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NOTE ON THE RATIONAL POINTS OF A PFAFF CURVE

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Abstract Let $X \subset \mathbb{R}^2$ be the graph of a Pfaffian function f in the sense of Khovanskii. Suppose that X is non-algebraic. This note gives an estimate for the number of rational points on X of height less than or equal to H; the estimate is uniform in the order and degree of f.

Keywords: rational point; Pfaff curve; plane algebraic curve

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1. Introduction

In [**8**] and [**10**] I have studied the distribution of rational points on the graph X of a transcendental real analytic function f on a *compact* interval. I have shown that the number of rational points of X of *height* (see Definition 1.2 below) less than or equal to H is $O_{f,\varepsilon}(H^{\varepsilon})$ for all positive ε.

Suppose that X is the graph of a function that is analytic on a *non-compact* domain, such as $\mathbb{R}$ or $\mathbb{R}^+$. To bound the number of rational points of height less than or equal to H on X requires controlling the implied constant in the above estimate over the enlarging intervals $[-H, H]$.

In the estimate in [**10**], the implied constant depends on a bound for the number of solutions of an algebraic equation in $P(x, f)$, where $P \in \mathbb{R}[x, y]$ is a polynomial (of degree depending on ε), as well as a bound for the number of zeros of derivatives of f (of order depending on ε). In general, these quantities may not behave at all well over different intervals.

However, these numbers are globally bounded for the so-called *Pfaffian functions* (see [**3**, **6**] and also Definition 1.1 below), indeed they are bounded uniformly in terms of the *order* and *degree* of the function (see Definition 1.1). For this class of functions, a uniform estimate on the number of rational points of bounded height may be obtained by adapting the methods of [**2**, **8**, **10**].

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Definition 1.1 (see Definition 2.1 in [3]). Let $U \subset \mathbb{R}^n$ be an open domain. A *Pfaffian chain* of order $r \geqslant 0$ and degree $\alpha \geqslant 1$ in U is a sequence of real analytic functions $f_1, \dots, f_r$ in U satisfying differential equations

$$\mathrm{d}f_j = \sum_{i=1}^{n} g_{ij}(\boldsymbol{x}, f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_j(\boldsymbol{x}))\,\mathrm{d}x_i$$

for $j = 1, \dots, r$, where $\boldsymbol{x} = (x_1, \dots, x_n)$ and $g_{ij} \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_r]$ of degree less than or equal to α. A function f on U is called a *Pfaffian function* of order r and degree (α, β) if $f(\boldsymbol{x}) = P(\boldsymbol{x}, f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x}))$, where P is a polynomial of degree at most $\beta \geqslant 1$.

The usual elementary functions e^x, $\log x$ (but not $\sin x$ on all $\mathbb{R}$), algebraic functions, combinations and compositions of these are Pfaffian functions (see [**3**, **6**]). In this paper n is always equal to 1, so $\boldsymbol{x} = x$. A *Pfaff curve* X is the graph of a Pfaffian function f on some connected subset of its domain. The order and degree of X will be taken to be the order and degree of f.

Definition 1.2. For a point $P = (a_1/b_1, a_2/b_2, \dots, a_n/b_n) \in \mathbb{Q}^n$, where $a_j, b_j \in \mathbb{Z}$, $b_j \geqslant 1$ and $(a_j, b_j) = 1$ for all $j = 1, 2, \dots, n$, define the *height* $H(P) = \max\{|a_j|, b_j\}$. Note that this is not the projective height. If $X \subset \mathbb{R}^n$ let $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$ and let $X(\mathbb{Q}, H)$ be the subset of points P with $H(P) \leqslant H$. Finally, put

$$N(X, H) = \#X(\mathbb{Q}, H) = \#\{P \in X(\mathbb{Q}),\ H(P) \leqslant H\}.$$

Theorem 1.3. *There is an explicit function $c(r, \alpha, \beta)$ with the following property. Suppose X is a non-algebraic Pfaff curve of order r and degree (α, β). Let $H \geqslant c(r, \alpha, \beta)$. Then*

$$N(X, H) \leqslant \exp(5\sqrt{\log H}).$$

Now, in certain cases where the polynomials defining the chain have rational (or algebraic) coefficients, results in transcendence theory show that the number of *algebraic* points of X is *finite*, indeed explicitly bounded (see, for example, [**7**, **11**]). On the other hand, the example $X = \{(x, y) : y = 2^x,\ x \in \mathbb{R}\}$ shows that the set $X(\mathbb{Q})$ is not finite in general. For many X, e.g. the graph of $y = \mathrm{e}^{\mathrm{e}^x}$, finiteness is unknown.

For the example $X = \{(x, y) : y = 2^x,\ x \in \mathbb{R}\}$, $N(X, H) = O(\log H)$ of course. I know of no examples in which the growth of $N(X, H)$ is faster than this, so the above bound might be very far from the truth. Note, however, that elementary considerations do not suffice to establish better bounds on $N(X, H)$ for, for example, $X = \{(x, y),\ y = \log\log(\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^x)\}$, for which finiteness of $X(\mathbb{Q})$ is presumably expected.

The methods herein are also applicable to algebraic curves: indeed the fact that Pfaffian functions have finiteness properties analogous to algebraic functions was the impetus for applying those methods to them. Since algebraic functions are Pfaffian [**6**], it is appropriate to record here the following improvement to the result obtained in [**10**].

Theorem 1.4. *Let $b, c \geqslant 2$ be integers and let $H \geqslant 3$. Let $F(x, y) \in \mathbb{R}[x, y]$ be irreducible of bidegree (b, c). Let $d = \max(b, c)$ and let $X = \{(x, y) \in \mathbb{R}^2,\ F(x, y) = 0\}$. Then*

$$N(X, H) \leqslant (6d)^{10} 4^d H^{2/d} (\log H)^5.$$

The improvement over the result of [**10**] is that the exponent of $\log H$ is here independent of d; the exponent $2/d$ of H is best possible. I refer to [**10**] for discussion of related results [**1**, **5**].

2. The main lemma

Let $M = \{x^h y^k : (h,k) \in J\}$ be a finite set of monomials in the indeterminates x, y. Put

$$D = \#M, \qquad R = \sum_{(h,k)\in J} (h+k), \qquad s = \max_{(h,k)\in J}(h), \qquad t = \max_{(h,k)\in J}(k),$$

$$S = D(s+t), \qquad \rho = \frac{2R}{D(D-1)}, \qquad \sigma = \frac{2S}{D(D-1)}, \qquad C = (D!D^R)^{2/D(D-1)} + 1.$$

Note that $S \geqslant R$ for any M. If Y is a plane algebraic curve defined by $G(x,y) = 0$, say Y is *defined in* M if all the monomials appearing in G belong to M.

Lemma 2.1. *Let M be a set of monomials with $D \geqslant 2$ and $S \geqslant 2R$. Let $H \geqslant 1$, $L \geqslant 1/H^2$ and let I be a closed interval of length less than or equal to L. Let $f \in C^D(I)$ with $|f'| \leqslant 1$ and $f^{(j)}$ either non-vanishing in the interior of I or identically zero for $j = 1, 2, \ldots, D$. Let X be the graph of $y = f(x)$ on I. Then $X(\mathbb{Q}, H)$ is contained in the union of at most*

$$(4CD4^{1/\rho} + 2)L^\rho H^\sigma$$

real algebraic curves defined in M.

Proof. Fix M, H. If f is a function satisfying the hypotheses on some interval I, and X is the graph of f on I, then the set $X(\mathbb{Q}, H)$ is contained in some minimal number $G(f,I)$ of algebraic curves of degree less than or equal to d; let $G(L)$ be the maximum of $G(f,I)$ over all intervals and functions satisfying the hypotheses.

Now suppose that f is such a function on an interval $I = [a,b]$, and $A \geqslant 1$. An equation $f^{(2)}(x) = \pm 2AL^{-1}$ has at most one solution in the interior I, unless it is satisfied identically. Suppose c is a solution. Since $f^{(2)}$, $f^{(3)}$ are one-signed throughout I, it follows that $|f^{(2)}(x)| \leqslant 2A^{2/(D-1)}L^{-1}$ in either $[a,c]$ or $[c,b]$, and $|f^{(2)}(x)| \geqslant 2A^{2/(D-1)}L^{-1}$ in (respectively) either $[b,c]$ or $[a,c]$. Now an interval with the latter condition has length less than or equal to $2A^{1/(D-1)}$ by [**10**, 2.6] (or [**2**, Lemma 7]) applied with $A = A^{2/(D-1)}$.

Continuing to split the interval at points where

$$f^{(\kappa)} = \kappa! A^{\kappa/(D-1)} L^{1-\kappa}, \quad \kappa = 2, 3, \ldots, D,$$

yields a (possibly empty) subinterval $[s,t]$ in which $|f^{(\kappa)}| \leqslant \kappa! A^{\kappa/(D-1)} L^{1-\kappa}$ for all $\kappa = 1, 2, 3, \ldots, D$, while the intervals $[a,s]$, $[t,b]$ comprise fewer than or equal to D subintervals of length less than or equal to $2A^{-1/(D-1)}L$ (by [**10**, 2.6] (or [**2**, Lemma 7]) applied with $A = A^{\kappa/(D-1)}$), and so have length at most $2DA^{-1/(D-1)}L$. (If $[s,t]$ is empty, take $s = t = b$.)

On $[s,t]$, the points of height less than or equal to H lie on at most $CA^{1/(D-1)}H^\sigma L^\rho$ curves in M by [**10**, 2.4]. Therefore, the function $G(L)$ satisfies the recurrence

$$G(L) \leqslant CA^{1/(D-1)}H^\sigma L^\rho + 2G(\lambda L)$$

when $L \geqslant 1/H^2$, where $\lambda = 2DA^{-1/(D-1)}$. Thus, provided $\lambda^{n-1}L \geqslant 1/H^2$,

$$G(L) \leqslant CA^{1/(D-1)}H^\sigma L^\rho(1 + 2\lambda^\rho + \cdots + (2\lambda^\rho)^{n-1}) + 2^n G(\lambda^n L).$$

Choose A such that $2\lambda^\rho = \frac{1}{2}$, that is, $A^{1/(D-1)} = 2D4^{1/\rho}$ (so $A \geqslant 1$), and choose n such that

$$\frac{\lambda}{LH^2} \leqslant \lambda^n < \frac{1}{LH^2}.$$

Then $G(\lambda^n L) \leqslant 1$, while

$$2^n = \lambda^{-n\rho/2} \leqslant \left(\frac{LH^2}{\lambda}\right)^{\rho/2} = 2(LH^2)^{\rho/2} \leqslant 2L^\rho H^\sigma.$$

Therefore, $G(L) \leqslant (4CD4^{1/\rho} + 2)H^\rho L^\sigma$ as required. □

3. Pfaff curves

Since a Pfaffian function of order $r = 0$ is a polynomial, to which Theorem 1.3 is inapplicable, it is convenient now to assume that $r \geqslant 1$.

Proposition 3.1. *Let $f_1, \dots, f_r$ be a Pfaffian chain of order $r \geqslant 1$ and degree α on an open domain $U \subset \mathbb{R}$, and let f be a Pfaffian function on U having this chain and degree (α, β).*

(a) *Let $k \in \mathbb{N}$. Then $f^{(k)}$ is a Pfaffian function with the same chain as f (so of order r) and degree $(\alpha, \beta + k(\alpha - 1))$.*

(b) *Let $P(x, y)$ be a polynomial of degree d. Suppose that f is not algebraic. Then the equation $P(x, f(x)) = 0$ has at most*

$$2^{1+r(r-1)/2} d\beta(r\alpha + d\beta)^r$$

solutions.

(c) *Let $V \subset U$ be an open set on which $f' \neq 0$ and $k \geqslant 1$. Then on V there is an inverse function g of f, and the number of zeros of $g^{(k)}$ on V is at most*

$$2^{1+r(r-1)/2}((k-1)(\beta + k(\alpha - 1)))(r\alpha + ((k-1)(\beta + k(\alpha - 1))))^r.$$

Proof. Part (a) follows from [**3**, 2.5].

For part (b), observe that $P(x, f(x))$ is a Pfaffian function of order $r \geqslant 1$ and degree $(\alpha, d\beta)$. Since f is not algebraic, all the solutions are isolated and the result is in [**3**, 3.3].

Part (c). By differentiating the relation $g(f(x)) = x$ and simple induction, for $k \geqslant 1$,

$$g^{(k)}(y) = \frac{Q_k(f^{(1)}, f^{(2)}, \dots, f^{(k)})}{(f'(x))^{2k-1}},$$

where $Q_k(z_1, z_2, \dots, z_k)$ is a polynomial of degree $\gamma_k = k - 1$. Since $f^{(j)}$ are Pfaffian functions with the same chain, the function $Q_k(f^{(1)}, f^{(2)}, \dots, f^{(k)})$ is a Pfaffian function of order r and degree $(\alpha, \gamma_k(\beta + k(\alpha - 1)))$. The statement now follows from (b). □

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Proof of Theorem 1.3. Let $d \geqslant 2$ and let $M = M(d)$ be the set of monomials of degree d in x, y. Then, elementarily (see [**10**]),

$$D = \frac{d(d-1)}{2}, \qquad \rho = \frac{8}{3(d+3)}, \qquad \sigma = 3\rho, \qquad C \leqslant 6.$$

Subdivide the connected domain U into at most

$$2^{2+r(r-1)/2}(\beta+\alpha-1)(r\alpha+\beta+\alpha-1)+1 \leqslant 2^{1+r(r-1)/2}((r+1)(\alpha+\beta))^{r+1}$$

intervals on which $f' \leqslant -1$, $-1 \leqslant f' \leqslant 1$ or $f' \geqslant 1$, and then divide further into subintervals on which the inverse g has non-vanishing derivatives up to order D in the first and third cases, or f has non-vanishing derivatives up to order D in the second case. The total number of intervals is at most

$$2^{2+r(r-1)/2}((r+1)(\alpha+\beta))^{r+1}D^2 2^{r(r-1)}(\beta+D(\alpha-1))(r+D(\beta+D(\alpha-1)))^r.$$

Intersecting with the interval $[-H, H]$ of the appropriate axis, these intervals are of length less than or equal to $2H$. By Lemma 2.1, in each interval the points of $X(\mathbb{Q}, H)$ lie on at most

$$(24D4^{1/\rho}+2)(2H)^{\rho}H^{3\rho} \leqslant 6d^2 4^{1/\rho}2^{\rho}$$

real algebraic curves of degree d; the number of points of X on a curve of degree d is at most

$$2^{1+r(r-1)/2}d\beta(r\alpha+d\beta)^r.$$

Combining these estimates yields

$$N(X, H) \leqslant c'(r, \alpha, \beta, d, D)4^{3(d+3)/8}H^{32/(3(d+3))}.$$

Let $t = \frac{3}{8}(d+3)$. Choose d so that t is as near as possible to (and so within $\frac{1}{2}$ of) $\sqrt{4\log H/\log 4}$. Then $4\sqrt{\log 4} < 5$, and noting that d, D appear polynomially in c' completes the proof. □

Remark 3.2. Note that the constant 5 in Theorem 1.3 can be improved by further optimizing the proof. However, a bound of the shape $\exp(c\sqrt{\log H})$ seems to be the best obtainable by the present method.

Remark 3.3. A result can be formulated for any real analytic (or even smooth) function f with suitable finiteness properties (zeros of derivatives, derivatives of the inverse, and algebraic relations). An example of such a function that is not Pfaffian is exhibited in [**4**]. (Indeed, the given example $e^x + \sin x$ does not belong to any *o-minimal structure* (see [**4**]).)

Remark 3.4. I expect that a similar result would hold in higher dimensions for Pfaff manifolds: that is, a uniform (in 'complexity') H^{ε} bound for rational points that do not lie on some semi-algebraic subset of positive dimension (cf. the conjectures for subanalytic sets made in [**9, 10**]). A similar result should hold for sets definable in an o-minimal structure.

4. Algebraic curves

For integers $\beta, \gamma \geqslant 2$ let

$$M(\beta,\gamma) = \{x^h y^k : 0 \leqslant h \leqslant \beta - 1,\ 0 \leqslant k \leqslant \gamma - 1\}.$$

Then [**10**], for $M = M(\beta,\gamma)$,

$$D = \beta\gamma, \qquad R = \tfrac{1}{2}D(\gamma+\beta-2), \qquad S = D(\beta-1+\gamma-1) = 2R, \qquad C \leqslant 2D,$$

and (elementarily)

$$\max\left(\frac{1}{\beta}, \frac{1}{\gamma}\right) \leqslant \rho \leqslant \frac{1}{\beta} + \frac{1}{\gamma}.$$

Proof of Theorem 1.4. The proof adapts the proof of [**10**, 1.4] using Lemma 2.1 instead of [**10**, 4.2].

Consider first a C^∞ function f on a subinterval of $[-1,1]$ with $|f'| \leqslant 1$, with $f^{(j)}$ either non-vanishing or identically vanishing for $j = 0, \ldots, D$. Suppose that f satisfies an irreducible algebraic relation of degree (b,c), $d = \max(b,c)$. If $d = b$, take $M = (d,\delta)$ with $\delta \geqslant d$; if $d = c$, take $M = M(\delta, d)$ with $\delta \geqslant d$. Then, by Lemma 2.1, $X(\mathbb{Q}, H)$ is contained in the union of at most

$$10d^2\delta^2 4^d 2^\rho H^{2\rho} \leqslant 20d^2\delta^2 4^d H^{2/d+2/\delta}$$

curves defined in M. The intersections are proper, X is of degree less than or equal to $b + c \leqslant 2d$, the curves in M are of degree less than or equal to 2δ, so

$$N(X,H) \leqslant 80d^3\delta^3 4^d H^{2/d+2/\delta}.$$

Next consider an algebraic curve X defined by $F(x,y) = 0$ in the box $B = [-1,1]^2$, where F is irreducible of bidegree (b,c) and $d = \max(b,c)$. Then X has at most $2d(2d-1)$ singular points, and at most $4d(d-1)$ points with slope ± 1. So $X \cap B$ consists of at most $20d^3$ graphs of C^∞ functions f with slope $|f'| \leqslant 1$ relative to one of the coordinate axes.

For each such function, the domain can be divided into at most $8d^2D^2$ subintervals (see [**2**, Lemmas 5 and 6]) in which $f^{(j)}$ is non-vanishing or identically zero, $j = 1, 2, \ldots, D$. So

$$N(X,H) \leqslant 25 \cdot 2^{10} d^{10} \delta^5 4^d H^{2/d+2/\delta}.$$

Finally, let $F(x,y)$ be of bidegree (b,c), $d = \max(b,c)$, $X = \{(x,y) \in \mathbb{R}^2 : F(x,y) = 0\}$. Let $P = (x,y) \in X(\mathbb{Q})$ with $H(P) \leqslant H$. Then one of the following holds:

(i) $|x|, |y| \leqslant 1$,

(ii) $|x| \leqslant 1$, $|y| > 1$,

(iii) $|x| > 1$, $|y| \leqslant 1$,

(iv) $|x| > 1$, $|y| > 1$.

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In case (i), P lies in the box $[-1,1]^2 \subset \mathbb{R}^2$. In case (ii), the point $Q = (x, 1/y)$ is on the curve $Y : y^c F(x, 1/y) = 0$. This curve is also irreducible and of bidegree (b, c) (because F must have a term independent of y). The point Q is then in the box $[-1,1]^2$ and has $H(Q) \leqslant H$. Likewise, in cases (iii) and (iv) the corresponding points $R = (1/x, y)$, $S = (1/x, 1/y)$ lie on irreducible curves $x^b F(1/x, y) = 0$, $x^b y^c F(1/x, 1/y) = 0$ of bidegree (b, c) in the box $[-1,1]^2$ and have height less than or equal to H.

Therefore, up to a factor 4, it suffices to consider the points of F inside the box $[-1,1]^2$, so

$$N(X, H) \leqslant 100(2d)^{10} 4^d \delta^5 H^{2/d+2/\delta}.$$

Take δ to be the least integer exceeding $\log H$. Then, provided $H \geqslant \mathrm{e}^d$ (so that $\delta \geqslant d$),

$$N(X, H) \leqslant 100\mathrm{e}^2 2^{15} d^{10} (\log H)^5 4^d H^{2/d}.$$

However, for $\log H \leqslant d$ the bound is easily seen to hold. □

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