

ON COMPARISON OF CLUSTERING PROPERTIES OF POINT PROCESSES

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Abstract

In this paper, we propose a new comparison tool for spatial homogeneity of point processes, based on the joint examination of void probabilities and factorial moment measures. We prove that determinantal and permanental processes, as well as, more generally, negatively and positively associated point processes are comparable in this sense to the Poisson point process of the same mean measure. We provide some motivating results on percolation and coverage processes, and preview further ones on other stochastic geometric models, such as minimal spanning forests, Lilypond growth models, and random simplicial complexes, showing that the new tool is relevant for a systemic approach to the study of macroscopic properties of non-Poisson point processes. This new comparison is also implied by the directionally convex ordering of point processes, which has already been shown to be relevant to the comparison of the spatial homogeneity of point processes. For this latter ordering, using a notion of lattice perturbation, we provide a large monotone spectrum of comparable point processes, ranging from periodic grids to Cox processes, and encompassing Poisson point processes as well. They are intended to serve as a platform for further theoretical and numerical studies of clustering, as well as simple models of random point patterns to be used in applications where neither complete regularity nor the total independence property are realistic assumptions.

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1. Introduction

The usual statistical approach to the study of clustering in point processes (PPs) consists of the evaluation of Ripley's K function, pair-correlation function, or contact distribution function (also called the empty space function). However, such a comparison of local characteristics seems a weak tool for the study of the impact of clustering on some macroscopic properties of PPs such as those required in continuum percolation models. We are particularly motivated by heuristics indicating that PPs exhibiting more clustering should have larger critical radius for the percolation of its spherical-grain Boolean model than a spatially homogeneous PP.

It was observed in [4] that the directionally convex (DCX) order on a PP implies the ordering of K functions as well as pair-correlation functions, in the sense that PPs larger in the DCX order

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connections to other stochastic geometric models and the modeling applications. Lemma A.1, which is of independent interest and used in this paper for showing DCX ordering of perturbed lattices and determinantal and permanental PPs (on mutually disjoint simultaneously observable sets) is proved in Appendix A.

2. Notions, notation, and basic facts

2.1. Point processes

We assume the usual framework for random measures and PPs on d -dimensional Euclidean space \mathbb{R}^d ($d \geq 1$), where these are considered as random elements on the space $\mathbb{M}(\mathbb{R}^d)$ of nonnegative Radon measures on \mathbb{R}^d (see [17]). A PP Φ is *simple* if almost surely $\Phi(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$. We denote the void probabilities of the PP Φ by $\nu(B) = \mathbb{P}(\Phi \cap B = \emptyset)$ and the *factorial moment measure* of Φ by $\alpha^{(k)}(\cdot)$. Recall that for a simple PP, $\alpha^{(k)}(B_1 \times \dots \times B_k) = \mathbb{E}(\prod_{i=1}^k \Phi(B_i))$ for pairwise disjoint BBSs B_i ($i = 1, \dots, k$). Here, BBS denotes a bounded Borel subset. The *kth joint intensity*, $\rho^{(k)}: (\mathbb{R}^d)^k \rightarrow [0, \infty)$, is the density (if it exists) of $\alpha^{(k)}(\cdot)$ with respect to the Lebesgue measure $dx_1 \dots dx_k$. Recall that the joint intensities $\rho^{(k)}$, $k \geq 1$, characterize the distribution of a PP. The above facts remain true even when the densities $\rho^{(k)}$ are considered with respect to $\prod_{i=1}^k \mu(dx_k)$ for an arbitrary Radon measure μ on \mathbb{R}^d . As always, a PP or a random measure on \mathbb{R}^d is said to be *stationary* if its distribution is invariant with respect to translation by vectors in \mathbb{R}^d .

2.2. Directionally convex ordering

A Lebesgue-measurable function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is said to be DCX if, for every $x \in \mathbb{R}^k$, $\epsilon, \delta > 0$, and $i, j \in \{1, \dots, k\}$, we have that $\Delta_\epsilon^i \Delta_\delta^j f(x) \geq 0$, where $\Delta_\epsilon^i f(x) := f(x + \epsilon e_i) - f(x)$ is the discrete differential operator, with $\{e_i\}_{1 \leq i \leq k}$ denoting the canonical basis vectors for \mathbb{R}^k . We abbreviate *increasing* and DCX by IDCX and *decreasing* and DCX by DDCX (see [25, Chapter 3]). For real-valued random vectors of the same dimension X and Y , X is said to be *less than Y in DCX order* (denoted by $X \leq_{\text{DCX}} Y$) if $\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))$ for all f DCX such that both the expectations are finite. For two PPs on \mathbb{R}^d , we say that $\Phi_1(\cdot) \leq_{\text{DCX}} \Phi_2(\cdot)$ if, for any B_1, \dots, B_k BBSs in \mathbb{R}^d , $(\Phi_1(B_1), \dots, \Phi_1(B_k)) \leq_{\text{DCX}} (\Phi_2(B_1), \dots, \Phi_2(B_k))$; see [4]. The definition is similar for other orders, i.e. those defined by IDCX and DDCX functions. It is enough to verify the above conditions for B_i mutually disjoint. In order to avoid technical difficulties, we will consider here only PPs whose *mean measures* $\mathbb{E}(\Phi(\cdot))$ are Radon (finite on bounded sets). For such PPs, DCX order is a transitive order. This is owing to the fact that each DCX function can be monotonically approximated by DCX functions $f_i(\cdot)$ that satisfy $f_i(x) = O(\|x\|_\infty)$ at infinity, where $\|x\|_\infty$ is the L_∞ norm on the Euclidean space; see [25, Theorem 3.12.7].

It is easy to see that $\Phi_1(\cdot) \leq_{\text{DCX}} \Phi_2(\cdot)$ implies the *equality of their mean measures*, i.e. $\mathbb{E}(\Phi_1(\cdot)) = \mathbb{E}(\Phi_2(\cdot))$. Moreover, as shown in [4], higher-order moment measures are nondecreasing in DCX order on PPs, provided that they are σ -finite. (The σ -finiteness condition is missing in [4]; see [36, Proposition 4.2.4] for the correction.) In addition, DCX ordering allows us also to compare the void probabilities as stated in the following new result.

Proposition 2.1. *Denote the void probabilities of PPs Φ_1 and Φ_2 on \mathbb{R}^d by $\nu_1(\cdot)$ and $\nu_2(\cdot)$, respectively. If $\Phi_1 \leq_{\text{DDCX}} \Phi_2$ then $\nu_1(B) \leq \nu_2(B)$ for all BBSs $B \subset \mathbb{R}^d$.*

Proof. This follows directly from the definition of DCX ordering of PPs, expressing $\nu_j(B) = \mathbb{E}(f(\Phi_j(B)))$, $j = 1, 2$, with the function $f(x) = \max(0, 1 - x)$ that is decreasing and convex (so DDCX is in one dimension).

In particular, Proposition 2.1 implies ordering of all contact distribution functions (empty space functions) for PPs comparable in DCX order and not having fixed atoms (i.e. satisfying $\mathbb{P}(x \notin \Phi) = 1$ for all $x \in \mathbb{R}^d$). We see in the joint comparison of moment measures and void probabilities of PPs having equal mean measures, a new tool for comparison of their clustering properties, weaker than DCX order but more easy to verify.

2.3. Positive and negative association

Denote the covariance of random variables X, Y by $\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. A point process Φ is called *associated* if

$$\text{cov}(f(\Phi(B_1), \dots, \Phi(B_k)), g(\Phi(B_1), \dots, \Phi(B_k))) \geq 0$$

for any finite collection of BBSs $B_1, \dots, B_k \subset \mathbb{R}^d$, and f, g are continuous and increasing functions taking values in $[0, 1]$; see [7]. This property is also called *positive association*, or the *FKG property*. The theory for the opposite property is more tricky, see [26], but we can call Φ *negatively associated* if

$$\text{cov}(f(\Phi(B_1), \dots, \Phi(B_k)), g(\Phi(B_{k+1}), \dots, \Phi(B_l))) \leq 0$$

for any finite collection of BBSs $B_1, \dots, B_l \subset \mathbb{R}^d$ such that $(B_1 \cup \dots \cup B_k) \cap (B_{k+1} \cup \dots \cup B_l) = \emptyset$ and f, g are increasing functions. Both definitions can be straightforwardly extended to random measures.

3. Comparison of clustering to Poisson point processes

We call a PP *sub-Poisson* if it is smaller in DCX order than the Poisson PP and we call a PP *super-Poisson* if it is larger in DCX order than the Poisson PP (necessarily of the same mean measure in both cases). (More precisely, we should have called these processes DCX-sub-Poisson or DCX-super-Poisson PPs, but we omit the word DCX for simplicity.) Examples of such PPs are given in Section 4. A weaker notion of sub-Poissonianity and super-Poissonianity can be defined when comparing only moment measures or void probabilities. Bearing in mind that a Poisson PP can be characterized as having void probabilities of the form $v(B) = \exp(-\alpha(B))$, where $\alpha(\cdot)$ is its mean measure, we say that a PP Φ is *weakly sub-Poisson in the sense of void probabilities* (v -weakly sub-Poisson) if

$$\mathbb{P}(\Phi(B) = 0) \leq e^{-\mathbb{E}(\Phi(B))}, \tag{3.1}$$

for all Borel sets $B \subset \mathbb{R}^d$. Similarly, we say that a PP Φ is *weakly sub-Poisson in the sense of moment measures* (α -weakly sub-Poisson) if

$$\alpha^{(k)}(B_i) \leq \prod_{i=1}^k \alpha^{(1)}(B_i) = \prod_{i=1}^k \mathbb{E}(B_i), \tag{3.2}$$

for all mutually disjoint BBSs $B_i \subset \mathbb{R}^d$. When the inequalities in (3.1) and (3.2) are reversed, we will say that Φ is v -weakly super-Poisson or α -weakly super-Poisson, respectively.

Finally, we will say that Φ is *weakly sub-Poisson* if Φ is α -weakly sub-Poisson and v -weakly sub-Poisson. Similarly, we define *weakly super-Poisson PPs*. Examples of weakly sub-Poisson and super-Poisson PPs are given in Section 5.

The fact that DCX ordering implies ordering of moment measures and void probabilities lends credence to our usage of the terms weak sub-Poissonianity and super-Poissonianity.

Interestingly, these inequalities are also implied by negative and positive association. The following result is a key observation in this matter.

Proposition 3.1. *Consider a PP Φ with Radon mean measure $\alpha(\cdot) = \mathbb{E}(\Phi(\cdot))$. If Φ is simple, has Radon second-order factorial moment measure $\alpha^{(2)}(\cdot)$, and*

$$\mathbb{P}(\Phi(B_1) = 0, \Phi(B_2) = 0) \leq \mathbb{P}(\Phi(B_1) = 0)\mathbb{P}(\Phi(B_2) = 0), \tag{3.3}$$

for any two disjoint BBSs B_1 and B_2 , then Φ is ν -weakly sub-Poisson.

If the mean measure $\alpha(\cdot)$ of Φ is diffuse (without atoms) and Φ satisfies (3.3) with the reversed inequality (\geq) for any two disjoint BBSs B_1 and B_2 , then Φ is ν -weakly super-Poisson.

Proof. Define a set function $Q(B) = -\log(\mathbb{P}(\Phi(B) = 0))$. Regarding the first statement, it is immediate to see that Q is nonnegative and, under assumption (3.3), superadditive; i.e. for any finite $k \geq 1$ and any pairwise disjoint BBSs $B_j, j = 1, \dots, k, Q(B_1 \cup \dots \cup B_k) \geq \sum_{j=1}^k Q(B_j)$. In order to prove the result, we need to show that $Q(B) \geq \alpha(B)$, for any BBS B . In this regard, note that, by the superadditivity of Q , for any BBS B we have

$$Q(B) = \sup_J \sum_{j \in J} Q(B_j), \tag{3.4}$$

where the supremum is taken over all finite partitions of B into BBSs B_i . Moreover, for any BBS B ,

$$\begin{aligned} \mathbb{P}(\Phi(B) = 0) &= 1 - \mathbb{E}(\Phi(B)) + \mathbb{E}(\Phi(B) \mathbf{1}_{\{\Phi(B) \geq 2\}}) - \mathbb{P}(\Phi(B) \geq 2) \\ &\leq 1 - \mathbb{E}(\Phi(B)) + \mathbb{E}(\Phi(B)(\Phi(B) - 1)^+) \\ &= 1 - \alpha(B) + \alpha^{(2)}(B \times B); \end{aligned}$$

hence, $Q(B) = -\log(\mathbb{P}(\Phi(B) = 0)) \geq \alpha(B) - \alpha^{(2)}(B \times B)$. Consequently, by (3.4), for any BBS B

$$Q(B) \geq \sup_J \sum_{j \in J} (\alpha(B_j) - \alpha^{(2)}(B_j \times B_j)) = \alpha(B) - \inf_J \sum_{j \in J} \alpha^{(2)}(B_j \times B_j),$$

owing to the finiteness of all terms. In order to complete the proof it is enough to show that the infimum term is equal to zero. In this regard, for a given $\epsilon > 0$ define $\Delta_B^\epsilon = \{B \times B \ni (x, y) : |(x, y) - (z, z)| \leq \epsilon \text{ for some } z \in B\}$. Note that Δ_B^ϵ can be seen as some neighborhood of the intersection of the diagonal with $B \times B$. Note also that for any $\epsilon > 0$ there exists a suitable fine partition I of B such that $\sum_{j \in J} \alpha^{(2)}(B_j \times B_j) \leq \alpha^{(2)}(\Delta_B^\epsilon)$. (For example, take a finite coverage of B by balls of radius ϵ , which exists by local compactness of the space, and refine it to have disjoint partition of B .) By the local finiteness and σ -additivity of $\alpha^{(2)}$, $\lim_{\epsilon \rightarrow 0} \alpha^{(2)}(\Delta_B^\epsilon) = \alpha^{(2)}(\{(z, z) : z \in B\}) = 0$, where the last equality follows from the assumption that Φ is simple. This completes the proof of the first statement.

For the second statement, we will show that $Q(B) \leq \alpha(B)$. In this regard, note that the reversed inequality in (3.3) implies that $Q(\cdot)$ is subadditive and, consequently, for any BBS B ,

$$Q(B) = \inf_J \sum_{j \in J} Q(B_j),$$

where the infimum is over all finite partitions of B . Moreover, observe that $\mathbb{P}(\Phi(B) = 0) \geq 1 - \alpha(B)$ and that, for $0 \leq x \leq \epsilon, -\log(1 - x) \leq x(1 + \delta(\epsilon))$, where $\delta(\epsilon) = \epsilon/(2(1 - \epsilon)^2)$,

4.1. Perturbation operator

Let Φ be a PP on \mathbb{R}^d and $\mathcal{N}(\cdot, \cdot)$ and $\mathcal{X}(\cdot, \cdot)$ be two probability kernels from \mathbb{R}^d to nonnegative integers \mathbb{Z}^+ and \mathbb{R}^d , respectively. Consider the following *independently marked* version of the PP Φ : $\Phi^{\text{pert}} = \{(X, N_X, \mathbf{Y}_X)\}_{X \in \Phi}$. Here, given Φ we have

- $N_X, X \in \Phi$, are independent, nonnegative integer-valued random variables with distribution $\mathbb{P}(N_X \in \cdot \mid \Phi) = \mathcal{N}(X, \cdot)$,
- $\mathbf{Y}_X = (Y_{iX} : i = 1, 2, \dots), X \in \Phi$, are independent vectors of independent and identically distributed (i.i.d.) elements of \mathbb{R}^d , with the Y_{iX} s having the conditional distribution $\mathbb{P}(Y_{iX} \in \cdot \mid \Phi) = \mathcal{X}(X, \cdot)$,
- the random elements N_X, \mathbf{Y}_X are independent for all $X \in \Phi$.

Consider the following subset of \mathbb{R}^d :

$$\Phi^{\text{pert}} = \bigcup_{X \in \Phi} \bigcup_{i=1}^{N_X} \{X + Y_{iX}\}, \tag{4.1}$$

where the inner sum is interpreted as \emptyset when $N_X = 0$. The set Φ^{pert} can (and will) be considered as a PP on \mathbb{R}^d , provided that it is locally finite. In what follows, in accordance with our general assumption for this paper, we will assume that the mean measure of Φ^{pert} is locally finite (Radon measure), i.e.

$$\int_{\mathbb{R}^d} n(x) \mathcal{X}(x, B - x) \alpha(dx) < \infty, \quad \text{for all BBSs } B \subset \mathbb{R}^d, \tag{4.2}$$

where $\alpha(\cdot)$ is the mean measure of the PP Φ and $n(x) = \sum_{k=1}^{\infty} k \mathcal{N}(x, \{k\})$ is the mean value of the distribution $\mathcal{N}(x, \cdot)$.

The PP Φ^{pert} can be seen as *independently replicating and translating points from the PP Φ* , with the number of replications of the point $X \in \Phi$ having distribution $\mathcal{N}(X, \cdot)$ and the independent translations of these replicas from X by vectors having distribution $\mathcal{X}(X, \cdot)$. For this reason, we call Φ^{pert} a *perturbation of Φ driven by the replication kernel \mathcal{N} and the translation kernel \mathcal{X}* .

An important observation for us is that the *operation of perturbation of Φ is DCX monotone with respect to the replication kernel* in the following sense.

Proposition 4.1. *Consider a PP Φ with Radon mean measure $\alpha(\cdot)$ and its two perturbations $\Phi_j^{\text{pert}}, j = 1, 2$, satisfying (4.2), having the same translation kernel \mathcal{X} and possibly different replication kernels $\mathcal{N}_j, j = 1, 2$, respectively. If $\mathcal{N}_1(x, \cdot) \leq_{\text{CX}} \mathcal{N}_2(x, \cdot)$ (convex ordering of the conditional distributions of the number of replicas) for α -almost all $x \in \mathbb{R}^d$, then $\Phi_1^{\text{pert}} \leq_{\text{DCX}} \Phi_2^{\text{pert}}$.*

Proof. We will consider some particular coupling of the two perturbations $\Phi_j^{\text{pert}}, j = 1, 2$. Given Φ and $\mathbf{Y}_X = (Y_{iX} : i = 1, \dots)$ for each $X \in \Phi$, let $\Phi_{jX} = \bigcup_{i=1}^{N_X^j} \{X + Y_{iX}\}$, where N_X^j has distribution $\mathcal{N}_j(X, \cdot), j = 1, 2$, respectively. Thus, $\Phi_j^{\text{pert}} = \sum_{X \in \Phi} \Phi_{jX}, j = 1, 2$, are the two considered perturbations. Note that, given $\Phi, \Phi_j^{\text{pert}}$ can be seen as independent superpositions of Φ_{jX} for $X \in \Phi$. Hence, by [4, Proposition 3.2(4)] (superposition preserves DCX order) and [25, Theorem 3.12.8] (weak and L_1 convergence jointly preserve DCX order), it is enough to show that conditioned on $\Phi, \Phi_{1X} \leq_{\text{DCX}} \Phi_{2X}$ for every $X \in \Phi$. In this regard,

given Φ , consider $X \in \Phi$ and let B_1, \dots, B_k be mutually disjoint BBSs and $f: \mathbb{R}^k \rightarrow \mathbb{R}$, a DCX function. Define a real-valued function $g: \mathbb{Z} \rightarrow \mathbb{R}$, as

$$g(n) := \mathbb{E} \left(f \left(\text{sgn}(n) \sum_{i=1}^{|n|} (\mathbf{1}_{\{Y_{iX} \in B_1 - X\}}, \dots, \mathbf{1}_{\{Y_{iX} \in B_k - X\}}) \right) \middle| \Phi \right),$$

where $\text{sgn}(n) = n/|n|$ for $n \neq 0$ and $\text{sgn}(0) = 0$. By Lemma A.1, $g(\cdot)$ is a convex function on \mathbb{Z} and by Lemma A.2 it can be extended to a convex function $\tilde{g}(\cdot)$ on \mathbb{R} . Moreover, $\mathbb{E}(\tilde{g}(N_X^j) \mid \Phi) = \mathbb{E}(g(N_X^j) \mid \Phi) = \mathbb{E}(f(\Phi_{jX}(B_1), \dots, \Phi_{jX}(B_k)) \mid \Phi)$ for $j = 1, 2$. Thus, the result follows from the assumption $N_X^1 \leq_{\text{CX}} N_X^2$.

Remark 4.1. The above proof remains valid for an extension of the perturbation model in which the distribution $\mathcal{X}(X, \cdot)$ of the translations Y_{iX} depends not only on the location of the point $X \in \Phi$ but also on the entire configuration Φ ; $\mathcal{X}(X, \cdot) = \mathcal{X}(X, \Phi, \cdot)$, provided that (4.2) is replaced by the finiteness of $\int_{\mathbb{M}^d} \int_{\mathbb{R}^d} n(x) \mathcal{X}(x, \phi, B - x) C(d(x, \phi))$, where $C(d(x, \phi))$ is the Campbell measure of Φ .

4.2. Examples

4.2.1. *Perturbed Poisson point process.* Let Φ be a (possibly inhomogeneous) Poisson PP of mean measure $\alpha(dx)$ on \mathbb{R}^d . Let $\mathcal{N}(x, \cdot) = \varepsilon_1 = \mathbf{1}_{\{1 \in \cdot\}}$ be the Dirac measure on \mathbb{Z}^+ concentrated at 1 for all $x \in \mathbb{R}^d$ and assume an arbitrary translation kernel \mathcal{X} satisfying $\alpha^{\text{pert}}(A) = \int_{\mathbb{R}^d} \mathcal{X}(x, A - x) \alpha(dx) < \infty$ for all BBSs A . Then, by the displacement theorem for Poisson PPs, Φ^{pert} is also a Poisson PP with mean measure $\alpha^{\text{pert}}(dx)$. Assume any replication kernel $\mathcal{N}_2(x, \cdot)$, with mean number of replications $n_2(x) = \sum_{k=1}^{\infty} k \mathcal{N}_2(x, \{k\}) = 1$ for all $x \in \mathbb{R}^d$. Then, by Jensen’s inequality and Proposition 4.1, we obtain a super-Poisson PP Φ_2^{pert} . In the special case, when $\mathcal{N}_2(x, \cdot)$ is the Poisson distribution with mean 1 for all $x \in \mathbb{R}^d$, Φ_2^{pert} is a Poisson–Poisson cluster PP which is a special case of a Cox (doubly stochastic Poisson) PP with (random) intensity measure $\Lambda(A) = \sum_{X \in \Phi} \mathcal{X}(x, A - x)$. The fact that it is super-Poisson was already observed in [4]. Note that, for a general distribution of Φ , its perturbation Φ_2^{pert} is also a Cox PP of the intensity Λ given above.

4.2.2. *Perturbed-lattice point process.* Assuming a deterministic lattice Φ (e.g. $\Phi = \mathbb{Z}^d$) gives rise to the perturbed-lattice PP of the type considered in [31]. Surprisingly enough, starting from such a Φ , we can also construct a Poisson PP and both super-Poisson and sub-Poisson perturbed PPs. In this regard, assume for simplicity that $\Phi = \mathbb{Z}^d$, and the translation kernel $\mathcal{X}(x, \cdot)$ is uniform on the unit cube $[0, 1)^d$. Let $\mathcal{N}(x, \cdot)$ be the Poisson distribution with mean λ ($\text{Poi}(\lambda)$). It is easy to see that such a perturbation Φ^{pert} of the lattice \mathbb{Z}^d gives rise to a homogeneous Poisson PP with intensity λ .

4.2.3. *Sub-Poisson perturbed lattices.* Assuming for \mathcal{N}_1 some distribution convexly (CX) smaller than $\text{Poi}(\lambda)$, we obtain a sub-Poisson perturbed-lattice PP. Examples are the *hypergeometric* $\text{HGeo}(n, m, k)$, $m, k \leq n, km/n = \lambda$, and *binomial* $\text{Bin}(n, \lambda/n)$, $\lambda \leq n$, distributions, which can be ordered as follows:

$$\text{HGeo} \left(n, m, \frac{\lambda n}{m} \right) \leq_{\text{CX}} \text{Bin} \left(m, \frac{\lambda}{m} \right) \leq_{\text{CX}} \text{Bin} \left(r, \frac{\lambda}{r} \right) \leq_{\text{CX}} \text{Poi}(\lambda), \tag{4.3}$$

for $\lambda \leq m \leq \min(r, n)$; see [34]. ($\text{Bin}(n, p)$ has probability mass function $p_{\text{Bin}(n,p)}(i) = \binom{n}{i} p^i (1 - p)^{n-i}$, where $i = 0, \dots, n$, while $\text{HGeo}(n, m, k)$ has probability mass function $p_{\text{HGeo}(n,m,k)}(i) = \binom{m}{i} \binom{n-m}{k-i} / \binom{n}{k}$, $\max(k - n + m, 0) \leq i \leq m$.) (We show the logarithmic

concavity of the ratio of the respective probability mass functions, which implies increasing convex order and, consequently, CX, provided that the distributions have the same means.) Specifically, taking $\mathcal{N}_1(x, \cdot)$ to be binomial $\text{Bin}(n, \lambda/n)$ for $n \geq \lambda$, we obtain a DCX monotone increasing family of sub-Poisson PPs. Taking $\lambda = n = 1$ (equivalent to $\mathcal{N}(x, \cdot) = \varepsilon_1$), we obtain a *simple perturbed lattice* that is DCX smaller than the Poisson PP of intensity 1.

4.2.4. *Super-Poisson perturbed lattices.* Assuming for \mathcal{N}_2 some distribution convexly larger than $\text{Poi}(\lambda)$, we obtain a super-Poisson perturbed lattice. Examples are the *negative binomial* $\text{NBin}(r, p)$ distribution, with $rp/(1 - p) = \lambda$, and the *geometric* $\text{Geo}(p)$ distribution, with $1/p - 1 = \lambda$, which can be ordered in the following way:

$$\begin{aligned} \text{Poi}(\lambda) &\leq_{\text{CX}} \text{NBin}\left(r_2, \frac{\lambda}{r_2 + \lambda}\right) \\ &\leq_{\text{CX}} \text{NBin}\left(r_1, \frac{\lambda}{r_1 + \lambda}\right) \\ &\leq_{\text{CX}} \text{Geo}\left(\frac{1}{1 + \lambda}\right) \\ &\leq_{\text{CX}} \sum_j \lambda_j \text{Geo}(p_j), \end{aligned} \tag{4.4}$$

with $r_1 \leq r_2, 0 \leq \lambda_j \leq 1, \sum_j \lambda_j = 1$, and $\sum_j \lambda_j/p_j = \lambda + 1$, where the largest distribution above is a mixture of geometric distributions having mean λ ; see [34]. ($\text{Geo}(p)$ has probability mass function $p_{\text{Geo}(p)}(i) = p(1 - p)^i$ and $\text{NBin}(r, p)$ has probability mass function $p_{\text{NBin}(r,p)}(i) = \binom{r+i-1}{i} p^i (1 - p)^r$.) Specifically, taking $\mathcal{N}_2(x, \cdot)$ to be negative binomial $\text{NBin}(n, \lambda/(n + \lambda))$ for $n = 1, \dots$, we obtain a DCX monotone decreasing family of super-Poisson PPs. Recall that $\text{NBin}(r, p)$ is a mixture of $\text{Poi}(x)$ with parameter x distributed as a gamma distribution with scale parameter $p/(1 - p)$ and shape parameter r .

From [22, Lemma 2.18], we know that any *mixture of Poisson distributions* having mean λ is CX larger than $\text{Poi}(\lambda)$. Thus, the super-Poisson perturbed lattice with such a replication kernel (translation kernel being the uniform distribution) again gives rise to a Cox PP.

4.2.5. *Associated point processes.* From [7, Theorem 5.2], we know that any *Poisson center cluster PP* is (positively) associated. This is a generalization of our perturbation (4.1) of a Poisson PP Φ (see Section 4.2) having the form $\Phi^{\text{cluster}} = \sum_{X \in \Phi} \{X + \Phi_X\}$ with Φ_X being arbitrary i.i.d. (cluster) point measures. Other examples of associated PPs given in [7] are Cox PPs with intensity measures being associated. (It is easy to see by Jensen’s inequality that all Cox PPs are ν -weakly super-Poisson.)

It is easy to see that the PP formed by throwing n i.i.d. points in a bounded region forms a negatively associated PP. Further, we can show that independent superposition of a negatively associated PP is a negatively associated PP. Hence, simple perturbed lattices (see Section 4.2.3) are negatively associated.

5. Determinantal and permanental point processes

In this section, we focus on spatial determinantal and permanental PPs. We will show that they are, respectively, weakly sub-Poisson and super-Poisson PPs. Some partial DCX comparison of these PPs with respect to Poisson PPs, namely on mutually disjoint, simultaneously observable sets, will be proved as well.

Proof. It is known that, for each compact D , $\Phi^{\det}(D) \stackrel{D}{=} \sum_j \text{Bin}(1, \lambda_j^D)$ and $\Phi^{\text{perm}}(D) \stackrel{D}{=} \sum_j \text{Geo}(1/(1 + \lambda_j^D))$, where the summation is taken over all eigenvalues λ_j^D of \mathcal{K}_D and $\text{Bin}(1, \lambda_j^D)$ and $\text{Geo}(1/(1 + \lambda_j^D))$ are independent Bernoulli random variables, while $\text{Geo}(1/(1 + \lambda_j^D))$ are independent geometric random variables; see [14, Theorems 4.5.3 and 4.9.4]. Consequently,

$$\Phi^{\det}(D) \leq_{\text{CX}} \text{Poi}\left(\sum_j \lambda_j^D\right) \leq_{\text{CX}} \Phi^{\text{perm}}(D),$$

with the left-most inequality holding provided that Φ^{\det} exists (i.e. $\lambda_j^D \in [0, 1]$ for all compact D). Noting that convex order of integer-valued random variables implies ordering of probabilities taking the value 0 concludes the proof (see the proof of Proposition 2.1).

Alternatively, we can prove the above result via Proposition 3.1, as the inequality (3.3) has been proved for determinantal PPs in [10, Corollary 3.3.].

Corollary 5.1. *Combining results of Proposition 5.1 and 5.2, we conclude that Φ^{\det} is weakly sub-Poisson, while Φ^{perm} is weakly super-Poisson.*

In the next result, we will strengthen the above corollary, proving DCX ordering of finite-dimensional distributions of Φ^{\det} and Φ^{perm} on mutually disjoint *simultaneously observable* sets D_1, \dots, D_k . Simultaneous observability means that the eigenfunctions of $\mathcal{K}_{\cup D_i}$, restricted to D_i , are also eigenfunctions of \mathcal{K}_{D_i} for every $i = 1, \dots, k$.

Proposition 5.3. *Let Φ^{\det} and Φ^{per} be, respectively, the determinantal PP and the permanental PP with a trace-class integral kernel K and with Φ^{\det} being defined only if the spectrum of $\mathcal{K}_{\mathbb{R}^d}$ is in $[0, 1]$. Denote by Φ^{Poi} the Poisson PP of mean measure $\alpha(\cdot)$ given by $\alpha(D) = \sum_j \lambda_j^D$ for all compact D , where the summation is taken over all eigenvalues λ_j^D of \mathcal{K}_D . Let D_1, \dots, D_k be mutually disjoint, simultaneously observable (with respect to the kernel K) compact subsets of \mathbb{R}^d , and $D = \cup D_i$. Then we have*

$$\begin{aligned} (\Phi^{\det}(D_1), \dots, \Phi^{\det}(D_k)) &\leq_{\text{DCX}} (\Phi^{\text{Poi}}(D_1), \dots, \Phi^{\text{Poi}}(D_k)) \\ &\leq_{\text{DCX}} (\Phi^{\text{per}}(D_1), \dots, \Phi^{\text{per}}(D_k)). \end{aligned}$$

Proof. Let $\{\lambda_{j,i}\}_{j=1, \dots, J}$ denote the eigenvalues of \mathcal{K}_{D_i} and $\lambda_j^D = \sum_{i=1}^k \lambda_{j,i}^D$, $j = 1, \dots, J$, are the eigenvalues of \mathcal{K}_D with J denoting the number of eigenvalues of \mathcal{K}_D ($J = \infty$ and $J = 0$ are allowed, and in the latter case the sum is understood to be 0). From [14, Proposition 4.5.9], we know that $(\Phi^{\det}(D_1), \dots, \Phi^{\det}(D_k)) \stackrel{D}{=} \sum_{j=1}^J \sum_{i=1}^{N_j} \xi'_{i,j}$, where $N_j \sim \text{Bin}(1, \lambda_j^D)$ and, given N_j s, $\xi'_{i,j}$, $i \geq 1$, are independent *multinomial* vectors $\text{Mul}(1, \lambda_{j,1}^D/\lambda_j^D, \dots, \lambda_{j,k}^D/\lambda_j^D)$. ($\text{Mul}(k, p_1, \dots, p_k)$, with $0 \leq p_i \leq 1$ and $\sum_{i=1}^k p_i = 1$, has probability mass function $p_{\text{Mul}(n, p_1, \dots, p_k)}(n_1, \dots, n_k) = (n!/n_1! \dots n_k!) p_1^{n_1} \dots p_k^{n_k}$ for $n_1 + \dots + n_k = n$ and 0 otherwise.) It is easy to see that $(\Phi^{\text{Poi}}(D_1), \dots, \Phi^{\text{Poi}}(D_k)) \stackrel{D}{=} \sum_{j=1}^J \sum_{i=1}^{M_j} \xi'_{i,j}$, where $M_j \sim \text{Poi}(\lambda_j^D)$ with $\xi'_{i,j}$ s and $\lambda_{i,j}^D$ s as defined above. Due to the independence of the $\xi'_{i,j}$ s and the assumption $\sum_{j=1}^J \lambda_j < \infty$ (local trace-class property of \mathcal{K}_D), it is enough to prove for each j that $\sum_{i=1}^{N_j} \xi'_{i,j} \leq_{\text{DCX}} \sum_{i=1}^{M_j} \xi'_{i,j}$. Define $g(n) := \mathbb{E}(f(\text{sgn}(n) \sum_{i=1}^{|n|} \xi'_{i,j}))$ for $n \in \mathbb{Z}$ and f a DCX function. From Lemmas A.1 and A.2, we know that $g(\cdot)$ can be extended to a convex function on \mathbb{R} . Since we know from (4.3) that $\text{Bin}(1, \lambda_j^D) \leq_{\text{CX}} \text{Poi}(\lambda_j^D)$; hence, it follows that $\mathbb{E}(g(N_j)) \leq \mathbb{E}(g(M_j))$, as required. This completes the proof of the inequality for the determinantal PP.

Regarding the permanental PP, $(\Phi^{\text{per}}(D_1), \dots, \Phi^{\text{per}}(D_k)) \stackrel{D}{=} \sum_{j=1}^J \sum_{i=1}^{K_j} \xi'_{i,j}$, where $K_j \sim \text{Geo}(1/(1 + \lambda_j^D))$ and the $\xi'_{i,j}$ s are as defined above; see [14, Theorem 4.9.7]. Similar to the above proof, the required inequality follows from the ordering $\sum_{i=1}^{M_j} \xi'_{i,j} \leq_{\text{DCX}} \sum_{i=1}^{K_j} \xi'_{i,j}$ for all j , which follows from the fact that $\text{Poi}(\lambda) \leq_{\text{CX}} \text{Geo}(1/(1 + \lambda))$ (see (4.4)) and Lemmas A.1 and A.2. This completes the proof.

Remark 5.1. The key observation used in the above proof was that the number of points in disjoint, simultaneously observable sets can be represented as a sum of independent vectors, which themselves are binomial (for determinantal) or Poisson (for Poisson) or geometric (for permanental) sums of some further independent vectors. This is exactly the same representation as for the perturbed PP of Section 4.1 (available for any disjoint sets); compare with the proof of Proposition 4.1. In both cases, this representation and Lemmas A.1 and A.2 allow us to conclude DCX ordering of the corresponding vectors.

5.2.1. *Example of the Ginibre process.* Let Φ^G be the determinantal PP on \mathbb{R}^2 with kernel $K((x_1, x_2), (y_1, y_2)) = \exp[(x_1 y_1 + x_2 y_2) + i(x_2 y_1 - x_1 y_2)]$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, with respect to the measure $\mu(d(x_1, x_2)) = \pi^{-1} \exp[-x_1^2 - x_2^2] dx_1 dx_2$. This process is known as the *infinite Ginibre* PP. It is an important example of the determinantal PP recently studied on the theoretical ground (see e.g. [11]) and considered in modeling applications (see [24]). Denote by $\Psi^G = \{|X_i|^2 : X_i \in \Phi^G\}$, the PP on \mathbb{R}^+ of the squared radii of the points of Φ^G . This process has an interesting representation in terms of exponential random variables, similar to but different from this of a homogeneous one-dimensional Poisson PP Φ^1 ; see [11, Theorem 8]. An interesting questions posed in [4] is whether these two processes are DCX ordered. A partial result given in [4] is that $\Psi^G([0, r]) \leq_{\text{CX}} \Phi^1([0, r])$ for all $r \geq 0$, was proved in [4]. Full DCX ordering of these two PPs is possible by studying the simultaneously observable sets for the Ginibre process.

Corollary 5.2. *The process of the squared radii of the Ginibre process is sub-Poisson; i.e. $\Psi^G \leq_{\text{DCX}} \Phi^1$.*

Proof. We know that an arbitrary finite collection of the annuli centered at the origin $D_i = \{(x_1, x_2) : r_i \leq x_1^2 + x_2^2 \leq R_i\}$ is simultaneously observable for this PP; see [14, Example 4.5.8]. Using this observation, Proposition 5.3, and the fact that DCX order of PP on \mathbb{R} is generated by the semi-ring of intervals, we conclude that Ψ^G is DCX smaller than the Poisson PP Φ^1 of unit intensity on \mathbb{R}^+ .

6. Applications and further research

In what follows we will give some motivating results and preview further ones inspiring the ideas presented in this paper.

6.1. Continuum percolation

The Boolean model on a PP Φ with radius r is defined as $C(\Phi, r) := \bigcup_{X \in \Phi} B_X(r)$, where $B_X(r)$ denotes the ball of radius r centered at X . By percolation, we mean the existence of an unbounded connected subset of the Boolean model. The critical radius for percolation is defined as $r_c(\Phi) := \inf\{r : \mathbb{P}(C(\Phi, r) \text{ percolates}) > 0\}$. We mentioned in Section 1 a heuristic saying that clustering worsens percolation. Now we can use some family of perturbed-lattice PPs (see Section 4.2), monotone in DCX order, to illustrate this heuristic. Indeed, Figure 1 hints at ordering of the critical radii of DCX ordered PPs in $d = 2$. However, as shown in [5], this conjecture is not true in general: there exists a super-Poisson PP with the critical radius equal

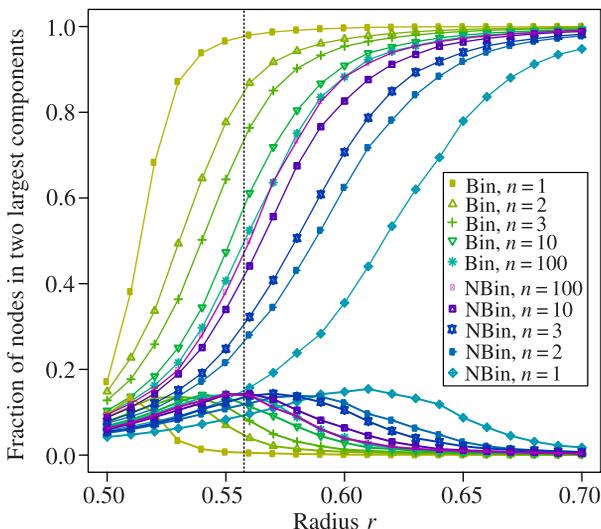


FIGURE 1: Mean fractions of nodes in the two largest components of the Boolean models generated by perturbed-lattice PPs in $d = 2$ with $\text{Bin}(n, 1/n)$ and $\text{NBin}(n, 1/(1 + n))$ as replication kernels, having fixed spherical grains of radius r . The replication kernels converge in n , from below and from above in DCX, respectively, to Poisson PPs whose critical radius is depicted by the dashed line.

to 0. What was also shown there, is that weakly sub-Poisson PPs exhibit a (uniformly) nontrivial phase transition in their continuum percolation model (i.e. admit uniformly nondegenerate lower and upper bounds for the critical radius). Similar results regarding k -percolation and SINR-percolation models (arising in modeling of connectivity of wireless networks) hold for DCX-sub-Poisson PPs.

In what follows, we will present some intuition leading to the above results and motivating our special focus on moment measures and void probabilities in the previous sections. Specifically, we will introduce two newer critical radii, r_c and \bar{r}_c , which act as lower and upper bounds, respectively, for the usual critical radius: $r_c \leq r_c \leq \bar{r}_c$. We will show that clustering acts differently on these new radii, i.e.

$$r_c(\Phi_2) \leq r_c(\Phi_1) \leq r_c(\Phi_1) \leq \bar{r}_c(\Phi_1) \leq \bar{r}_c(\Phi_2),$$

for Φ_1 having smaller voids and moment measures than Φ_2 . This sandwich inequality tells us that Φ_1 exhibits the usual phase transition $0 < r_c(\Phi_1) < \infty$, provided that Φ_2 satisfies a stronger condition $0 < r_c(\Phi_2)$ and $\bar{r}_c(\Phi_2) < \infty$. Conjecturing that it holds for the Poisson PP Φ_2 , we obtain the result on (uniformly) nontrivial phase transition for all weakly sub-Poisson Φ_1 , i.e. the one proved in [5] in a slightly different way.

6.1.1. *Moment measures and percolation.* Let $W_m = [-m, m]^d$ and define $h_{m,k}: (\mathbb{R}^d)^k \rightarrow \{0, 1\}$ to be the indicator of the event that $x_1, \dots, x_k \in (\Phi \cap W_m)^k$, $|x_i| \leq r$, $\inf_{x \in \partial W_m} |x - x_k| \leq r$, $|x_{i+1} - x_i| \leq r$ for all $1 \leq i \leq (k - 1)$, where ∂W_n denotes the boundary of set W_n . Let $N_{m,k}(\Phi, r) = \sum_{x_1, \dots, x_k \in \mathbb{R}^d}^{\neq} h_{m,k}(X_1, \dots, X_k)$ denote the number of distinct self-avoiding paths of length k from the origin $O \in \mathbb{R}^2$ to the boundary of the box W_m in the Boolean model and $N_m(\Phi, r) = \sum_{k \geq 1} N_{m,k}(\Phi, r)$ to be the total number of distinct self-avoiding paths to the

6.1.2. *Void probabilities and percolation.* Though we are interested in the percolation of Boolean models (continuum percolation models), but as is the wont in the subject we will use discrete percolation models as approximations. For $r > 0$ and $x \in \mathbb{R}^d$, define the following subsets of \mathbb{R}^d : $Q^r := (-1/2r, 1/2r]^d$ and $Q^r(x) := x + Q^r$. We will consider the following discrete graph parametrized by $n \in \mathbb{N}$: $\mathbb{L}_n^{*d} = (\mathbb{Z}_n^d, \mathbb{E}_n^{*d})$ is the usual *close-packed lattice graph* scaled down by the factor $1/n$. It has $\mathbb{Z}_n^d = (1/n)\mathbb{Z}^d$, where \mathbb{Z} is the set of integers, as the set of vertices and the set of edges $\mathbb{E}_n^{*d} := \{(z_i, z_j) \in (\mathbb{Z}_n^d)^2 : Q^{n/2}(z_i) \cap Q^{n/2}(z_j) \neq \emptyset\}$.

A *contour* in \mathbb{L}_n^{*d} is a minimal collection of vertices such that any infinite path in \mathbb{L}_n^{*d} from the origin has to contain one of these vertices (the minimality condition implies that the removal of any vertex from the collection will lead to the existence of an infinite path from the origin without any intersection with the remaining vertices in the collection). Let Γ_n be the set of all contours around the origin in \mathbb{L}_n^{*d} . For any subset of points $\gamma \subset \mathbb{R}^d$, in particular for paths $\gamma \in \Upsilon_m^n(K)$, Γ_n , we define $Q_\gamma = \bigcup_{z \in \gamma} Q^n(z)$.

With this notation, we can define the ‘upper’ critical radius $\bar{r}_c(\Phi)$ as follows:

$$\bar{r}_c = \bar{r}_c(\Phi) := \inf \left\{ r > 0 : \text{for all } n \geq 1, \sum_{\gamma \in \Gamma_n} \mathbb{P}(C(\Phi, r) \cap Q_\gamma = \emptyset) < \infty \right\}.$$

It might be seen as the critical radius corresponding to the phase transition when the discrete model $\mathbb{L}_n^{*d} = (\mathbb{Z}_n^d, \mathbb{E}_n^{*d})$, approximating $C(\Phi, r)$ with an arbitrary precision, starts percolating through the Peierls argument. As a consequence, $\bar{r}_c(\Phi) \geq r_c(\Phi)$ (see [6, Lemma 4.1]). The following ordering result follows immediately from the definition.

Corollary 6.1. *Let $C_j = C(\Phi_j, r)$, $j = 1, 2$, be two Boolean models with simple PPs of germs Φ_j , $j = 1, 2$. If Φ_1 has smaller void probabilities than Φ_2 then $\bar{r}_c(\Phi_1) \leq \bar{r}_c(\Phi_2)$.*

Remark 6.1. Even if the finiteness of \bar{r}_c is not clear for Poisson PPs and hence Corollary 6.1 cannot be directly used to prove the finiteness of the critical radii of ν -weakly sub-Poisson PPs, the approach based on void probabilities can be refined, as shown in [5], to conclude the aforementioned property.

6.2. Multiple coverage

For a PP Φ , define the k -covered set $C_k(\Phi, r) := \{x \in \mathbb{R}^d : \text{there exists } X_1, \dots, X_m \in \Phi, m \geq k \ni x \in \bigcap_{i=1}^m B_{X_i}(r)\}$. Heuristically, clustering should reduce the 1-covered region but increase the k -covered region for large k and we present a more formal statement of the same. The expected volume of the k -covered region is one of the important quantities of interest in sensor networks and our result has obvious implications regarding the choice of k or the PP in the context of sensor networks. We introduce another stochastic order to state the result. We say that two random variables X and Y are ordered in *uniformly convex variable order* (UCVO) ($X \leq_{uv} Y$) if their respect density functions f and g satisfy the following conditions: $\text{supp}(f) \subset \text{supp}(g)$ and $f(\cdot)/g(\cdot)$ is a unimodal function but their respective distribution functions are not ordered (i.e. $F(\cdot) \not\leq G(\cdot)$ or vice-versa (see [34]). Here, $\text{supp}(\cdot)$ denotes the support of a function. Denote by $\|A\|$ the Lebesgue’s measure of BBSs $A \subset \mathbb{R}^d$.

Proposition 6.2. *Let Φ_1 and Φ_2 be two simple, stationary PPs such that $\Phi_1(B_O(r)) \leq_{uv} \Phi_2(B_O(r))$ for $r \geq 0$. Then there exists $k_0 \geq 1$ such that, for any BBSs $W \subset \mathbb{R}^d$,*

$$\text{for all } k : 1 \leq k \leq k_0, \quad \mathbb{E}(\|C_k(\Phi_1, r) \cap W\|) \geq \mathbb{E}(\|C_k(\Phi_2, r) \cap W\|),$$

and

$$\text{for all } k > k_0, \quad \mathbb{E}(\|C_k(\Phi_1, r) \cap W\|) \leq \mathbb{E}(\|C_k(\Phi_2, r) \cap W\|).$$

Proof. First, note that

$$\mathbb{E}(\|C_k(\Phi_i, r) \cap W\|) = \int_W \mathbb{P}(\Phi_i(B_x(r)) \geq k) \, dx = \|W\| \mathbb{P}(\Phi_i(B_O(r)) \geq k)$$

for $i = 1, 2$. Now it suffices to show that $\mathbb{P}(\Phi_1(B_O(r)) \geq k) - \mathbb{P}(\Phi_2(B_O(r)) \geq k)$ changes sign exactly once in k for $k \geq 1$. This is implied by the UCVO order (see [34, Section 2 and Theorem 1]).

It is known that log-concavity of f/g implies UCVO order as well as convex ordering. We have used the latter implication in our examples for sub-Poisson perturbed lattices (see Section 4.2.3) and super-Poisson perturbed lattices (see Section 4.2.4). We can take for Φ_1 any of the sub-Poisson perturbed lattices presented in this paper or a determinantal PP and Φ_2 as a Poisson PP. We can also take Φ_1 to be a Poisson PP and Φ_2 to be any of the super-Poisson perturbed lattices presented in this paper or a permanental PP.

6.3. Further applications

6.3.1. *Minimal spanning forest.* In a recent work [12], the authors showed the connectivity of some approximations of the minimal spanning forest for the weakly sub-Poisson PPs (see the conjecture in [1] that the minimal spanning forest of Poisson PPs is almost surely connected, proved in [2] for dimension $d = 2$).

6.3.2. *First passage percolation.* Existence of arbitrarily large voids in Poisson PP was shown in [3] to be a reason of infinite end-to-end packet-delivery delays in a time-space SINR model, studied in the framework of a first passage percolation problem. Superposing the Poisson PP with an independent lattice of arbitrarily small intensity makes the delays finite. The latter result remains true when using a simple perturbed lattice, in which case the superposition is an example of a (DCX) sub-Poisson PP. Generalization to an arbitrary sub-Poisson PP is an open question. An interesting connection exists to the work in [19] and [33] on inequalities for time constants in first passage percolation on \mathbb{Z}^d with differing distributions for edge-passage times. More precisely, it was shown that more variable (in the sense of convex order) edge-passage times lead to faster transmission, i.e. a smaller time constant. An analogous result for time constants in the continuum case would be a welcome addition to the subject.

6.3.3. *Lilypond growth model.* In [9, Section 4.2], it was shown that the Lilypond growth model exists for sub-Poisson PPs, though not using the same terminology. Our examples of sub-Poisson PPs adds to the list of examples given in [9] for which a Lilypond growth model exists. Further, it was shown that sub-Poisson PPs with absolutely continuous (with respect to Lebesgue measure) $\alpha^{(k)}(\cdot)$ s do not percolate. The absolute continuity condition also holds true for our examples.

6.3.4. *Random geometric complexes.* This topological extension of random geometric graphs [28] was introduced and studied in [16] on Poisson PPs exploiting the connection between the Betti numbers of a random geometric complex and component counts of the corresponding random geometric graph [28, Chapter 3]. The motivation lies in the recent subject of topological data analysis. In an upcoming work [37], we study these models on more general stationary PPs using tools of stochastic ordering as well as asymptotic analysis of joint intensities and void probabilities. In particular, if we denote $r_n^{\text{con}}(\Phi)$ as the critical contractibility radius for the Čech complex on $\Phi \cap [-n^{1/d}/2, n^{1/d}/2]$ (i.e. the least radius r above which the Boolean

model $C(\Phi \cap [-n^{1/d}/2, n^{1/d}/2], r)$ becomes homotopic to a single point) then $r_n^{\text{con}}(\Phi) = O((\log n)^{1/d})$ for a ν -weakly sub-Poisson PP, whereas the critical contractibility radius of a Poisson PP is $\Theta((\log n)^{1/d})$. For Čech and Vietoris–Rips complexes on α -weakly sub-Poisson PPs, it is shown that order of the radii for existence of nonzero k th Betti numbers ($k \geq 1$) are $\Omega(n^{-1/d(k+1)})$ and $\Omega(n^{-1/d(2k+1)})$, respectively, i.e. at least that of the Poisson PP. For specific weak sub-Poisson PPs such as the Ginibre determinantal PP for which we have more accurate information about its joint intensities and void probabilities, it is shown that the correct orders differ significantly from that of the Poisson PP. The stronger assumption of negative association allows us to obtain variance bounds and hence derive asymptotics for existence of Betti numbers with high probability in the intermediate regime.

Subgraph counts and connective constants of random geometric graphs constitute specific instances of order statistics of PPs. Scaling limits of order statistics of Poisson PPs has garnered some interest in recent times; see [30]. Our techniques can easily yield that first moments of the order statistics are ordered for α -weakly ordered PPs but the question of further asymptotics remains open.

6.3.5. *Applications in modeling.* In the context of wireless networks, PPs are used to model locations of emitters/receivers. An ubiquitous assumption when modeling base stations in cellular networks is to consider deterministic lattices (usually hexagonal). On the other hand, mobile users are usually modeled by a Poisson PP. Both the assumptions are too simplistic. In reality, patterns of base stations are neither perfectly periodic, owing to various locational constants, nor completely independent because of various interactions: social, human interactions typically introduce more clustering, while the medium access protocols implemented in mobile wireless devices (as, e.g. CSMA used in the popular WiFi technology) tend to separate active users. We clearly see the interest in perturbed-lattice models in this context. We also believe that our work may lay the groundwork in other domains, e.g. in social and economic sciences, where the impact of clustering on the macroscopic properties of models is studied (see e.g. [8]).

6.3.6. *Further research.* Another motivation to study sub-Poisson perturbed lattices comes from their relations to zeros of *Gaussian analytic functions* [32], whose points exhibit repulsion at smaller distances and independence over large distances. However, the points seem more regularly distributed than in Poisson PPs [29]. This asks the question whether zeros of Gaussian analytic functions are comparable in some sense to Poisson PPs. *Gibbsian PP* is another well-known class of PPs which, depending on the nature of the potential, would be more or less clustering. Super-Poissonianity and sub-Poissonianity (even in the weak sense) have not been studied yet for Gibbsian PPs. Devising *statistical tests* for sub-Poissonianity would be desirable.

Appendix A

The following result, similar to [22, Lemma 2.17] is used in the proof of Propositions 4.1 and 5.3.

Lemma A.1. *Let $\xi_i = (\xi_i^1, \dots, \xi_i^k) \in \mathbb{R}^k$ ($i \in \mathbb{Z}$) be i.i.d. vectors of (possibly dependent) nonnegative random variables. Suppose that f is a DCX function on \mathbb{R}^k . Then, the function g defined on \mathbb{Z} by $g(n) = \mathbb{E}(f(\text{sgn}(n) \sum_{i=1}^{|n|} \xi_i))$ for $n \neq 0$ and $g(0) = 0$ is convex on \mathbb{Z} .*

Proof. We will prove that $g(n)$ has nonnegative second differences

$$g(n - 1) + g(n + 1) - 2g(n) \geq 0, \quad \text{for all } n \in \mathbb{Z}, \tag{A.1}$$

Note that for any $\lambda_1, \dots, \lambda_{n-1}$ we have $\sum_{i=1}^{n+1} \lambda_i = 1$ and $\sum_{i=1}^{n+1} \lambda_i k_i = k$. Consider the following subset of the $n - 1$ -dimensional unit cube $C := \{(\lambda_1, \dots, \lambda_{n-1}) \in [0, 1]^{n-1} : 0 \leq \lambda_n \leq 1, 0 \leq \lambda_{n+1} \leq 1\}$. The proof of the inductive step will be completed if we show that $F(\cdot) \geq g(k)$ on C . In this regard, note that C is *closed* and *convex*. Assume moreover that C is not empty; otherwise the condition (A.2) is trivially satisfied. Note also that $F(\cdot)$ is an affine, real-valued function defined on \mathbb{R}^{n-1} . Hence, by the maximum principle, the affine (hence convex) function $-F$ attains its maximum relative to C on some point $(\lambda_1^0, \dots, \lambda_{n-1}^0) \in \partial C$ of the boundary of C . Consequently, we have $F(\cdot) \geq F(\lambda_1^0, \dots, \lambda_{n-1}^0)$ on C and the proof of the inductive step will be completed if we show that $F(\lambda_1^0, \dots, \lambda_{n-1}^0) \geq g(k)$. In this regard, let $\lambda_n^0 = \lambda_n(\lambda_1^0, \dots, \lambda_{n-1}^0)$ and $\lambda_{n+1}^0 = \lambda_{n+1}(\lambda_1^0, \dots, \lambda_{n-1}^0)$. Using the continuity of the functions $\lambda_n(\cdot)$ and $\lambda_{n+1}(\cdot)$ is not difficult to verify that $(\lambda_1^0, \dots, \lambda_{n-1}^0) \in \partial C$ implies $\lambda_j^0 = 0$ for some $j = 1, \dots, n+1$. Thus, by our inductive assumption, $F(\lambda_1^0, \dots, \lambda_{n-1}^0) = \sum_{i=1, i \neq j}^{n+1} \lambda_i^0 g(k_i) \geq g(k)$, which completes the proof of (A.2) for all $n \geq 2$.

For the second statement, we recall the arguments used in [35] to show that a function satisfying (A.2) for all $n \geq 2$ (called *globally convex function* there) has a convex extension on \mathbb{R} . In this regard, consider the epigraph $\text{epi}(g) := \{(k, \mu) \in \mathbb{Z} \times \mathbb{R} : \mu \geq g(k)\}$ of g and its convex envelope $\text{epi}^{\text{co}}(g)$. It is easy to see that $\text{epi}^{\text{co}}(g) = \{(x, \mu) \in \mathbb{R}^2 : \mu \geq \sum_{i=1}^n \lambda_i g(k_i) \text{ for some } k_i \in \mathbb{Z}, 0 \leq \lambda_i \leq 1, \sum_{i=1}^n \lambda_i = 1, \text{ and } \sum_{i=1}^n \lambda_i k_i = x\}$. Define $\tilde{g}(x) := \inf\{\mu : (x, \mu) \in \text{epi}^{\text{co}}(g)\}$ for all $x \in \mathbb{R}$. The convexity of $\text{epi}^{\text{co}}(g)$ implies that \tilde{g} is convex on \mathbb{R} and the global convexity (A.2) of g implies that \tilde{g} is an extension of g . This completes the proof.

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