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A SHARP LOWER BOUND FOR THE RICCI CURVATURE OF BOUNDED HYPERSURFACES IN SPACE FORMS

ALAIN R. VEERAVALLI

DEDICATED TO LAMIAE AND LUCAS ZAKARIA WITH GREAT AFFECTION.

We give a sharp lower bound for the Ricci curvature of bounded complete hypersurfaces of space forms. This leads to several applications.

1. INTRODUCTION AND THE MAIN RESULT.

It is easy to see that if a closed smooth plane curve is included in a disk of radius $r > 0$, then there exists a point of the curve for which the curvature is in absolute value greater than or equal to $1/r$. A similar result holds for surfaces: any compact surface of $\mathbb{R}^3$ included in a ball of radius r admits a point for which the Gaussian curvature is greater than or equal to $1/r^2$. In 1983, Leung extended these results by showing the following.

THEOREM. [4] *If M is a complete hypersurface of $\mathbb{R}^{n+1}$ $(n \geqslant 2)$ included in a ball of radius $r > 0$ with sectional curvature bounded away from $-\infty$, then*

$$\limsup_{\xi \in UM} \operatorname{Ric}(\xi, \xi) \geqslant \frac{n-1}{r^2}$$

where Ric *is the Ricci curvature of M and UM the unit tangent bundle of M.*

Note that for the sphere of radius r in $\mathbb{R}^{n+1}$ the above inequality is in fact an equality. A natural question is to search for the Ricci curvature lower bound when replacing the Euclidean space by any space form. In two recent papers, Beltagy [1] and Erdŏgan [3] tried to give an answer but infortunately some estimates are false and the others are not sharp. In a previous paper [8], the author dealt with a close problem whose ideas can be used to settle the question. The main result of this work is the following:

THEOREM. *Let $\widetilde{S}_{n+1}(c)$ be the simply connected space form of constant sectional curvature c ($c \in \mathbb{R}$, $n \geqslant 2$), M a complete hypersurface of $\widetilde{S}_{n+1}(c)$ with sectional curvature*

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bounded away from $-\infty$ and included in a closed normal ball of radius $r > 0$ in $\widetilde{S}_{n+1}(c)$, with $r < \pi/(2\sqrt{c})$ if $c > 0$. Then

$$\limsup_{\xi \in UM} \operatorname{Ric}(\xi, \xi) \geqslant (n-1)\left(c + k_c^2(r)\right)$$

where Ric *is the Ricci curvature of M, UM the unit tangent bundle of M and*

$$k_c(r) = \begin{cases} \sqrt{c}\cot(r\sqrt{c}) & \text{if } c > 0 \\ 1/r & \text{if } c = 0 \\ \sqrt{-c}\coth\left(r\sqrt{-c}\right) & \text{if } c < 0 \end{cases}.$$

As one can see, this generalises Leung's theorem. Moreover, the function k_c is well-known by Riemannian geometers: the distance sphere of radius r in $\widetilde{S}_{n+1}(c)$, with $r < \pi/\sqrt{c}$ if $c > 0$, is an umbilical hypersurface with principal curvatures being precisely $k_c(r)$. It also shows by the Gauss formula that its Ricci curvature is constant and equal to $(n-1)\left(c + k_c^2(r)\right)$. Therefore the inequality given in our theorem is sharp.

Before giving the proof, one can remark that $(n-1)\left(c + k_c^2(r)\right)$ is positive for any constant c and positive r (with $r < \pi/\sqrt{c}$ if $c > 0$). This leads to criteria of unboundness:

Corollary 1. *Let M be a complete hypersurface of $\widetilde{S}_{n+1}(c)$ ($n \geqslant 2$) with sectional curvature bounded away from $-\infty$ and nonpositive Ricci curvature. If $c \leqslant 0$, then M is unbounded. If $c > 0$, then the diameter of M (for both Riemannian distances on M and $\widetilde{S}_{n+1}(c)$) satisfies* $\operatorname{diam}(M) \geqslant \pi/(2\sqrt{c})$.

When M is compact, these results can be reformulated in an easier way:

Corollary 2.

(i) *Let M be a compact hypersurface of $\widetilde{S}_{n+1}(c)$. If c is nonpositive, then there exists a point $q \in M$ and a unit tangent vector u to M at q such that*

$$\operatorname*{Ric}_q(u, u) \geqslant (n-1)\left(c + k_c^2(r)\right) \quad (> 0)$$

where r is the radius of any ball in $\widetilde{S}_{n+1}(c)$ containing M. Therefore, if c is nonpositive, there is no compact hypersurfaces in $\widetilde{S}_{n+1}(c)$ with nonpositive Ricci curvature. In particular, if c is nonpositive, there is no compact minimal hypersurfaces in $\widetilde{S}_{n+1}(c)$.

(ii) *If c is positive, then the diameter of any compact hypersurface M of $\widetilde{S}_{n+1}(c)$ with nonpositive Ricci curvature satisfies* $\operatorname{diam}(M) \geqslant \pi/(2\sqrt{c})$.

2. Preliminary results.

Let $\left(\widetilde{M}, \langle,\rangle\right)$ be a Riemannian manifold, $\widetilde{\nabla}$ its Levi-Civita connection and $\tilde{f} : \widetilde{M} \to \mathbb{R}$ a smooth function on $\widetilde{M}$. Recall that the gradient of $\tilde{f}$ is a smooth vector field $\widetilde{\nabla}\tilde{f}$

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on $\widetilde{M}$ defined by $\langle \widetilde{\nabla}\tilde{f}, X\rangle = X\tilde{f}$ and the Hessian of $\tilde{f}$ is the $(0,2)$-symmetric tensor $\widetilde{\nabla}^2\tilde{f}$ defined by $\widetilde{\nabla}^2\tilde{f}(X,Y) = \left\langle \widetilde{\nabla}_X\big(\widetilde{\nabla}\tilde{f}\big), Y\right\rangle = \widetilde{X}\big(\widetilde{Y}\tilde{f}\big) - \big(\widetilde{\nabla}_X Y\big)\tilde{f}$, X and Y being smooth vectors fields on $\widetilde{M}$. If M is a submanifold of $\widetilde{M}$ with the induced metric and the induced connection ∇, we can also define the gradient ∇f and the Hessian $\nabla^2 f$ of any smooth function f on M. For the particular case of smooth functions f on M of the form $f = \tilde{f}\circ i$ (where $i : M \to \widetilde{M}$ is the canonical injection), these operators are related: for any vector fields X, Y on M, we have

$$
\begin{aligned}
\widetilde{\nabla}^2\tilde{f}(X,Y) &= X\big(Y\tilde{f}\big) - \big(\widetilde{\nabla}_X Y\big)\tilde{f} \\
&= X\big(Y\tilde{f}\big) - \big\{(\nabla_X Y) + S(X,Y)\big\}f \\
&= \nabla^2 f(X,Y) - \big\langle \widetilde{\nabla}\tilde{f}, S(X,Y)\big\rangle \qquad (1)
\end{aligned}
$$

where S is the second fundamental form of M.

The gradient and Hessian are used in the classical Hopf lemma which says that for a smooth function $f : M \to \mathbb{R}$ on a compact Riemannian manifold M, there exists a point $q \in M$ such that $\nabla f(q) = 0$ and $\nabla^2 f(q)(X,X) \leqslant 0$ for any vector $X \in T_qM$. The proof of our theorem uses a rather technical result due to Omori which can be seen as a generalisation of the Hopf lemma:

Theorem [5] *Let M be a complete Riemannian manifold with sectional curvature bounded away from $-\infty$ and $f : M \to \mathbb{R}$ a smooth function on M bounded from above. Then for any $q_0 \in M$ and any $\varepsilon > 0$, there exists a point $q \in M$ such that $f(q) \geqslant f(q_0)$, $\left|\nabla f(q)\right| < \varepsilon$ and $\nabla^2 f(q)(X,X) < \varepsilon$ for any unit tangent vector X at q.*

Another trick used in the proof is an algebraic lemma due to Otsuki:

Lemma [7] *Let $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ be a symmetric bilinear form on $\mathbb{R}^n$ $(n, k > 0)$. If S^{n-1} denotes the unit sphere of $\mathbb{R}^n$, then the function $S^{n-1} \to \mathbb{R} : x \mapsto \left|S(x,x)\right|^2$ achieves its minimum at a point x_0 and we have the following properties*

(i) $x_0 \perp \operatorname{Ker} S(x_0,\cdot)$

(ii) $\left\langle S(x_0,x_0), S(x,x)\right\rangle \geqslant \left|S(x_0,x_0)\right|^2$ *for any unit vector $x \in \operatorname{Ker} S(x_0,\cdot)$.*

At last, the crucial point in the proof is the following result which uses classical material of Riemannian geometry:

Proposition. *Let $\left(\widetilde{M}, \langle,\rangle\right)$ be a Riemannian manifold, p a point of $\widetilde{M}$, d the Riemannian distance of $\widetilde{M}$ and $\tilde{f} : \widetilde{M} \to \mathbb{R} : q \mapsto d_p^2(q)/2$. Then*

(i) *$\tilde{f}$ is smooth on $\widetilde{M} \setminus \operatorname{Cut}(p)$ where $\operatorname{Cut}(p)$ is the cut point of p.*

(ii) *For any $q \in \widetilde{M}$ and any tangent vector v to $\widetilde{M}$ at q, we have $\widetilde{\nabla}\tilde{f}(q) = d_p(q)\dot{\gamma}\big(d_p(q)\big)$ and $\widetilde{\nabla}^2\tilde{f}(q)(v,v) = d_p(q)\Big\langle \widetilde{\nabla}X\big(d_p(q)\big), X\big(d_p(q)\big)\Big\rangle$ where $\gamma : \big[0, d_p(q)\big] \to \widetilde{M}$ is the unique normal geodesic joining p to q, X the unique Jacobi field along γ with the boundary condition $\Big(X(0), X\big(d_p(q)\big)\Big) = (0, v)$ and $\widetilde{\nabla}X$ the covariant derivative of X along γ.*

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(iii) *In particular, if $\widetilde{M} = \widetilde{S}_{n+1}(c)$ is a space form of constant sectional curvature c and p is a point of $\widetilde{M}$, then for any $q \in B_p(r)$, with $r < \pi/(2\sqrt{c})$ if $c > 0$, and any tangent vector v to $\widetilde{M}$ at q with $v \perp \widetilde{\nabla} d_p(q)$, we have $\nabla^2 \tilde{f}(q)(v,v) = d_p(q) \cdot k_c\big(d_p(q)\big) \cdot |v|^2$.*

The first two points come from the first and second variation formulae for length and energy (see, for example, [**2**] or [**6**]). For the last point, it suffices to solve the differential equation satisfied by the Jacobi field.

3. Proof of the theorem.

The manifold M is endowed with the metric $\langle , \rangle$ induced by $\widetilde{S}_{n+1}(c)$. The Riemannian distance in $\widetilde{S}_{n+1}(c)$ will be noted d and $UM = \bigcup_{q \in M} U_q M$ will mean the unit tangent bundle of M. We shall also consider the function $\tilde{f} = d_p^2/2 : \widetilde{S}_{n+1}(c) \to \mathbb{R}$ and its restriction $f = d_p^2/2 : M \to \mathbb{R}$. As M is included in a closed normal ball $\overline{B}_p(r)$, the manifold M avoids the cut point of p in $\widetilde{S}_{n+1}(c)$ and therefore the function f is smooth (by the above proposition) and bounded by $r^2/2$. Choose a point q_0 in M different from p. By Omori's theorem, for any positive integer m, there exists a point $q_m \in M$ such that $f(q_m) \geqslant f(q_0)$, $\big|\nabla f(q_m)\big| < 1/m$ and $\nabla^2 f(q_m)(u,u) < 1/m$ for any $u \in U_{q_m} M$. Remark that $0 < d_p(q_0) \leqslant d_p(q_m) \leqslant r$. For any integer $m \geqslant 0$, we shall write for convenience ℓ_m for $d_p(q_m)$ and γ_m for the unique normal geodesic joining p to q_m. Fix now a positive integer m and a vector $u \in U_{q_m} M$. By Omori's theorem, equation (1), the previous proposition and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 1/m > \nabla^2 f(q_m)(u,u) &= \widetilde{\nabla}^2 \tilde{f}(q_m)(u,u) + \ell_m \big\langle \dot{\gamma}_m(\ell_m), S_{q_m}(u,u) \big\rangle \\ &\geqslant \widetilde{\nabla}^2 \tilde{f}(q_m)(u,u) - \ell_m \cdot \big| S_{q_m}(u,u) \big|. \end{aligned} \tag{2}$$

Our next work will be to estimate the first term of the right-hand side of equation (2). We remark that u need not to be normal to $\widetilde{\nabla} \tilde{f}(q_m)$ $\big(= \ell_m \dot{\gamma}_m(\ell_m)\big)$. In order to apply the preceding proposition, share u in two parts: $u = u^t + u^n$ where u^t (respectively u^n) is normal (respectively parallel) to $\widetilde{\nabla} \tilde{f}(q_m)$. Then

$$\widetilde{\nabla}^2 \tilde{f}(q_m)(u,u) = \widetilde{\nabla}^2 \tilde{f}(q_m)\big(u^t,u^t\big) + 2 \cdot \widetilde{\nabla}^2 \tilde{f}(q_m)\big(u^t,u^n\big) + \widetilde{\nabla}^2 \tilde{f}(q_m)(u^n,u^n). \tag{3}$$

Of course, we have by the above proposition

$$\widetilde{\nabla}^2 \tilde{f}(q_m)\big(u^t,u^t\big) = \ell_m \cdot k_c(\ell_m) \cdot |u^t|^2.$$

Since $u^n = \big\langle u, \widetilde{\nabla} \tilde{f}(q_m) \big\rangle \dot{\gamma}_m(\ell_m)/\ell_m$ and since $\big\langle u, \widetilde{\nabla} \tilde{f}(q_m) \big\rangle = \big\langle u, \nabla f(q_m) \big\rangle$, we obtain by Omori's theorem and the Cauchy-Schwarz inequality

$$|u^n| < \frac{1}{m\ell_m} \leqslant \frac{1}{m\ell_0} \tag{4}$$

and so

$$|u^t| > 1 - \frac{1}{m\ell_m} \geqslant 1 - \frac{1}{m\ell_0}. \tag{5}$$

From now, we shall assume that m is choosen sufficiently large to ensure that $1 - (1/m\ell_0) > 0$.

Finally, note that the linear map $L_{q_m} : T_{q_m}\widetilde{S}_{n+1}(c) \to T_{q_m}\widetilde{S}_{n+1}(c) : w \mapsto \widetilde{\nabla}_w\big(\widetilde{\nabla}\tilde{f}\big)$ is continuous. So $\|L_{q_m}\| < \infty$ and we can write that for any tangent vectors w_1 and w_2 to $\widetilde{S}_{n+1}(c)$ at q_m,

$$\left|\widetilde{\nabla}^2\tilde{f}(q_m)(w_1, w_2)\right| = \left|\left\langle L_{q_m}(w_1), w_2\right\rangle\right| \leqslant \|L_{q_m}\| \cdot |w_1| \cdot |w_2|.$$

By using the continuity of the map $\overline{B}_p(r) \to \mathbb{R} : q \mapsto \|L_q\|$ on the compact set $\overline{B}_p(r)$, there exists a positive constant a such that on $\overline{B}_p(r)$,

$$\left|\widetilde{\nabla}^2\tilde{f}(q)(w_1, w_2)\right| \leqslant a \cdot |w_1| \cdot |w_2|. \tag{6}$$

Combining inequalities (3), (4), (5), (6), $|u^t| \leqslant 1$, $|u^n| \leqslant 1$ and since k_c is a decreasing function, we obtain

$$\begin{aligned}
\widetilde{\nabla}^2\tilde{f}(q_m)(u, u) &\geqslant \ell_m \cdot k_c(\ell_m) \cdot |u^t|^2 - 2a|u^t| \cdot |u^n| - a|u^n|^2 \\
&\geqslant \ell_m \cdot k_c(r)\left\{1 - \frac{1}{m\ell_0}\right\}^2 - 3a|u^n| \\
&> \ell_m \cdot k_c(r)\left\{1 - \frac{1}{m\ell_0}\right\}^2 - \frac{3a}{m\ell_0}.
\end{aligned}$$

By inequality (2), we conclude that

$$\left|S_{q_m}(u, u)\right| > k_c(r)\left\{1 - \frac{1}{m\ell_0}\right\}^2 - \frac{3a}{m\ell_0^2} - \frac{1}{m\ell_0}.$$

Since k_c is a positive function, one sees that, for sufficiently large m, $\left|S_{q_m}(u, u)\right|$ is positive.

Among all vectors of $U_{q_m}M$, let u_1 be one which makes $|S_{q_m}|$ minimal on the diagonal of $U_{q_m}M$. By the last remark, $\left|S_{q_m}(u_1, u_1)\right| > 0$ for sufficiently large m and so the kernel of the linear map $S_{q_m}(u_1, \cdot) : T_{q_m}M \to (T_{q_m}M)^\perp$ is $(n-1)$-dimensional. If $\{u_2, \dots, u_n\}$ is an orthonormal basis of this kernel, the first part of Otsuki's lemma asserts that $\{u_1, \dots, u_n\}$ is an orthonormal basis of $T_{q_m}M$. By the above inequality, the Gauss formula and the second part of Otsuki's lemma, we have

$$\begin{aligned}
\operatorname*{Ric}_{q_m}(u_1) &= (n-1)c + \sum_{i=2}^{n}\left\langle S_{q_m}(u_1, u_1), S_q(u_i, u_i)\right\rangle \\
&\geqslant (n-1)c + \sum_{i=2}^{n}\left|S_{q_m}(u_1, u_1)\right|^2 \\
&\geqslant (n-1)c + (n-1)\left\{k_c(r)\left\{1 - \frac{1}{m\ell_0}\right\}^2 - \frac{3a}{m\ell_0^2} - \frac{1}{m\ell_0}\right\}^2.
\end{aligned}$$

By letting m going to $+\infty$, this leads to the announced inequality. $\square$

REMARKS

1. It was asserted in [**3**] without proof that for the elliptic space form $S^{n+1} = \widetilde{S}_{n+1}(1)$, we have

$$\limsup_{\xi \in UM} \operatorname{Ric}(\xi, \xi) \geqslant (n-1)\Big(1 + \cos^4(r/2)/\sin^2(r/2)\Big).$$

An easy computation shows that

$$(n-1)\Big(1 + k_1^2(r)\Big) - (n-1)\Big(1 + \cos^4(r/2)/\sin^2(r/2)\Big) < 0$$

for any positive $r < \pi/2$. So, the assertion in [**3**] cannot be true as was shown above by taking the hypersphere. For the hyperbolic space form $\mathbb{H}^{n+1} = \widetilde{S}_{n+1}(-1)$, our result sharpens the inequality given in [**3**].

2. If M is assumed to be compact, the proof can be shortened of course by using the Hopf lemma.

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Département de Mathématiques
Université d'Evry-Val d'Essonne
Boulevard des Coquibus
91025 Evry Cedex
France
e-mail: Alain.Veeravalli@maths.univ-evry.fr