that for $e = 0, 1$, $\zeta_E^2 \otimes \mathscr{L}^{e+2}$ cannot be very ample.

(1.3) THEOREM. *There exist surfaces $\hat{X}$ which are $\mathbf{P}^1$ bundles over an elliptic curve ϵ and such that there are very ample line bundles $\hat{L}$ on them with $g(\hat{L}) = 4$, $\hat{d} = 12$ and $h^0(\hat{L}) = 9$. Examples are*:

(1) *$\hat{X}$ is $\mathbf{P}^1 \times \epsilon$ and $\hat{L} \equiv \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(3)$,*

(2) *$\hat{X} = \mathscr{P}(E)$ with $e = -1$ and $\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^2$,*

(3) *$\hat{X} = \mathscr{P}(E)$ with $e = 0$ and $\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^3$.*

Proof. Since $\hat{d} = 12$ by the long cohomology sequence associated to the short exact sequence

$$(1.3.1) \quad 0 \to \mathcal{O}_{\hat{X}} \to \hat{L} \to \hat{L}|_{\hat{C}} \to 0$$

we obtain $h^0(\hat{L}) = 9, 10$. It is known, see [**5**], that the very ample line bundles on $\mathbf{P}^1 \times \epsilon$ are of the type $\mathcal{O}_{\mathbf{P}^1}(q) \otimes \mathcal{O}_\epsilon(p)$ with $q \geqq 1$ and $p \geqq 3$ and that

$$K_{\mathbf{P}^1 \times \epsilon} \equiv \epsilon^{-2}.$$

Moreover, using the adjunction formula,

$$g(\mathcal{O}_{\mathbf{P}^1}(q) \otimes \mathcal{O}_\epsilon(p)) = pq - p + 1.$$

Therefore if $q = 2$ and $p = 3$, then

$$g(\mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(3)) = 4.$$

Furthermore

$$h^0(\mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(3)) = 9$$

since by Kunneth theorem

$$h^0(\mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(3)) = h^0(\mathcal{O}_{\mathbf{P}^1}(2)) \cdot h^0(\mathcal{O}_\epsilon(3)).$$

Now we will give another example such that $h^0(\hat{L})$ is again nine. To do this consider $\hat{X}$ given by $\mathcal{P}(E)$ with $e = -1$ and $\hat{L} \equiv \zeta_E^a \otimes \mathcal{L}^b$. If $g = 4$, then, by the adjunction formula, it follows that

(1.3.2) $(a - 1) \cdot (a + 2b) = 6.$

Moreover, since $\hat{d} = 12$,

(1.3.3) $a \cdot (a + 2b) = 12.$

Dividing (1.3.3) by (1.3.2), we obtain

$$\frac{a}{a-1} = 2$$

i.e., $a = 2$. Hence $b = 2$ and $\hat{L} \equiv \zeta_E^2 \otimes \mathcal{L}^2$. To show that $\zeta_E^2 \otimes \mathcal{L}^2$ is very ample, consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \zeta_E \rightarrow \zeta_{E|_\epsilon} \rightarrow 0$$

and tensor it with $\zeta_E \otimes \mathcal{L}^2$. We get

$$0 \rightarrow \zeta_E \otimes \mathcal{L}^2 \rightarrow \zeta_E^2 \otimes \mathcal{L}^2 \rightarrow \zeta_E^2 \otimes \mathcal{L}^2|_\epsilon \rightarrow 0.$$

Applying Proposition (0.5.6) it follows that $\zeta_E \otimes \mathcal{L}^2$ is very ample. Furthermore $\zeta_E^2 \otimes \mathcal{L}^2|_\epsilon$ is a very ample line bundle on the elliptic curve ϵ. Then, by Proposition (0.4), it follows that $\zeta_E^2 \otimes \mathcal{L}^2$ is very ample. By Proposition (0.5.7) it follows that

$$h^0(\zeta_E^2 \otimes \mathcal{L}^2) = 9.$$

To complete the proof of the theorem we have to show that if $\hat{X} = \mathcal{P}(E)$ with $e = 0$, then

$$\hat{L} \equiv \zeta_E^2 \otimes \mathcal{L}^3.$$

By (0.5.2)

$$12 = -a^2e + 2ab$$

and

$$6 = -a^2 e + ae + 2ab - 2b.$$

So $a = 2$ and $b = e + 3$. If $e = -1$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^2$$

which gives the example which we have already studied. If $e = 0$, then

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^3$$

which is very ample by Proposition (0.4). Again by Proposition (0.5.7) we have

$$h^0(\zeta_E^2 \otimes \mathscr{L}^3) = 9.$$

If $e > 0$ then

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^{3+e}$$

but

$$\deg(\zeta_E^2 \otimes \mathscr{L}^{3+e}|_\epsilon) = 3 - e < 3.$$

Then by the sequence

$$0 \to \zeta_E \otimes \mathscr{L}^{3+e} \to \zeta_E^2 \otimes \mathscr{L}^{3+e} \to \zeta_E^2 \otimes \mathscr{L}^{3+e}|_\epsilon \to 0$$

we have that $\zeta_E^2 \otimes \mathscr{L}^{3+e}$ is not very ample.

(1.4) THEOREM. *Let $\hat{X}$ be a $\mathbf{P}^1$-bundle over an elliptic curve ϵ and $\hat{L}$ a very ample line bundle on it. Then*

$$(\hat{X}, \hat{L}) = (\mathbf{P}^1 \times \epsilon, \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(4)) \quad \textit{and}$$

$$(\hat{X}, \hat{L}) = (\mathbf{P}^1 \times \epsilon, \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(5))$$

give examples respectively for $g = 5$, $\hat{d} = 16$, $h^0(\hat{L}) = 12$ and $g = 6$, $\hat{d} = 20$, $h^0(\hat{L}) = 15$.

Proof. Proceed as in the previous theorem.

(1.5) THEOREM. *Let $\hat{X}$ be a $\mathbf{P}^1$-bundle over an elliptic curve ϵ and $\hat{L}$ a very ample line bundle on it. Then for $g = 5$, $\hat{d} = 16$, $h^0(\hat{L}) = 12$, examples are given by*

(i) $(\hat{X}, \hat{L}) = (\mathscr{P}(E), \zeta_E^2 \otimes \mathscr{L}^3)$ *with* $e = -1$

(ii) $(\hat{X}, \hat{L}) = (\mathscr{P}(E), \zeta_E^2 \otimes \mathscr{L}^4)$ *with* $e = 0$ *while for* $g = 6$, $\hat{d} = 20$, $h^0(\hat{L}) = 15$, *examples are given by*

(iii) $(\hat{X}, \hat{L}) = (\mathscr{P}(E), \zeta_E^2 \otimes \mathscr{L}^4)$ *with* $e = -1$

(iv) $(\hat{X}, \hat{L}) = (\mathscr{P}(E), \zeta_E^2 \otimes \mathscr{L}^5)$ *with* $e = 0$.

Proof. If $g = 5$, by (0.5.3), we get that $a = 2$ and $b = 4 + e$. Thus if $e = -1$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^3,$$

if $e = 0$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^4$$

and if $e > 0$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^{4+e}.$$

As in the proof of Theorem (1.3) we see that $\zeta_E^2 \otimes \mathscr{L}^3$, $\zeta_E^2 \otimes \mathscr{L}^4$ are very ample and that

$$h^0(\zeta_E^2 \otimes \mathscr{L}^3) = 12, \quad h^0(\zeta_E^2 \otimes \mathscr{L}^4) = 12.$$

If $g = 6$, by (0.5.3), we get $a = 2$ and $b = 5 + e$. Thus if $e = -1$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^4,$$

if $e = 0$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^5$$

and if $e > 0$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^{5+e}.$$

As before we see that

$$h^0(\zeta_E^2 \otimes \mathscr{L}^4) = 15 \quad \text{and} \quad h^0(\zeta_E^2 \otimes \mathscr{L}^5) = 15.$$

Now we would like to prove the following theorem.

(1.6) THEOREM. *Let $X = \mathscr{P}(E)$ be a ruled surface over an elliptic curve ϵ. Then $L \equiv \zeta_E^a \otimes \mathscr{L}^b$ is very ample if*

$$a \geqq 1 \quad \text{and} \quad b \geqq \max_{1 \leqq k \leqq a} \{3 + ke\}.$$

Proof. By Proposition (0.5.6) we have that $\zeta_E \otimes \mathscr{L}^b$ is very ample for $b \geqq e + 3$ and is only spanned for $b = e + 2$. Suppose $b \geqq e + 3$ and consider the long cohomology sequence of

$$(1.6.1) \quad 0 \to \zeta_E \otimes \mathscr{L}^b \to \zeta_E^2 \otimes \mathscr{L}^b \to \zeta_E^2 \otimes \mathscr{L}^b|_\epsilon \to 0.$$

$\zeta_E \otimes \mathscr{L}^b$ is very ample by Proposition (0.5.6). Moreover

$$\deg(\zeta_E^2 \otimes \mathscr{L}^b|_\epsilon) = (2\zeta_E + b\mathscr{L}) \cdot \zeta_E = -2e + b$$

and since the degree of a very ample line bundle on an elliptic curve has to be at least three, we have that, for $b \geqq 3 + 2e$, the line bundle $\zeta_E^2 \otimes \mathscr{L}^b|_\epsilon$ is very ample. Now using the long cohomology sequence of

$$0 \to \mathscr{L}^b \to \zeta_E \otimes \mathscr{L}^b \to \zeta_E \otimes \mathscr{L}^b|_\epsilon \to 0$$

and the fact that by the Leray spectral sequence

$$h^i(\mathscr{L}^b) = h^i(\mathscr{L}^b|_\epsilon),$$

we have that

$$H^1(\zeta_E \otimes \mathscr{L}^b) = 0.$$

Hence by Proposition (0.4), $\zeta_E^2 \otimes \mathscr{L}^b$ is very ample for

$$b \geqq \max\{e + 3, 3 + 2e\}.$$

By inductive argument, using sequences obtained by tensoring the short exact sequence (1.6.1) by powers of ζ_E, we obtain that $\zeta_E^a \otimes \mathscr{L}^b$ is very ample for $a \geqq 1$, and

$$b \geqq \max_{1 \leqq k \leqq a} \{3 + ke\}.$$

Thus if $e = -1$, $b \geqq 2$; if $e = 0$, $b \geqq 3$ and if $e > 0$ then $b \geqq 3 + ae$.

Now we study the cases in which $h^{1,0}(X) = 2$. Consider the case in which $g = 5$, $\hat{d} = 8$, $h^{1,0}(X) = 2$. Let $h^0(\hat{L}) = 5$. Since $d \geqq 7$ we have to consider the two cases in which either $X = \hat{X}$ or X is gotten by blowing up one point. If $X = \hat{X}$, by Proposition (0.2) $\hat{c}_1^2 = c_1^2 = -14$ which contradicts

$$\hat{c}_1^2 = 8 - 8h^{1,0}(\hat{X}) = -8.$$

In the other case $d = 7$ and by Proposition (0.2) $c_1^2 = -1$ which contradicts again the fact that $\hat{c}_1^2 = -8$. Thus $h^0(\hat{L}) = 6$ and $X = \hat{X}$ by Castelnuovo's Inequality and by a degree consideration. Let, as usual

$$\hat{L} \equiv \zeta_E^a \otimes \mathscr{L}^b.$$

By [**12**, Proposition (2.1)], it follows that $a = 2$ while, using (0.5.3), we obtain $b = e + 3$. Since $\hat{L}$ is an ample line bundle, applying Proposition (0.5.4) and Proposition (0.5.5), we get $e = -2, \dots, 1$. Moreover

$$\deg(\hat{L}|_{\zeta_E}) = 2 - e$$

which, in our case, is always less than or equal to 4. Therefore, since ζ_E is a genus two curve, $\hat{L}|_{\zeta_E}$ is not very ample and consequently $\hat{L}$ is not very ample. But this contradicts $L = \hat{L}$.

It remains to study the case:

$$g = 6,\ h^{1,0}(X) = 2,\ \hat{d} = 12,\ \hat{c}_1^2 = -8,\ h^0(\hat{L}) = 6, 7, 8.$$

Let $\hat{L} \equiv \zeta_E^a \otimes \mathscr{L}^b$. By [**12**, Proposition (2.1)] it follows $a = 2$, while using (0.5.3), we obtain $b = e + 3$. Since $\hat{L}$ is an ample line bundle, applying Proposition (0.5.4) and Proposition (0.5.5), we get $-2 \leqq e \leqq 2$. Moreover

$$\deg(\hat{L}|_{\zeta_E}) = 3 - e = 1, \ldots, 5.$$

Therefore for $e = -1, \ldots, 2$,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^{e+3}$$

is ample but not very ample, while for $e = -2$, i.e.,

$$\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}$$

is ample but we don't know if it is very ample.

$\dim \phi_{\bar{L}}(X)$	g	$h^{1,0}(X)$	$h^{2,0}(X)$	$\hat{d}$	$\hat{c}_1^2$	$h^0(\hat{L})$	$h^0(K_X \otimes L)$	Examples of $(\hat{X}, \hat{L})$ with $\hat{L}$ very ample
1	3	1	0	8	0	6	2	$(e = -1, \zeta_E^2 \otimes \mathscr{L})$
1	4	1	0	12	0	9	3	$(\mathbf{P}^1 \times \epsilon, \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(3))$ $(e = -1, \zeta_E^2 \otimes \mathscr{L}^2)$ $(e = 0, \zeta_E^2 \otimes \mathscr{L}^3)$
1	5	1	0	16	0	12	4	$(\mathbf{P}^1 \times \epsilon, \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(4))$ $(e = -1, \zeta_E^2 \otimes \mathscr{L}^3)$
1	6	1	0	20	0	15	5	$(\mathbf{P}^1 \times \epsilon, \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_\epsilon(5))$ $(e = -1, \zeta_E^2 \otimes \mathscr{L}^4)$

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