Can. J. Math., Vol. XXXII, No. 2, 1980, pp. 385–394

THE DUALITY OF DISTRIBUTIVE CONTINUOUS LATTICES

B. BANASCHEWSKI

Various aspects of the prime spectrum of a distributive continuous lattice have been discussed extensively in Hofmann-Lawson [**7**]. This note presents a perhaps optimally direct and self-contained proof of one of the central results in [**7**] (Theorem 9.6), the duality between distributive continuous lattices and locally compact sober spaces, and then shows how the familiar dualities of complete atomic Boolean algebras and bounded distributive lattices derive from it, as well as a new duality for all continuous lattices. As a biproduct, we also obtain a characterization of the topologies of compact Hausdorff spaces.

Our approach, somewhat differently from [**7**], takes the open prime filters rather than the prime elements as the points of the dual space. This appears to have conceptual advantages since filters enter the discussion naturally, besides being a well-established tool in many similar situations. Moreover, it helps to emphasize the strong analogy between the present setting and the representation of (ordinary) distributive lattices by rings of sets.

We recall the relevant basic concepts, using standard lattice and partial order terminology as in [**3**]. For the categorical notions appearing later on, see [**10**].

A *continuous lattice* ([**11**]) is a complete partially ordered set L in which

(*) $\quad x = \bigvee \bigwedge U(x \in U,\ U\ d\text{-open end}) \qquad (\text{all } x \in L)$

where (i) an *end* is a subset E of L such that $x \in E$ whenever $x \geqq y$ for some $y \in E$, and (ii) $W \subseteq L$ is called *d-open* if, and only if, for any up-directed $D \subseteq L$, $\bigvee D \in W$ implies $D \cap W \neq \emptyset$. The d-open ends of any complete lattice L constitute a T_0-topology $\mathfrak{D}L$ on L ([**11**]), and continuity of L means $\mathfrak{D}L$ is sufficiently sizeable to ensure (*). In the following, for any continuous lattice L, topological notions always refer to $\mathfrak{D}L$, and $\mathfrak{D}(x)$ will be the collection of all open neighbourhoods of $x \in L$.

A *filter* $P \subseteq L$ is an end such that $x \wedge y \in P$ for any $x, y \in P$. P is called *prime* if and only if $\bigvee F \in P$ implies $F \cap P \neq \emptyset$ for any finite

Received May 10, 1978 and in revised form June 13, 1979. This paper was presented at the 2nd Workshop on Continuous Lattices at the Technische Hochschule Darmstadt, June 1978. Financial assistance from the National Science and Engineering Research Council of Canada is gratefully acknowledged.

$F \subseteq L$. Note that any such P is proper, i.e., $P \neq L$ (the case $F = \emptyset$). An *open prime* filter P is *completely prime* in the sense that $\bigvee A \in P$ implies $A \cap P \neq \emptyset$ for any $A \subseteq L$: $\bigvee A \in P$ implies $\bigvee F \in P$ for some finite $F \subseteq A$ since P is open, and then $F \cap P \neq \emptyset$ by primeness. Conversely, of course, any completely prime filter in L is open and prime; we shall use the two terms interchangibly. The *prime spectrum* ΣL of L is the space whose points are the open prime filters $P \subseteq L$, and whose open sets are the sets $\Sigma_a = \{P | a \in P \in \Sigma L\}$ $(a \in L)$ (which clearly form a topology).

Of particular interest here are the distributive continuous lattices; since a continuous lattice always satisfies the distribution law

$$a \wedge \bigvee B = \bigvee (a \wedge b)\ (b \in B)$$

for up-directed sets B ([**4**]), distributivity implies this law for arbitrary B, and thus distributive continuous lattices are, in particular, complete Heyting algebras, or what are called *frames* ([**5**]) or *local lattices* ([**2**], [**8**]). For the present purposes, the relevant maps between these are the $\bigvee\wedge$-homomorphisms, preserving arbitrary joins and finite meets; **ContF** will be the category of continuous frames (= distributive continuous lattices) with these maps. Our motivation for this terminology lies in the fact that the $\bigvee\wedge$-homomorphisms are exactly the significant maps for frames ([**5**]), whereas continuous lattices also have to be considered with various other classes of maps (e.g. [**4**], [**7**], [**11**]).

A topological space X is called *sober* if and only if it is T_0 and any $\bigcup$-irreducible closed subset of X is the closure of a singleton. It is easy to see the second part of the latter is equivalent to the condition that every completely prime filter in the lattice $\mathfrak{O}X$ of open sets of X is the neighbourhood filter $\mathfrak{O}(x) = \{U | x \in U \in OX\}$ of some point $x \in X$, and it is in this form that we shall use the notion of sobriety. Further, a space X is called *locally compact* (we omit the Bourbakian prefix "quasi") if and only if each point of X has arbitrarily small compact neighbourhoods ("compact" intended without separation assumption). **SobLc** will be the category of locally compact sober spaces and continuous maps. Note that, in any topological space X, the set of all open neighbourhoods of a compact set K is a d-open end in $\mathfrak{O}X$ for which the interior $\mathbf{I}K$ of K is a lower bound in $\mathfrak{O}X$. Since local compactness of X means $U = \bigcup \mathbf{I}K (K \subseteq U$ compact) for each $U \in \mathfrak{O}X$, it follows that $\mathfrak{O}X$ is a continuous lattice for any locally compact X ([**7**]). Obviously, the correspondence $X \rightsquigarrow \mathfrak{O}X$ is the object part of a contravariant functor $\mathfrak{O} : \mathbf{SobLc} \rightarrow \mathbf{ContF}$ whose effect on maps is given by taking inverse images; this is the functor which has to be shown to provide a dual equivalence.

An important aspect of a continuous lattice L is its "way below" relation $\ll$, $x \ll y$ defined to mean that, for any up-direct set $D \subseteq L$, $\bigvee D \geqq y$ implies $d \geqq x$ for some $d \in D$. Obviously, for any $U \in \mathfrak{O}(x)$,

https://doi.org/10.4153/CJM-1980-030-3 Published online by Cambridge University Press

$\bigwedge U \ll x$; moreover, for any $U \in \mathfrak{D}L$, if $x \in U$ then also $\bigwedge W \in U$ for some $W \in \mathfrak{D}(x)$ by (*), and hence for any $x \in U$ there exists a $y \ll x$ in U. Actually, this is the only property of $\ll$ needed here, in particular for our proof of the following familiar fact:

LEMMA 1. *In any continuous lattice L, any open set is the union of open filters.*

Proof. If $x \in U$ and $U \in \mathfrak{D}L$ then, by the preceding remark, there exists a sequence $x = x_1 \gg x_2 \gg \ldots$ in U, and

$$W = \{z | z \in L, z \geqq x_k \text{ for some } k\}$$

is then obviously on open filter such that $x \in W \subseteq U$.

The next lemma describes the compact subsets of the prime spectrum of a continuous lattice by what might be viewed as a size restriction on their intersections: intuitively, open means large whereas compact means small.

LEMMA 2. *For any continuous lattice L, a subset Δ of ΣL is compact if and only if the filter $F = \bigcap P(P \in \Delta)$ is open.*

Proof. If Δ is compact and $\bigvee D \in F$ for some up-directed set D then $\Delta \subseteq \Sigma_d$ $(d \in D)$, hence $\Delta \subseteq \Sigma_d$ for some $d \in D$, and thus $d \in F$. Conversely, if F is open and $\Delta \subseteq \bigcup \Sigma_a$ $(a \in A) = \Sigma_{\vee A}$ then $\bigvee A \in F$, hence $a_1 \vee \ldots \vee a_n \in F$ for some $a_i \in A$, and therefore $\Delta \subseteq \Sigma_{a_1} \cup \ldots \cup \Sigma_{a_k}$.

We now turn to the particular properties of distributive continuous lattices.

LEMMA 3. *In any distributive continuous lattice L, every open filter is an intersection of open prime filters.*

Proof. If $F \subseteq L$ is an open filter and $a \notin F$ in L, let $G \subseteq L$ be an open filter maximal such that $G \supseteq F$ and $a \notin G$. Then, for any $b, c \notin G$ such that $b \vee c \in G$, $H = \{x | x \in L, b \vee x \in G\}$ is an open filter (the latter by distributivity) such that $G \subseteq H$ and $c \in H$, hence $a \in H$ so that $a \vee b \in G$ and therefore, applying this conclusion over again, $a = a \vee a \in G$, a contradiction. This shows G is prime, and the lemma follows.

Remark. The above proof remains valid for any distributive lattice with a topology for which all maps $x \rightsquigarrow b \vee x$ are continuous. In particular, it proves the analogous lemma for (discrete) distributive lattices, a familiar step in the proof of the representation of distributive lattices by rings of sets.

In view of Lemma 1, an immediate consequence of Lemma 3 is:

https://doi.org/10.4153/CJM-1980-030-3 Published online by Cambridge University Press

COROLLARY 1. *In any distributive continuous lattice, the open prime filters separate the points.*

Another consequence of Lemma 3, in this case resulting from Lemmas 1 and 2, is:

COROLLARY 2. *For any distributive continuous lattice L, ΣL is locally compact.*

Proof. If $a \in P \in \Sigma L$, let $U \subseteq P$ be an open filter such that $a \in U$ and $c = \bigwedge U \in P$, by (*) and Lemma 1. Then, $\Delta = \{Q | U \subseteq Q \in \Sigma L\}$ is compact by Lemmas 3 and 2, and since $P \in \Sigma_c \subseteq \Delta \subseteq \Sigma_a$ this proves the assertion.

The prime spectrum ΣL of a continuous lattice L is always sober since any completely prime filter F in $\mathfrak{O}\Sigma L$ is of the form $F = \{\Sigma_a | a \in P\}$, where $P = \{a | \Sigma_a \in F\}$ is an open prime filter in L and hence F its filter of open neighbourhoods. Further, the correspondence $L \rightsquigarrow \Sigma L$ is clearly contravariantly functorial, the continuous map $\Sigma h : \Sigma M \to \Sigma L$ determined by a $\bigvee\wedge$-homomorphism $h : L \to M$ being $P \rightsquigarrow h^{-1}(P)$; thus one also has a contravariant functor $\Sigma : \mathbf{ContF} \to \mathbf{SobLc}$. Moreover, for the two composites of Σ and $\mathfrak{O}$ there are the maps

$$\epsilon_L : L \to \mathfrak{O}\Sigma L,\ \epsilon_L(x) = \Sigma_x \quad (L \in \mathbf{ContF})$$

and

$$\eta_X : X \to \Sigma\mathfrak{O}X,\ \eta_X(x) = \mathfrak{O}(x) \quad (X \in \mathbf{SobLc})$$

which are easily seen to be natural in L and X, respectively, and such that the composites

$$\Sigma L \xrightarrow{\eta_{\Sigma L}} \Sigma\mathfrak{O}\Sigma L \xrightarrow{\Sigma\epsilon_L} \Sigma L$$
$$P \rightsquigarrow \{\Sigma_a | a \in P\} \rightsquigarrow P$$

and

$$\mathfrak{O}X \xrightarrow{\epsilon_{\mathfrak{O}X}} \mathfrak{O}\Sigma\mathfrak{O}X \xrightarrow{\mathfrak{O}\eta_x} \mathfrak{O}X$$
$$U \rightsquigarrow \Sigma_U \rightsquigarrow U$$

are identities. Finally, the ϵ_L are isomorphisms exactly because the open prime filters in any $L \in \mathbf{ContF}$ separate the points, and the η_X are homeomorphisms by the familiar fact that $x \rightsquigarrow \mathfrak{O}(x)$ is an embedding into the space of all filters in $\mathfrak{O}X$ for any space X ([**1**]), and the given hypothesis that X is sober.

In all this has shown:

PROPOSITION. *The contravariant functors Σ and $\mathfrak{O}$ form an adjoint dual equivalence between the categories* **ContF** *and* **SobLc**.

The following observations establish connections between this duality and certain others.

Remark 1. The Boolean $L \in$ **ContF** are, up to isomorphism, exactly the power set lattices $\mathfrak{P}E$: For a Boolean continuous lattice L, the open sets of ΣL are the same as the closed sets, and any T_0-space of this type is easily seen to be discrete so that $\mathfrak{D}\Sigma L$ is a power set lattice; conversely, any of the latter is obviously a Boolean continuous lattice. Moreover, $L \in$ **ContF** is Boolean if and only if ΣL is discrete, and for such L the open prime filters are exactly the principal prime filters, i.e., the prime filters determined by the atoms. Since the subcategory of **SobLc** of discrete spaces is essentially the same as the category of sets this shows that the duality between **ContF** and **SobLc** extends the familiar duality of the category of complete atomic Boolean algebras and complete Boolean homomorphisms with the category of sets.

Remark 2. The full subcategory of **ContF** given by the finite $L \in$ **ContF** is just the category **FinD** of finite distributive lattices; on the other hand, the corresponding subcategory of **SobLc** is the category $\mathbf{FinT_0}$ of finite T_0-spaces. Also, $L \in$ **ContF** is finite if and only if ΣL is finite. Moreover, the category $\mathbf{FinT_0}$ is well-known to be equivalent to the category **FinPoEns** of finite partially ordered sets and order preserving maps. Hence, the duality between **ContF** and **SobLc** extends the familiar duality between **FinD** and **FinPoEns**.

Remark 3. The full duality of the category **D** of bounded distributive lattices ([**12**]) which extends the duality of **FinD** just discussed can also be located in the present setting. Recall that a distributive continuous lattice L is called *arithmetical* if and only if it is algebraic and $\wedge F$ is compact for any finite set F of compact elements. If **ArF** is the subcategory of **ContF** consisting of the arithmetical $L \in$ **ContF** and the $\bigvee \wedge$-homomorphisms between them which preserve compact elements then the covariant ideal lattice functor $\mathfrak{J}$ is an equivalence between **D** and **ArF**. Here, $\mathfrak{J}D$ is the ideal lattice of D for any $D \in$ **D** (known to be distributive again ([**3**], p. 114) and clearly algebraic) and for any homomorphism $h : D \to E$ in **D** the map $\mathfrak{J}h : \mathfrak{J}D \to \mathfrak{J}E$ associates with each ideal in D the ideal generated by its image in E. It is clear that $\mathfrak{J}h$ is a $\bigvee \wedge$-homomorphism; moreover, it obviously preserves principal ideals, and these are exactly the compact elements of $\mathfrak{J}D$. This defines, then, a functor $\mathfrak{J} : \mathbf{D} \to \mathbf{ArF}$, and that this is indeed an equivalence easily results from familiar facts about algebraic lattices. Now, the objects in the image of **ArF** with respect to $\Sigma :$ **ContL**$\to$ **SobLc** are, up to isomorphism, the $X \in$ **SobLc** for which the compact-open subsets are closed under finite intersection and constitute a basis of X, which are exactly the spaces appearing in Stone's representation theorem for bounded

https://doi.org/10.4153/CJM-1980-030-3 Published online by Cambridge University Press

distributive lattices ([**12**]), otherwise known as the *spectral* spaces ([**6**]); moreover, the maps between spectral spaces corresponding to maps in **ArF** by Σ are precisely the continuous maps for which the inverse images of compact-open sets are compact. Hence, Σ induces a dual equivalence between **ArF** and the category **Spec** of spectral spaces and the maps just described. An additional simple consideration shows that the contravariant functor $\Sigma\mathfrak{J}$: **D**$\rightarrow$**Spec** is naturally equivalent to the contravariant functor Π : **D** $\rightarrow$ **Spec** where ΠD is the space of prime filters of D with the usual filter space topology (with each completely prime filter $\mathfrak{F}$ in $\mathfrak{J}D$, associate the prime filter of all $x \in D$ such that the principal ideal $[0, x]$ belongs to $\mathfrak{F}$) as well as to the contravariant functor P : **D** $\rightarrow$ **Spec** for which $\mathrm{P}D$ is the space of prime ideals of D with the Zariski (= hull-kernel) topology. Hence the result, essentially due to Stone [**12**]: The category **D** is dually equivalent to the category **Spec** via the contravariant functor $\mathrm{P} \simeq \Sigma\mathfrak{J} \simeq \Pi$.

Remark 4. Somewhat similar to the questions dealt with in the preceding remarks is that of identifying the subcategory of **ContF** corresponding to the locally compact Hausdorff spaces, which might be of interest since the notion of Hausdorffness for frames in general seems to present some problems. In the following, let x^* be the pseudocomplement, for any $x \in L$, i.e., the largest $y \in L$ such that $x \wedge y = 0$. With this, we have:

For any $L \in$ **ContF**, ΣL is Hausdorff if and only if
$$x = \bigvee z(x \vee z^* = e) \text{ for each } x \in L.$$

Let L satisfy this condition and consider any distinct P and Q in ΣL. Then there exists, say, an $x \in P$ not in Q, and therefore a $z \in P$ such that $x \vee z^* = e$; now $x \vee z^* \in Q$, and since $x \notin Q$ one has $z^* \in Q$, so that Σ_z and Σ_{z^*} are disjoint neighbourhoods of P and Q, respectively, in ΣL, i.e., ΣL is Hausdorff. Conversely, any locally compact Hausdorff space X is regular, and therefore any $U \in \mathfrak{D}X$ is the union of all open sets V with closure $\Gamma V \subseteq U$; however, for any open set V, V^* is the complement of ΓV, and thus $\Gamma V \subseteq U$ if and only if $U \cup V^* = X$. It follows that any $L \in$ **ContF** with Hausdorff ΣL indeed satisfies the stated condition. As a further use of this condition, we add a characterization of the compact Hausdorff topologies which naturally arises here:

A frame L is isomorphic to the topology of a compact Hausdorff space if and only if

$$\text{(i) } e \text{ is compact, and (ii) } x = \bigvee z(x \vee z^* = e) \text{ for each } x \in L.$$

By what has been said already, one direction of this is obvious, and for the converse it is enough to show that (i) and (ii) imply L is continuous. For this, consider $A_z = \{y | \, y \in L, y \vee z^* = e\}$. Clearly this is an end,

https://doi.org/10.4153/CJM-1980-030-3 Published online by Cambridge University Press

and if $(\bigvee D) \vee z^* = e$ for some updirected $D \subseteq L$ then $d \vee z^* = e$ for some $d \in D$ by (i), i.e., A_z is d-open. Moreover, $e = y \vee z^*$ implies $z = y \wedge z \leqq y$ and therefore $z \leqq \bigwedge A_z$. Now, (ii) shows L is continuous.

Finally, we have an analogous result concerning locally compact Hausdorff spaces. For any frame L let $K \subseteq L$ be the set of all $c \in L$ such that the set A_c is d-open. A combination of the above arguments then readily proves:

A frame L is isomorphic to the topology of a locally compact Hausdorff space if and only if

$$x = \bigvee z(z \vee z^* = e, z \in K) \text{ for each } x \in L.$$

In addition to the above observations regarding dualities contained in the duality between **ContF** and **SobLc**, it should be noted that the latter, in turn, is part of the contravariant adjointness between the category **F** of all frames and $\bigvee\wedge$-homomorphisms and the category **TOP** of all spaces and continuous maps of Dowker and Papert [**5**] given by taking the lattices of open sets (**TOP** $\to$ **F**) and, in our approach, the spaces of completely prime filters (**F** $\to$ **TOP**), respectively.

We now turn to the duality of all continuous lattices, or, more precisely, of the category **ContL** of continuous lattices and d-continuous (= up-directed join preserving) maps. That this appears in the present setting derives from the basic fact ([**11**]) that **ContL** is isomorphic to the category $\mathbf{InjT}_0$ of injective T_0-spaces and their continuous maps. Indeed, $\mathbf{InjT}_0$ is a full sucategory of **SobLc** since any injective T_0-space, having no proper essential extension ([**1**]), is evidently sober, and any continuous lattice L is locally compact in its topology $\mathfrak{O}L$ since any $U \in \mathfrak{O}(x)$ contains a $V \in \mathfrak{O}(x)$ such that $V \subseteq [a, \to] \subseteq U (a = \bigwedge V)$ where $[a, \to]$ is clearly compact. Hence the contravariant functor $\mathfrak{O}$: **SobLc** $\to$ **ContF** induces a duality for **ContL**. In order to make this more explicit, the image of $\mathbf{InjT}_0$ relative to $\mathfrak{O}$ has to be determined, or, dually, the $L \in$ **ContF** with injective prime spectrum have to be characterized, independently of $\mathfrak{O}$.

Definition. A continuous lattice L will be called *primal* if, and only if, it is sufficiently rich in open prime filters in the sense that

(P1) The filter join $P \vee Q$ of any open prime filters P and Q in L is an open prime filter, and

(P2) for any $a \in L$, $a = \bigvee \wedge P \ (a \in P \in \Sigma L)$.

Note that (P2) implies distributivity since it ensures that the open prime filters separate the points; thus, we have the (full) subcategory **PrimF** of **ContF** given by the primal continuous lattices.

LEMMA 4. *$L \in$ **ContF** is primal if and only if ΣL is injective.*

Proof. ($\Rightarrow$) For any continuous lattice L, an up-directed union of open prime filters is again an open prime filter. Hence, if L is primal then ΣL is closed under arbitrary filter joins by (P1), and therefore (i) the join of any neighbourhood filters of ΣL is a neighbourhood filter of ΣL. Moreover, if $a \in P \in \Sigma L$ then $\bigwedge Q \in P$ for some Q such that $a \in Q \in \Sigma L$, by (P2). Thus, (ii) for any neighbourhood Σ_a of P in ΣL there exists a neighbourhood Σ_b of P such that, for some $Q \in \Sigma_a$, every neighbourhood of Q contains Σ_b (take $b = \bigwedge Q$). Now, (i) and (ii) are precisely the conditions characterizing injectivity of T_0-spaces given in [**1**]; hence ΣL is injective.

($\Leftarrow$) Let X be any injective T_0-space, so that X is a continuous lattice in the partial order $\leqq$ for which $x \leqq y$ if and only if $\mathfrak{O}(x) \subseteq \mathfrak{O}(y)$ ([**11**]). Then

$$\mathfrak{O}(x) \vee \mathfrak{O}(y) = \{U \cap V | U \in \mathfrak{O}(x) \text{ and } V \in \mathfrak{O}(y)\}$$

is the neighbourhood filter of $x \vee y$ since $x \in U$ and $y \in V$ implies $x \vee y \in U \cap V$, and by the continuity of $\vee$ ([**11**]) this proves (P1) for $\mathfrak{O}X$ since X is sober. Further, in $\mathfrak{O}X$, $\bigwedge \mathfrak{O}(x) = \mathbf{I}[x, \rightarrow]$, the interior of the end $[x, \rightarrow]$ generated by x, and $U = \bigcup [x, \rightarrow]$ $(x \in U)$ for any $U \in \mathfrak{O}X$ by [**11**], which together show $\mathfrak{O}X$ satisfies (P2). Thus, $\mathfrak{O}X$ is primal for any injective T_0-space, and therefore any $L \in$ **ContF** is primal whenever ΣL is injective.

The discussion preceding Lemma 4 now leads to the following conclusion:

PROPOSITION. *The contravariant functors $\mathfrak{O}$ and Σ induce a dual equivalence between the categories **ContL** and **PrimF**.*

Remark. For any Hausdorff $X \in$ **SobLc**, (P1) clearly fails for $\mathfrak{O}X$, and (P2) holds if and only if X is discrete since $\bigcup \bigwedge \mathfrak{O}(x)$ $(x \in U)$ is exactly the set of isolated points in U. In particular, this shows that (P2) does not imply (P1). On the other hand, if D is any infinite distributive lattice with a smallest non-zero element a such that the sublattice of all $x \geqq a$ is Boolean then $L = \mathfrak{J}D$ satisfies (P1) but not (P2): (P1) for L means for D that the filter join of any two prime filters in D is again a prime filter, and since the prime filters of D are the principal end generated by a and its maximal proper subfilters this is indeed the case. Concerning (P2), one notes that the space ΣL consists of an infinite Boolean space X with a new point $*$ adjoined such that the non-void open sets are exactly the sets $U \cup \{*\}$, U open in X; since X is not discrete (P2) fails. Incidentally, for the finite $L \in$ **ContF**, (P2) always holds because every element of L is a join of join-irreducibles, and (P1) means that any meet of join-irreducibles is again join-irreducible.

We conclude with an outline of an alternative proof of the ($\Rightarrow$) part of Lemma 4 which does not use [**1**], and some features of which may be of independent interest.

The underlying set functor $|\cdot|$: **ContF** $\rightarrow$ **Ens** has a left adjoint: For any set X, let $\mathfrak{E}X$ be the set of all ends in the lattice of finite subsets of X, partially ordered by inclusion. Then, arbitrary joins and finite meets in $\mathfrak{E}X$ are given by set union and intersection, and any $\mathfrak{A} \in \mathfrak{E}X$ is the union of principal ends which are clearly compact; hence $\mathfrak{E}X$ is a distributive algebraic lattice and thus belongs to **ContF**. Moreover, $\mathfrak{E}X$ is evidently generated, with respect to arbitrary joins and finite meets, by the principal ends $\mathfrak{A}_x$ determined by the singletons $\{x\}$ in X. Now, if $u : X \rightarrow |L|$ is any map for some $L \in$ **ContF** then

$$h(\mathfrak{A}) = \bigvee_{F \in \mathfrak{A}} \bigwedge_{x \in F} u(x)$$

is easily checked to be a $\bigvee\wedge$-homomorphism; thus $\mathfrak{E}X$ is free with $\{\mathfrak{A}_x | x \in X\}$ as basis, and the correspondence $X \rightsquigarrow \mathfrak{E}X$ is the object part of a left adjoint $\mathfrak{E}$: **Ens** $\rightarrow$ **ContF** to $|\cdot|$.

Now, for any primal $L \in$ **ContF**, let $h : L \rightarrow \mathfrak{E}|\Sigma L|$ be the map for which $h(a)$ is the set of all $\{P_1, \ldots, P_n\}$ such that $a \in P_1 \vee \ldots \vee P_n$. It is easily checked that h is a $\bigvee\wedge$-homomorphism, and that its composite with the adjunction $\mathfrak{E}|\Sigma L| \rightarrow L$, which maps any end α of finite sets $\mathfrak{A}$ of prime filters in L to $\bigvee\bigwedge (\bigcup \mathfrak{A})(\mathfrak{A} \in \alpha)$, is the identity map on L. This makes L a retract of $\mathfrak{E}|\Sigma L|$ and hence ΣL a retract of a space of the type $\Sigma\mathfrak{E}X$, by duality. Now, for any set X, $\mathfrak{E}X$ is the coproduct of the $\mathfrak{E}\{x\}$, $x \in X$, and $\Sigma\mathfrak{E}\{x\} \simeq \mathbf{S}$, the Sierpinski space, so that $\Sigma\mathfrak{E}X \simeq \mathbf{S}^X$; this shows any $\Sigma\mathfrak{E}X$ is injective, and thus ΣL is injective, as a retract of an injective space.

It should be added that the $\mathfrak{E}X$ are actually free in the category **F** of all frames: the computation involved in showing their freeness in **ContF** only uses the identities between $\bigvee$ and $\wedge$ and not the continuity of the frame involved. The existence of free frames is already implicit in [**2**], and that they are, up to isomorphism, the topologies of the Sierpinski cubes occurs in [**8**]; so the main point here is that they are continuous. Since the underlying set functor of **F** is obviously monadic, i.e., **F** is a varietal category ([**9**]), this is not so for **ContF**; on the other hand, this may well be the case for the smaller category given by the maps in **ContF** which preserve all meets, in analogy with [**4**] where the category of continuous lattices and d-continuous $\wedge$-homomorphisms is shown to be monadic.

References

1. B. Banaschewski, *Essential extensions of T_0-spaces*, Gen. Top. Appl. *7* (1977), 233–246.

2. J. Benabou, *Treillis locaux et paratopologies*, Sém. C. Ehresmann 1957/58. Fac. de Sci. Paris, 1959.

3. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Coll. Publ. *25* (American Mathematical Society, Providence, Rhode Island, 1967).

4. R. A. Day, *Filter monads, continuous lattices, and closure systems*, Can. J. Math. *27* (1975), 50–59.

5. C. Dowker and D. Papert, *Quotient frames and subspaces*, Proc. London Math. Soc. *16* (1966), 275–296.

6. M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. *142* (1969), 43–60.

7. K. H. Hofmann and J. D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. Amer. Math. Soc. (to appear).

8. J. R. Isbell, *Atomless parts of spaces*, Math. Scand. *31* (1972), 5–32.

9. F. E. J. Linton, *Some aspects of equational theories*, Proc. Conference on Categorical Algebra La Jolla, 1965 (Springer-Verlag, Berlin, Heidelberg, New York, 1966), 84–94.

10. S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics *5* (Springer-Verlag, New York, Heidelberg, Berlin, 1971).

11. D. Scott, *Continuous lattices*, Lecture Notes in Mathematics *274* (Springer-Verlag, Berlin, Heidelberg, New York, 1972), 97–136.

12. M. H. Stone, *Topological representation of distributive lattices and Browerian logics*, Cas. Mat. Fys. *67* (1937), 1–25.

McMaster University,
Hamilton, Ontario