Can. J. Math., Vol. XXX, No. 6, 1978, 1228–1242

REAL SUBSPACES OF A QUATERNION VECTOR SPACE

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1. Introduction. If $U_{\mathbf{R}}$ is a real subspace of a finite dimensional vector space $V_{\mathbf{C}}$ over the field $\mathbf{C}$ of complex numbers, then there exists a basis $\{e_1, \ldots, e_n\}$ of $V_{\mathbf{C}}$ such that

$$U = \bigoplus_{t=1}^{n} (U \cap e_t\mathbf{C}),$$

where $U \cap e_t\mathbf{C}$ is either 0, $e_t\mathbf{R}$, or $e_t\mathbf{C}$. Thus, there are only three indecomposable types (U, V) of complex vector spaces V endowed with a real subspace U, namely $(0, \mathbf{C})$, $(\mathbf{R}, \mathbf{C})$ and $(\mathbf{C}, \mathbf{C})$; every pair (U, V) is the direct sum of copies of these types [**2**]. The classification problem of the right vector spaces over the division ring $\mathbf{H}$ of the quaternions endowed with a complex subspace, has a similar solution: the only indecomposable pairs are $(0, \mathbf{H}_{\mathbf{H}})$, $(\mathbf{C}_{\mathbf{C}}, \mathbf{H}_{\mathbf{H}})$ and $(\mathbf{H}_{\mathbf{C}}, \mathbf{H}_{\mathbf{H}})$, and every other pair is the direct sum of copies of these types.

The problem of classifying the $\mathbf{H}$-vector spaces with a *real* subspace is more involved. In fact, there is an infinite number of indecomposable pairs, some of which belong to continuous families; the latter are indexed by a set of quaternions in a way similar to indexing of indecomposable endomorphisms of a fixed complex vector space by the complex eigenvalues. The discrete indecomposable pairs are characterized, up to an isomorphism, by their dimension type. The dimension type of the pair $(U_{\mathbf{R}}, V_{\mathbf{H}})$ is, by definition, the pair of natural numbers $(\dim U_{\mathbf{R}}, \dim V_{\mathbf{H}})$, which we briefly denote by $\dim(U, V)$.

To be more specific, we want to consider pairs $(U_{\mathbf{R}}, V_{\mathbf{H}})$, where $V_{\mathbf{H}}$ is a right $\mathbf{H}$-vector space and $U_{\mathbf{R}}$ is an $\mathbf{R}$-subspace of the real vector space $V_{\mathbf{R}} (\approx V_{\mathbf{H}} \otimes {}_{\mathbf{H}}\mathbf{H}_{\mathbf{R}})$. Given two pairs (U, V) and (U', V'), a mapping $\alpha : (U, V) \to (U', V')$ is an $\mathbf{H}$-linear transformation $\alpha : V_{\mathbf{H}} \to V'_{\mathbf{H}}$ such that $\alpha(U) \subseteq U'$. An isomorphism is a mapping which has an inverse. Equivalently, an isomorphism is a bijective $\mathbf{H}$-linear transformation $\alpha : V_{\mathbf{H}} \to V'_{\mathbf{H}}$ such that $\alpha(U) = U'$. A pair (U, V) is the direct sum of (U_1, V_1) and (U_2, V_2) if $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$; (U, V) is decomposable if there is such a decomposition with $V_1 \neq 0$ and $V_2 \neq 0$. Thus, (U, V) is decomposable if and only if there are non-zero $\mathbf{H}$-subspaces V_1 and V_2 of V such that $V = V_1 \oplus V_2$ and $U = (U \cap V_1) \oplus (U \cap V_2)$.

We are going to introduce certain pairs $(U_{\mathbf{R}}, V_{\mathbf{H}})$ which will turn out to be representatives of the different isomorphism classes of indecomposable pairs.

Received July 8, 1977. This research was carried out during the stay of the first author at the Universität Bonn. It was also supported in part by the National Research Council of Canada (Grant A-7257).

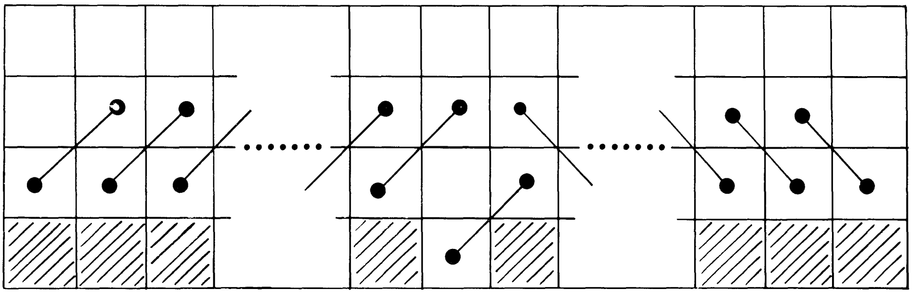
In all cases, $V_{\mathbf{H}}$ will be an n-dimensional **H**-vector space with a fixed basis $\{e_1, \ldots, e_n\}$, $n \geqq 1$. We denote by $\{1, i, j, k\}$ the standard basis of **H** over **R**.

$\boldsymbol{A}(n)$ is defined for all odd n, say $n = 2m - 1$, $m \geqq 1$. The subspace U is generated over **R** by the elements

$$\begin{aligned} &e_t, && 1 \leqq t \leqq m-1 \text{ and } m+1 \leqq t \leqq 2m-1, \\ &e_t i + e_{t+1} j, && 1 \leqq t \leqq m-1, \\ &e_m + e_{m+1} i, && \text{and} \\ &e_{t-1} j + e_t i, && m-2 \leqq t \leqq 2m-1. \end{aligned}$$

The dimension type is $\dim \boldsymbol{A}(n) = (2n-2, n) = (4m-4, 2m-1)$. Note that $\boldsymbol{A}(1) \approx (0, \mathbf{H}_{\mathbf{H}})$.

We may illustrate $\boldsymbol{A}(2m-1)$ in the following way:



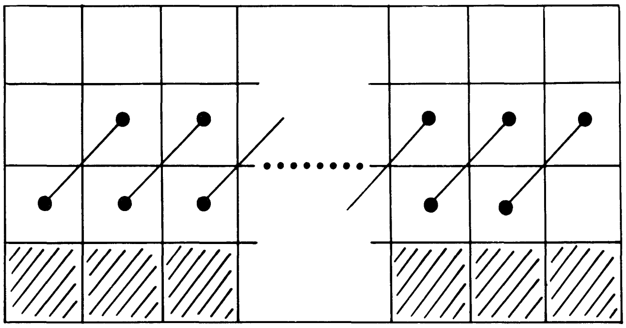
Here, the small squares should be considered as the elements f of a suitable **R**-basis of V, or as the corresponding one-dimensional **R**-subspace $f\mathbf{R}$ of V: In the bottom row, we have from left to right $e_1\mathbf{R}$, $e_2\mathbf{R}$, $e_3\mathbf{R}, \ldots, e_{m-1}\mathbf{R}$, $e_m\mathbf{R}$, $e_{m+1}\mathbf{R}, \ldots, e_{2m-3}\mathbf{R}$, $e_{2m-2}\mathbf{R}$, $e_{2m-1}\mathbf{R}$; in the second row, $e_1 i\mathbf{R}, \ldots e_m i\mathbf{R}, \ldots, e_{2m-1} i\mathbf{R}$; in the third row, $e_1 j\mathbf{R}, \ldots, e_m j\mathbf{R}, \ldots, e_{2m-1} j\mathbf{R}$; and finally, in the top row, $e_1 k\mathbf{R}, \ldots, e_m k\mathbf{R}, \ldots, e_{2m-1} k\mathbf{R}$. In particular, the subspaces corresponding to a single column generate a one-dimensional **H**-subspace. Now, the shaded area indicates those subspaces $f\mathbf{R}$ which are contained in U, and the diagonal connecting the squares $f\mathbf{R}$ and $f'\mathbf{R}$ indicates that $f + f'$ belongs to U, but that neither f nor f' belongs to U.

$\boldsymbol{B}(n)$ is defined for all $n \geqq 1$. The subspace U is generated over **R** by the elements

$$\begin{aligned} &e_t, && 1 \leqq t \leqq n, \text{ and} \\ &e_t i + e_{t+1} j, && 1 \leqq t \leqq n-1. \end{aligned}$$

The dimension type is $\dim \boldsymbol{B}(n) = (2n-1, n)$. Note that $\boldsymbol{B}(1) \approx (\mathbf{R}_{\mathbf{R}}, \mathbf{H}_{\mathbf{H}})$

and that $\boldsymbol{B}(n)$ may be illustrated (as above) by:



$\boldsymbol{C}(n, h)$ is defined for all $n \geqq 1$ and all $h \in \mathbf{H}\backslash\mathbf{R}$. If $h \notin \mathbf{R} + i\,\mathbf{R}$, $U_{\mathbf{R}}$ is generated by

$$e_t, \quad 1 \leqq t \leqq n,$$
$$e_1 h, \quad \text{and}$$
$$e_{t-1}i + e_t h, \quad 2 \leqq t \leqq n.$$

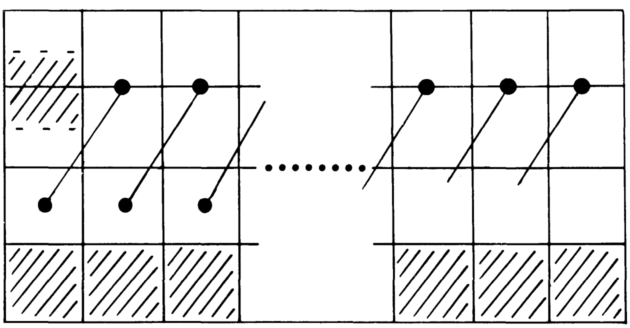
If $h \in (\mathbf{R} + i\,\mathbf{R})\backslash\mathbf{R}$, $U_{\mathbf{R}}$ is generated by

$$e_t, \quad 1 \leqq t \leqq n,$$
$$e_1 h, \quad \text{and}$$
$$e_{t-1}j + e_t h, \quad 2 \leqq t \leqq n.$$

The dimension type is dim $\boldsymbol{C}(n, h) = (2n, n)$. For $n = 1$, we get $\boldsymbol{C}(1, h) = (\mathbf{R} + h\mathbf{R}, \mathbf{H}_{\mathbf{H}})$. Obviously, $\boldsymbol{C}(1, h) \approx \boldsymbol{C}(1, h')$ if and only if $h' = hr + s$ for some $r \in \mathbf{R}^* = \mathbf{R}\backslash\{0\}$ and some $s \in \mathbf{R}$. The projective space $P(\mathbf{H}/\mathbf{R})$ is, by definition, the set of equivalence classes $h\mathbf{R}^* + \mathbf{R}$ with $h \in \mathbf{H}\backslash\mathbf{R}$. Thus, $\boldsymbol{C}(1, h) \approx \boldsymbol{C}(1, h')$ if and only if h and h' determine the same element in $P(\mathbf{H}/\mathbf{R})$. Denote by P a fixed set of representatives of these equivalence classes (for example, we may take

$$P = \{ia + jb + k | a, b \in \mathbf{R}\} \cup \{ia + j | a \in \mathbf{R}\} \cup \{i\}).$$

$\boldsymbol{C}(n, h)$ can be illustrated (as above) by:



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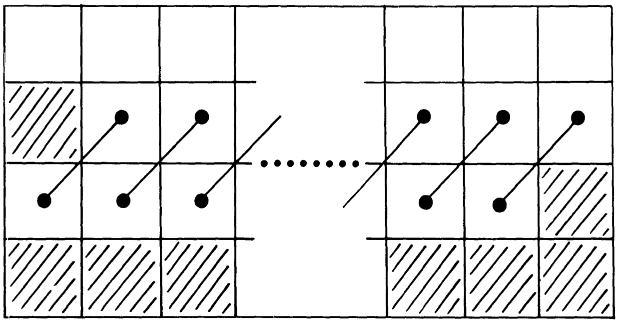
$\boldsymbol{D}(n)$ is defined for all $n \geqq 1$. The subspace $U_{\mathbf{R}}$ is generated by

$$e_t, \quad 1 \leqq t \leqq n,$$
$$e_1 j,$$
$$e_t i + e_{t+1} j, \quad 1 \leqq t \leqq n-1, \text{ and}$$
$$e_n i.$$

The dimension type is dim $\boldsymbol{D}(n) = (2n + 1, n)$. Note that

$$\boldsymbol{D}(1) \approx (\mathbf{R} + i\mathbf{R} + j\mathbf{R}, \mathbf{H_H})$$

and that $\boldsymbol{D}(n)$ may be illustrated (as above) by:

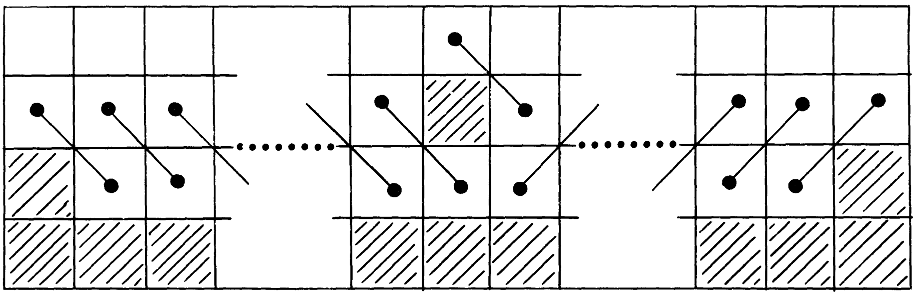


$\boldsymbol{E}(n)$ is defined for all odd n, say $n = 2m - 1$, $m \geqq 1$. If $m = 1$, the subspace $U_{\mathbf{R}}$ is, by definition, generated by e_1, $e_1 i$, $e_1 j$ and $e_1 k$; thus $\boldsymbol{E}(1) \approx (\mathbf{H_R}, \mathbf{H_R})$. If $m \geqq 2$, the subspace $U_{\mathbf{R}}$ is generated by

$$e_t, \quad 1 \leqq t \leqq 2m - 1,$$
$$e_1 i, \ e_m j, \ e_{2m-1} i,$$
$$e_t j + e_{t+1} i, \quad 1 \leqq t \leqq m - 1,$$
$$e_m k + e_{m+1} j, \quad \text{and}$$
$$e_{t-1} i + e_t j, \ m + 2 \leqq t \leqq 2m - 1.$$

The dimension type of $\boldsymbol{E}(n)$ is dim $\boldsymbol{E}(n) = (2n - 2, n) = (4m, 2m - 1)$.

$\boldsymbol{E}(2m - 1)$ may be illustrated (as above) by:



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THEOREM. *The pairs* $\boldsymbol{A}(2m-1)$, $\boldsymbol{B}(n)$, $\boldsymbol{C}(n, h)$, $\boldsymbol{D}(n)$, $\boldsymbol{E}(2m-1)$ *with* $m \geqq 1$, $n \geqq 1$ *and* $h \in P$, *form a complete set of indecomposable pairs* $(U_{\mathbf{R}}, V_{\mathbf{H}})$; *they are pairwise nonisomorphic and any indecomposable pair is isomorphic to one of them. Every pair is a direct sum of the indecomposable pairs* $\boldsymbol{A}(2m-1)$, $\boldsymbol{B}(n)$, $\boldsymbol{C}(n, h)$, $\boldsymbol{D}(n)$, $\boldsymbol{E}(2m-1)$, *and such a decomposition is unique (up to isomorphism).*

Let us remark that, in particular, the pairs $\boldsymbol{C}(n, h)$ and $\boldsymbol{C}(n, h')$ with $h, h' \in \mathbf{H}\backslash\mathbf{R}$ are isomorphic if and only if h and h' determine the same element of the projective space $P(\mathbf{H}/\mathbf{R})$.

The statement of the theorem can be easily reformulated in terms of matrices. Indeed, a real r-dimensional subspace $U_{\mathbf{R}}$ of a quaternion n-dimensional vector space $V_{\mathbf{H}}$ can be described, with respect to a choice of bases in $U_{\mathbf{R}}$ and $V_{\mathbf{H}}$, by a real $4n \times r$ matrix $\mathscr{A}$: each of the r basis vectors of $U_{\mathbf{R}}$ is a real combination of the $4n$ basis vectors $v \otimes h$ with $h = 1, i, j$ or k, of the real space $V_{\mathbf{H}} \otimes {}_{\mathbf{H}}\mathbf{H}_{\mathbf{R}}$. In this way, we can present the pairs $\boldsymbol{A}(n) - \boldsymbol{E}(n)$ as indecomposable real $4n \times r$ matrices. Thus, for example, one can find easily that $\boldsymbol{C}(n, h)$ will correspond to the $4n \times 2n$ $(n = 1, 2, \ldots)$ matrix

$$\begin{bmatrix} \mathscr{E}_h & \mathscr{F}_h & 0 & \cdots & 0 & 0 \\ 0 & \mathscr{E}_h & \mathscr{F}_h & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \mathscr{E}_h & \mathscr{F}_h \\ 0 & 0 & 0 & \cdots & 0 & \mathscr{E}_h \end{bmatrix},$$

where

$$\mathscr{E}_h = \begin{bmatrix} 1 & 0 \\ 0 & a \\ 0 & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad \mathscr{F}_h = \begin{bmatrix} 0 & 0 \\ 0 & s \\ 0 & t \\ 0 & 0 \end{bmatrix},$$

with $h = ia + jb + kc$ subject to the condition $c = 1$, or $c = 0$ and $b = 1$ (in which case $s = 1$, $t = 0$), or $c = b = 0$ and $a = 1$ (in which case $s = 0$, $t = 1$).

2. Homological and geometrical properties. It is perhaps of interest to provide more information on the different types of indecomposable pairs.

(1) *Endomorphism rings.* The endomorphism ring of the pairs $\boldsymbol{A}(2m-1)$ and $\boldsymbol{E}(2m-1)$ is always $\mathbf{H}$, the endomorphism ring of the pairs $\boldsymbol{B}(n)$ and $\boldsymbol{D}(n)$ is $\mathbf{R}$ and the endomorphism ring of $\boldsymbol{C}(n, h)$ is $\mathbf{C}[x]/(x^n)$.

(2) *Homomorphism.* The dimensions of the real vector spaces Hom $(\boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are indecomposable pairs, are listed in the following table. In the case that the entry in the table is negative, the respective Hom $(\boldsymbol{X}, \boldsymbol{Y})$ is zero. The symbol δ_{gh} is defined for $g, h \in \mathbf{H}\backslash\mathbf{R}$: $\delta_{gh} = 1$ if g and h determine the same element of $P(\mathbf{H}/\mathbf{R})$ and $\delta_{gh} = 0$ otherwise.

$X \backslash Y$	$A(2p-1)$	$B(q)$	$C(q, g)$	$D(q)$	$E(2p-1)$
$A(2m-1)$	$8(p-m)+4$	$4(q-m)+4$	$4q$	$4(q+m)-4$	$8(p+m)-12$
$B(n)$	$4(p-n)$	$2(q-n)+1$	$2q$	$2(q+n)-1$	$4(p+n)-4$
$C(n, h)$	0	0	$2\delta_{gh}$ min (q, n)	$2n$	$4n$
$D(n)$	0	0	0	$2(n-q)+1$	$4(n-p)+4$
$E(2m-1)$	0	0	0	$4(m-q)$	$8(m-p)+4$

(3) *Extensions.* As a consequence of the preceding table, one can also determine the dimension of the real vector space $\mathrm{Ext}^1(\boldsymbol{X}, \boldsymbol{Y})$ using the formula [**8**]

$$\dim_{\mathbf{R}} \mathrm{Ext}^1(\boldsymbol{X}, \boldsymbol{Y}) = \dim_{\mathbf{R}} \mathrm{Hom}(\boldsymbol{X}, \boldsymbol{Y}) - x_1y_1 - 4x_2y_2 + 4x_1y_2,$$

where $\dim \boldsymbol{X} = (x_1, x_2)$ and $\dim \boldsymbol{Y} = (y_1, y_2)$. These dimensions are listed in the following table.

$X \backslash Y$	$A(2p-1)$	$B(q)$	$C(q, g)$	$D(q)$	$E(2p-1)$
$A(2m-1)$	$8(m-p)-4$	$4(m-q)-4$	0	0	0
$B(n)$	$4(n-p)$	$2(n-q)-1$	0	0	0
$C(n, h)$	$4n$	$2n$	$2\delta_{gh}$ min (q, n)	0	0
$D(n)$	$4(n+p)$	$2(n+q)+1$	$2q$	$2(q-n)-1$	$4(p-n)-4$
$E(2m-1)$	$8(m+p)-4$	$4(m+q)$	$4q$	$4(q-m)$	$8(p-m)-4$

(4) *Threefold homological behaviour.* The above tables indicate that the indecomposable pairs can be divided into three classes:

(a) the class consisting of all $\boldsymbol{A}(2m-1)$ and $\boldsymbol{B}(n)$,

(b) the class consisting of all $\boldsymbol{D}(n)$ and $\boldsymbol{E}(2m-1)$, and

(c) the class containing all $\boldsymbol{C}(n, h)$.

Every pair which is a direct sum of copies of the various $\boldsymbol{A}(2m-1)$ and $\boldsymbol{B}(n)$ will be called *pre-projective*. Also, a pair will be called *pre-injective* if it is a direct sum of copies of the various $\boldsymbol{D}(n)$ and $\boldsymbol{E}(2m-1)$.

The rationale for this terminology stems from the following fact. Our category $\mathscr{S}$ of all pairs $(U_{\mathbf{R}}, V_{\mathbf{H}})$ is a subcategory of the category $\mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ of all triples $(U_{\mathbf{R}}, V_{\mathbf{H}}, \varphi)$, where $\varphi : U_{\mathbf{R}} \to V_{\mathbf{R}}$ is an arbitrary real linear transformation (that is not necessarily an embedding as in $\mathscr{S}$) which is an abelian category containing, besides the indecomposable objects of $\mathscr{S}$, only one additional indecomposable object $\boldsymbol{D}(0) = (\mathbf{R}_{\mathbf{R}}, 0, \varphi)$, where φ is the zero transformation. In $\mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$, the direct sums of copies of $\boldsymbol{A}(1)$ and $\boldsymbol{B}(1)$ are projective and the other $\boldsymbol{A}(2m-1)$ and $\boldsymbol{B}(n)$ have similar properties; also, $\boldsymbol{D}(0)$ and $\boldsymbol{E}(1)$ are the indecomposable injective objects and the other $\boldsymbol{D}(n)$ and $\boldsymbol{E}(2m-1)$ behave similarly (cf. Proposition C).

Finally, the direct sums of copies of the various $\boldsymbol{C}(n, h)$ are called *regular*. The regular pairs form an abelian exact subcategory which is uniserial [**5**; **8**].

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(5) *Threefold geometrical behaviour.* Here, we want to summarize some of the geometrical features of the pairs $(U_{\mathbf{R}}, V_{\mathbf{H}})$. By the geometry of $(U_{\mathbf{R}}, V_{\mathbf{H}})$, we understand the mutual position of the real subspaces Uh with $0 \neq h \in \mathbf{H}$ of $V_{\mathbf{H}}$. We may again observe three patterns of behaviour:

Given a pair $(U_{\mathbf{R}}, V_{\mathbf{H}})$, the following properties are equivalent:

(i) (U, V) is pre-projective.

(ii) For any one-dimensional subspace $W_{\mathbf{H}}$ of $V_{\mathbf{H}}$, $\dim (U \cap W)_{\mathbf{R}} \leqq 1$.

(iii) For any $h \in \mathbf{H}\backslash\mathbf{R}$, $U \cap Uh = 0$.

Similarly, the following properties are equivalent:

(i) (U, V) is pre-injective.

(ii) For any hyperplane $W_{\mathbf{H}}$ in $V_{\mathbf{H}}$, $\dim (V/U + W)_{\mathbf{R}} \leqq 1$.

(iii) For any $h \in \mathbf{H}\backslash\mathbf{R}$, $U + Uh = V$.

Furthermore, observe that, for any indecomposable pair $\boldsymbol{X} = (U, V)$ and any $0 \neq h \in \mathbf{H}$ such that $U \neq Uh$, the dimension of the intersection $\dim (U \cap Uh)_{\mathbf{R}}$ is always even. Indeed, for a pre-projective $\boldsymbol{X}$, $U \cap Uh = 0$; for a pre-injective $\boldsymbol{X}$, $U + Uh = V$ implies readily that $\dim (U \cap Uh)_{\mathbf{R}} = 2$ or 4; finally, for $\boldsymbol{X} = \boldsymbol{C}(n, g)$, $U \cap Uh \neq 0$ if and only if g and h define the same element of $P(\mathbf{H}/\mathbf{R})$ and then $\dim (U \cap Uh)_{\mathbf{R}} = 2$.

As a consequence, we conclude that for a (not necessarily indecomposable) regular pair $\boldsymbol{X} = (U, V)$, the number of direct summands of the form $\boldsymbol{C}(n, h)$ in a direct decomposition of X into indecomposable pairs equals $\frac{1}{2} \dim (U + Uh)_{\mathbf{R}}$.

(6) *Comparison of the pairs (U, V) and (Uh, V).* If (U, V) is pre-projective or pre-injective, then the pairs (U, V) and (Uh, V) are isomorphic for every $0 \neq h \in \mathbf{H}$. This follows immediately from the fact that every indecomposable pre-projective and pre-injective pair is uniquely determined by its dimension type. On the other hand, if $(U, V) = \boldsymbol{C}(n, g)$, then $(Uh, V) \approx \boldsymbol{C}(n, h^{-1} gh)$. This is clear for $n = 1$, and for $n > 1$, one can use the obvious embedding of $\boldsymbol{C}(1, g)$ into $\boldsymbol{C}(n, g)$. As a result, all the pairs (Uh, V), $0 \neq h \in \mathbf{H}$, are isomorphic if and only if the pair (U, V) has no non-zero regular direct summand.

3. Proof of the theorem. The classification theorem will be derived from two results in the representation theory of tame K-species (i.e. K-realizations of extended Dynkin diagrams) which we want to recall now in a form adapted to our particular situation.

PROPOSITION A [**5**]. *For every natural n, there is a unique indecomposable pair of dimension type $(2n - 1, n)$, and a unique indecomposable pair of dimension type $(2n + 1, n)$. For every odd n, there is a unique indecomposable pair of dimension type $(2n - 2, n)$, and a unique indecomposable pair of dimension type $(2n + 2, n)$. All other indecomposable pairs have dimension type $(2n, n)$.*

PROPOSITION B. [**5**; **8**]. *The direct sums of indecomposable pairs of dimension type $(2n, n)$ form an abelian, full and exact subcategory, which is, moreover,*

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uniserial (*that means, every indecomposable object has a unique composition series, and all of its composition factors are isomorphic*).

The reason that we may use the results from [**5**] and [**8**] stems from the fact that the category $\mathscr{L}(_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ in which the category $\mathscr{S}$ of all pairs $(U_{\mathbf{R}}, V_{\mathbf{H}})$ is embedded is the category of all representations of the **R**-species

$$\mathbf{R} \xrightarrow{\ _{\mathbf{R}}\mathbf{H}_{\mathbf{H}}\ } \mathbf{H} \quad \text{with the diagram} \quad \cdot \xrightarrow{\ (1,\,4)\ } \cdot$$

In the final Section 4, we shall indicate the proof of proposition A, because the Coxeter functors which play an essential role in the proof, will be used to establish some of the homological and geometrical properties mentioned in Section 2.

Let us outline the proof of the classification theorem from the above propositions. First, we shall show that the pairs $\boldsymbol{A}\ (2m - 1)$, $\boldsymbol{B}(n)$, $\boldsymbol{D}(n)$, $\boldsymbol{E}\ (2m - 1)$, with $m, n \in \boldsymbol{N}$, are indecomposable (Lemma 1). As a result, all indecomposable pairs of dimension type different from $(2n, n)$ will be determined. Second, we denote by $\mathscr{R}$ the subcategory of $\mathscr{S}$ of all direct sums of indecomposable pairs of dimension type $(2n, n)$, $n \in \boldsymbol{N}$ and call such pairs *regular* (it will turn out that this notion of regularity coincides with that introduced in Section 2). All simple regular pairs will be determined in Lemma 2. These are of the form $\boldsymbol{C}\ (1, h)$, $h \in \mathbf{H}\backslash\mathbf{R}$ and $\boldsymbol{C}\ (1, h) \approx \boldsymbol{C}\ (1, h')$ if and only if h and h' define the same element of $P(\mathbf{H}/\mathbf{R})$. Finally, we shall prove that all pairs $\boldsymbol{C}\ (n, h)$, $n \in \boldsymbol{N}$, are indecomposable (Lemma 3). In this way, all regular indecomposable pairs are classified. Indeed, let $\mathbf{X}$ be a regular indecomposable pair of composition length n, and let $\boldsymbol{Y}$ be one of its simple composition factors. Then $\boldsymbol{Y} \approx \boldsymbol{C}\ (1, h)$ for some $h \in \mathbf{H}\backslash\mathbf{R}$, and using the fact that $\mathscr{R}$ is uniserial (and thus contains at most one indecomposable object having a given composition factor and a given composition length), we conclude that $\boldsymbol{X} \approx \boldsymbol{C}\ (n, h)$.

LEMMA 1. *The pairs* $\boldsymbol{A}\ (2m - 1)$, $\boldsymbol{B}(n)$, $\boldsymbol{D}(n)$ *and* $\boldsymbol{E}\ (2m - 1)$ *are indecomposable.*

Proof. First, we recall some properties of decompositions which will be used in what follows. If $(U_{\mathbf{R}}, V_{\mathbf{H}})$ is a pair, let $\underline{U} = \{u \in U | u\mathbf{H} \subseteq U\}$ be the maximal **H**-subspace contained in U and let $\bar{U} = \sum_{u \in U} u\mathbf{H}$ be the **H**-subspace of V generated by U. A decomposition of $(U_{\mathbf{R}}, V_{\mathbf{H}})$ is given by an **H**-decomposition $V_{\mathbf{H}} = X_{\mathbf{H}} \oplus Y_{\mathbf{H}}$ which is compatible with the subspace $U : U = (U \cap X) + (U \cap Y)$. Obviously, any such decomposition is also compatible with the real subspaces Uh, $h \in \mathbf{H}$, with $\underline{U}$ and with $\bar{U}$. Also, if $V = X \oplus Y$ is compatible with two subspaces U_1 and U_2, then it is compatible with their sum $U_1 + U_2$ and intersection $U_1 \cap U_2$ (cf. [**4**]).

The statement that $\boldsymbol{A}\ (2m - 1)$ and $\boldsymbol{B}(n)$ are indecomposable will be proved by induction. It is obvious that $\boldsymbol{A}(1)$ and $\boldsymbol{B}(1)$ are indecomposable, but we will also need that $\boldsymbol{A}(3)$ is indecomposable. Thus, let $\boldsymbol{A}(3) = (U, V)$. We may extend the inclusion map $U_{\mathbf{R}} \to V_{\mathbf{R}}$ to an $\mathbf{H}$-homomorphism $\varphi : U_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}} \to V_{\mathbf{H}}$. By definition of $\boldsymbol{A}(3)$, there is an $\mathbf{R}$-Basis $\{u_1, u_2, u_3, u_4\}$ of $U_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{R}_{\mathbf{R}} = U_{\mathbf{R}}$ such that $\varphi(u_1) = e_1$, $\varphi(u_2) = e_1 i + e_2 j$, $\varphi(u_3) = e_2 + e_3 i$, $\varphi(u_4) = e_3$. The kernel of the map φ is therefore generated by the element $u = u_1 + u_2 i + u_3 k + u_4 j$. Now, assume there is given a decomposition of $\boldsymbol{A}(3)$, say $V = X \oplus Y$, with $U = (U \cap X) + (U \cap Y)$. Then $\varphi(U \cap X) \subseteq X$ and $\varphi(U \cap Y) \subseteq Y$, which, in turn implies that the kernel of φ is the direct sum of the kernels of the restrictions of φ to $(U \cap X)_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}}$ and to $(U \cap Y)_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}}$. However, ker φ is one-dimensional, and thus we may assume that u belongs to $(U \cap X)_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}}$. But this is possible only if $U \cap X = U$, which implies that $X = \bar{X} = \bar{U} = V$.

Now, we are going to show that $\boldsymbol{A}\,(2m-1) = (U, V)$ with $m > 2$ is indecomposable. Assume that $V = X \oplus Y$ is a decomposition of $V_{\mathbf{H}}$ compatible with U. Consequently, it is also compatible with the subspace

$$U + Ui = e_1(\mathbf{R} + i\mathbf{R}) + \sum_{t=2}^{2m-2} e_t\mathbf{H} + e_{2m-1}(\mathbf{R} + i\mathbf{R}),$$

and therefore with $V' = \underline{U + Ui} = \sum_{t=2}^{2m-2} e_t\mathbf{H}$ and $U' = U \cap V'$. Obviously, $(U'_{\mathbf{R}}, V'_{\mathbf{H}}) \approx \boldsymbol{A}\,(2m-3)$, which is, by induction hypothesis, indecomposable. Hence the decomposition $V' = (X \cap V') \oplus (Y \cap V')$ must be trivial and we may assume that $V' \subseteq X$. Using a similar argument, we deduce that

$$V'' = \underline{U + Uj} = \sum_{t=1}^{m-2} e_t\mathbf{H} + \sum_{t=m+2}^{2m-1} e_t\mathbf{H} + (e_{m-1} + e_{m+1}j)\mathbf{H}$$

has to be contained in one of the direct summands. Since, for $m > 2$, $V' \cap V'' \neq 0$, both V' and V'' are contained in the same summand X. Moreover, in view of $V' + V'' = V$, $X = V$ and $Y = 0$, as required.

To show that $\boldsymbol{B}(n) = (U, V)$ with $n > 1$ is decomposable, we can use a similar argument as for $\boldsymbol{A}\,(2m-1)$. Again, every decomposition $V = X \oplus Y$ which is compatible with U, is compatible also with

$$V' = \underline{U + Ui} = \sum_{t=2}^{n} e_t\mathbf{H},$$

$$V'' = \underline{U + Uj} = \sum_{t=1}^{n-1} e_t\mathbf{H},$$

and thus with $U' = U \cap V'$ and $U'' = U \cap V''$. Now, $(U', V') \approx (U'', V'') \approx \boldsymbol{B}\,(n-1)$, which is, by induction hypothesis, indecomposable. This implies that V', as well as V'', is contained in X or Y. In fact, they are both contained in the same direct summand, say X. This follows, for $n > 2$, from the fact that $V' \cap V'' \neq 0$ and, for $n = 2$, by a straight forward inspection of the decomposition $V = e_1\mathbf{H} \oplus e_2\mathbf{H}$ which is not compatible with U. Hence, $V' + V'' = V$ is contained in X, resulting in $X = V$ and $Y = 0$.

https://doi.org/10.4153/CJM-1978-102-6 Published online by Cambridge University Press

The proof that the pairs $\boldsymbol{D}(n)$ and $\boldsymbol{E}\,(2m - 1)$ are indecomposable can be given by dual arguments. In fact, a formal duality theory can be created in which $\boldsymbol{D}(n)$ is dual to $\boldsymbol{B}(n)$ and $\boldsymbol{E}\,(2m - 1)$ to $\boldsymbol{A}\,(2m - 1)$ (cf. [**4**]).

LEMMA 2. *The simple regular pairs are of the form* $(\mathbf{R} + h\mathbf{R}, \mathbf{H})$ *with* $h \in \mathbf{H}\backslash\mathbf{R}$, *and* $(\mathbf{R} + h\mathbf{R}, \mathbf{H})$ *and* $(\mathbf{R} + h'\mathbf{R}, \mathbf{H})$ *are isomorphic if and only if* $\mathbf{R} + h\mathbf{R} = \mathbf{R} + h'\mathbf{R}$. *Moreover,* End $(\mathbf{R} + h\mathbf{R}, \mathbf{H}) = \mathbf{R} + h\mathbf{R} \approx \mathbf{C}$.

Proof. A regular pair $(U_{\mathbf{R}}, \mathbf{H}_{\mathbf{R}})$ is evidently simple and satisfies dim $U_{\mathbf{R}} = 2$. The left multiplication by a non-zero element $u \in U$ yields an isomorphism between $(U, \mathbf{H})$ and $(u^{-1}\,U, \mathbf{H})$, and since $u^{-1}\,U$ contains $\mathbf{R}$, we get that

$$u^{-1}U = \mathbf{R} + h\mathbf{R} \quad \text{for some } h \in \mathbf{H}\backslash\mathbf{R}.$$

Observe that the $\mathbf{R}$-subspace $\mathbf{R} + h\mathbf{R}$ of $\mathbf{H}$ is a subfield of $\mathbf{H}$ which is isomorphic to the complex numbers $\mathbf{C}$. Now, the endomorphism ring of $(\mathbf{R} + h\mathbf{R}, \mathbf{H})$ consists of the set of left multiplications by elements g of $\mathbf{H}$ such that $g(\mathbf{R} + h\mathbf{R}) \subseteq \mathbf{R} + h\mathbf{R}$. Since $\mathbf{R} + h\mathbf{R}$ is closed under multiplication, every $g \in \mathbf{R} + h\mathbf{R}$ has this property, and conversely, every such left multiplication maps 1 into $\mathbf{R} + h\mathbf{R}$ and is therefore given by an element of $\mathbf{R} + h\mathbf{R}$.

Finally, assume that $(\mathbf{R} + h\mathbf{R}, \mathbf{H}) \approx (\mathbf{R} + h'\mathbf{R}, \mathbf{H})$. Such an isomorphism is given by the left multiplication on $\mathbf{H}$ by an element $g \in \mathbf{H}$. In particular, $g.\,1 \in \mathbf{R} + h'\mathbf{R}$, and therefore, making use of the fact that $\mathbf{R} + h'\mathbf{R}$ is a subfield again,

$$\mathbf{R} + h\mathbf{R} = g^{-1}(\mathbf{R} + h'\mathbf{R}) = \mathbf{R} + h'\mathbf{R}.$$

It remains to be seen that any regular pair which is simple is of the form $(U_{\mathbf{R}}, \mathbf{H}_{\mathbf{H}})$. The proof involves the complexification of pairs. This can be done for arbitrary $\mathbf{R}$-species (see the general theory of Galois descent in Gabriel [**7**]); however, we shall restrict ourselves to the classical notions of the theory of algebras here.

First, every pair $(U_{\mathbf{R}}, V_{\mathbf{H}})$ can be considered as an A-module, where A is the matrix algebra

$$\begin{bmatrix} \mathbf{R} & \mathbf{H} \\ 0 & \mathbf{H} \end{bmatrix},$$

namely as the right A-module with the additive structure $U \oplus V$. In this way, one gets an equivalence between the category $\mathscr{S}$ of all pairs, and the category of all right A-modules without simple projective submodules. Now, we may extend the scalars of the $\mathbf{R}$-algebra A to $\mathbf{C}$ by forming the $\mathbf{C}$-algebra $B = A \otimes_{\mathbf{R}} \mathbf{C}$. Similarly, every A-module M_A gives rise to the B-module $M \otimes_{\mathbf{R}} \mathbf{C}$. It is well-known and easy to see ([**3**, 29.5]) that the endomorphism ring of the B-module $M \otimes_{\mathbf{R}} \mathbf{C}$ is equal to End $(M_A) \otimes_{\mathbf{R}} \mathbf{C}$. Moreover, it is clear, that B is the matrix algebra

$$\begin{bmatrix} \mathbf{C} & M_2(\mathbf{C}) \\ 0 & M_2(\mathbf{C}) \end{bmatrix},$$

where $M_2(\mathbf{C})$ denotes the complex 2×2 matrix ring. Now, B is Morita equivalent to the matrix algebra B'

$$\begin{bmatrix} \mathbf{C} & \mathbf{C} \times \mathbf{C} \\ 0 & \mathbf{C} \end{bmatrix},$$

whose modules correspond to the representations of the $\mathbf{C}$-species

$$\mathbf{C} \xrightarrow{{}_{\mathbf{C}}\mathbf{C}_{\mathbf{C}} \oplus {}_{\mathbf{C}}\mathbf{C}_{\mathbf{C}}} \mathbf{C},$$

that is to the Kronecker modules over $\mathbf{C}$ (pairs of $\mathbf{C}$-linear transformations between two vector spaces)

$$P_{\mathbf{C}} \rightrightarrows Q_{\mathbf{C}}.$$

Such a Kronecker module is, in this way, identified with the B'-module $P \oplus Q$, and subsequently with the B-module $P \oplus Q \oplus Q$. We are going to exploit this relationship to determine the simple regular pairs.

Let (U, V) be a simple regular pair of dimension type (u, v). Its endomorphism ring has to be a division ring which is a finite dimension $\mathbf{R}$-algebra; thus, it is either $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. The endomorphism ring of the complexification $M \otimes_{\mathbf{R}} \mathbf{C}$ of $M_A = U \oplus V$ is End $(M_A) \otimes_{\mathbf{R}} \mathbf{C} \approx$ End $(U, V) \otimes_{\mathbf{R}} \mathbf{C}$; thus it is either $\mathbf{C}$, $\mathbf{C} \times \mathbf{C}$ or $M_2(\mathbf{C})$. The $\mathbf{R}$-dimension $u + 4v$ of M_A is also the $\mathbf{C}$-dimension of $M \otimes_{\mathbf{R}} \mathbf{C}$. Under the categorical equivalence specified above, the B-module $M \otimes_{\mathbf{R}} \mathbf{C}$ corresponds to a Kronecker module $P_{\mathbf{C}} \rightrightarrows Q_{\mathbf{C}}$ satisfying

$$P_{\mathbf{C}} = U \otimes_{\mathbf{R}} \mathbf{C} \quad \text{and} \quad Q_{\mathbf{C}} \oplus Q_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}.$$

From here, dim $P_{\mathbf{C}} = u$ and dim $Q_{\mathbf{C}} = 2v$. Moreover, since (U, V) is a regular pair, $u = 2v$.

Thus, we have obtained a Kronecker module $P_{\mathbf{C}} \rightrightarrows Q_{\mathbf{C}}$ satisfying dim $P_{\mathbf{C}} =$ dim $Q_{\mathbf{C}} = 2v$, whose endomorphism ring is either $\mathbf{C}$, $\mathbf{C} \times \mathbf{C}$ or $M_2(\mathbf{C})$. An easy consideration of the normal forms of the Kronecker modules [**5**; **7**] reveals that this is possible only for $v = 1$. Indeed, the only indecomposable Kronecker modules with dim $P_{\mathbf{C}} =$ dim $Q_{\mathbf{C}}$ whose endomorphism ring is $\mathbf{C}$, satisfies dim $P_{\mathbf{C}} = 1$; accordingly, our Kronecker module cannot be indecomposable. Hence, its endomorphism ring is either $\mathbf{C} \times \mathbf{C}$ or $M_2(\mathbf{C})$. It follows that the Kronecker module is the direct sum of two indecomposable modules

$$(P_1 \rightrightarrows Q_1) \oplus (P_2 \rightrightarrows Q_2)$$

which have either no non-zero homomorphisms between each other or are isomorphic. In the first case, the condition implies that dim $P_t =$ dim Q_t $(t = 1, 2)$ and since the endomorphism ring of $P_t \rightrightarrows Q_t$ is $\mathbf{C}$, it turns out that $v = 1$. Also, in the second case dim $P_1 =$ dim $Q_1 =$ dim $P_2 =$ dim $Q_2 = 1$, and thus $v = 1$. The proof of Lemma 2 is completed.

LEMMA 3. *Every pair* $\boldsymbol{C}(n, h)$ *with* $n \in \mathbf{N}$ *and* $h \in P(\mathbf{H}/\mathbf{R})$ *is indecomposable and contains* $\boldsymbol{C}(1, h) = (\mathbf{R} + h\mathbf{R}, \mathbf{H})$.

Proof. There are obvious embeddings

$$\boldsymbol{C}(n-2, h) \hookrightarrow \boldsymbol{C}(n-1, h) \hookrightarrow \boldsymbol{C}(n, h)$$

with $\boldsymbol{C}(m, h)$ generated by the first m base vectors e_ι $(m = n-2, n-1)$,

$$\boldsymbol{C}(n, h)/\boldsymbol{C}(n-1, h) \cong \boldsymbol{C}(1, h) \quad \text{and}$$
$$\boldsymbol{C}(n, h)/\boldsymbol{C}(n-2, h) \cong \boldsymbol{C}(2, h).$$

Since $\boldsymbol{C}(1, h)$ is a simple object of the category $\mathscr{R}$ and since $\mathscr{R}$ is closed under extensions, $\boldsymbol{C}(n, h)$ belongs to $\mathscr{R}$ and has a composition series with all factors isomorphic to $\boldsymbol{C}(1, h)$. Since $\mathscr{R}$ is uniserial, it suffices to prove that $\boldsymbol{C}(2, h)$ is indecomposable.

Now, if $\boldsymbol{C}(2, h) = (U, V)$ decomposes, then it decomposes in $\mathscr{R}$ and therefore has to be a direct sum of two copies of $\boldsymbol{C}(1, h)$. This implies, the equality $U = Uh$, because $\mathbf{R} + h\mathbf{R}$ is a subring of $\mathbf{H}$. However, it is easy to check that this is not the case : If $h \notin \mathbf{R} + i\mathbf{R}$, then $e_1 i + e_2 h \in U \backslash Uh$ and if $h = i$, then $e_1 j + e_2 i \in U \backslash Ui$.

4. Proof of the statements in Section 2. In this final section, we wish to outline the proof of some of the homological and geometrical properties of the pairs (U, V). In order to avoid a case-by-case inspection and complicated calculations, we have to go deeper into the theory of representations of the extended Dynkin diagrams (tame species). The fact that the category $\mathscr{R}$ is uniserial yields immediately, in view of Lemma 2, that the endomorphism ring End $(\boldsymbol{C}(n, h))$ is $\mathbf{C}[x]/(x^n)$ and that the dimension of the $\mathbf{R}$-vector space Hom $(\boldsymbol{C}(n, h), \boldsymbol{C}(q, g))$ equals 2 min $(q, n)\ \delta_{gh}$.

However, to derive the properties involving pre-projective and pre-injective pairs, we have to recall the Coxeter functors. These were first introduced in the "classical" situation by Bernštejn, Gel'fand and Ponomarev [**1**] and then fully explored in [**5**]. Let c be the linear transformation of $\mathbf{R} \times \mathbf{R}$ given by the matrix $\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ operating from the right-hand side. In what follows, we consider the abelian category $\mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{R}})$ of all triples

$$(U_{\mathbf{R}}, V_{\mathbf{H}}, \varphi : U_{\mathbf{R}} \to V_{\mathbf{H}} \otimes {}_{\mathbf{H}}\mathbf{H}_{\mathbf{R}} \approx V_{\mathbf{R}}).$$

$\mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ contains the category $\mathscr{S}$ as a full subcategory (of all triples with φ monic); it contains also one additional indecomposable object: $\boldsymbol{D}(0) = (\mathbf{R}, 0, 0)$.

PROPOSITION C [**5**]. *There exist two endofunctors* C^+ *and* C^- *of* $\mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ *possessing the following properties:*

(i) *C^+ is left exact and C^- is right exact.*

(ii) *If $\boldsymbol{X} \in \mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ is indecomposable and non-isomorphic to $\boldsymbol{A}(1)$ or $\boldsymbol{B}(1)$, then*

$$C^-C^+\boldsymbol{X} \approx \boldsymbol{X} \quad \textit{and} \quad \dim C^+\boldsymbol{X} = c(\dim \boldsymbol{X}),$$

whereas $C^+\boldsymbol{A}(1) = C^+\boldsymbol{B}(1) = \boldsymbol{O}$.

(ii′) *If $\boldsymbol{Y} \in \mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ is indecomposable and non-isomorphic to $\boldsymbol{D}(0)$ or $\boldsymbol{E}(1)$, then*

$$C^+C^-\boldsymbol{Y} \approx \boldsymbol{Y} \quad \textit{and} \quad \dim C^-\boldsymbol{Y} = c^{-1}(\dim \boldsymbol{Y}),$$

whereas $C^-\boldsymbol{D}(0) = C^-\boldsymbol{E}(\mathbf{1}) = \boldsymbol{O}$.

(iii) If $\boldsymbol{X}, \boldsymbol{Y} \in \mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ *and $\boldsymbol{X}$ is non-isomorphic to $\boldsymbol{A}(1)$ or $\boldsymbol{B}(1)$, then the* $\mathbf{R}$-*vector spaces* Hom $(\boldsymbol{X}, \boldsymbol{Y})$ *and* Hom $(C^+\boldsymbol{X}, C^+\boldsymbol{Y})$ *are canonically isomorphic.*

(iii′) *If $\boldsymbol{X}, \boldsymbol{Y} \in \mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ and $\boldsymbol{Y}$ is non-isomorphic to $\boldsymbol{D}(0)$ or $\boldsymbol{E}(1)$, then the* $\mathbf{R}$-*vector spaces* Hom $(\boldsymbol{X}, \boldsymbol{Y})$ *and* Hom $(C^-\boldsymbol{X}, C^-\boldsymbol{Y})$ *are canonically isomorphic.*

As a consequence of Proposition C, we can write down the complete list of images under C^+ and C^-. Thus,

$$C^+\boldsymbol{A}(2m-1) = \boldsymbol{A}(2m-3) \quad \text{for } m > 1, \quad C^+\boldsymbol{A}(1) = \boldsymbol{O};$$
$$C^+\boldsymbol{B}(n) = \boldsymbol{B}(n-1) \quad \text{for } n > 1, \quad C^+\boldsymbol{B}(1) = \boldsymbol{O};$$
$$C^+\boldsymbol{C}(n, h) = \boldsymbol{C}(n, h);$$
$$C^+\boldsymbol{D}(n) = \boldsymbol{D}(n+1); \quad \text{and}$$
$$C^+\boldsymbol{E}(2m-1) = \boldsymbol{E}(2m+1).$$

It may be perhaps of some interest to give an explicit description of one of the functors, say of C^+. Let $(U_{\mathbf{R}}, V_{\mathbf{H}}, \varphi) \in \mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$. Then the mapping $\varphi : U_{\mathbf{R}} \to V_{\mathbf{R}}$ can be extended, using the $\mathbf{H}$-vector space structure of $V_{\mathbf{H}}$, to an $\mathbf{H}$-linear mapping $\varphi_{\mathbf{H}} : U_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}} \to V_{\mathbf{H}}$. Denoting ker $\varphi_{\mathbf{H}}$ by $V'_{\mathbf{H}}$, we have a mapping $\kappa : V'_{\mathbf{H}} \to U_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}}$. Define the epimorphism $\epsilon : U_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{H}_{\mathbf{R}} \to U_{\mathbf{R}}$ by

$$\epsilon(u \otimes 1) = u \quad \text{and} \quad \epsilon(u \otimes i) = \epsilon(u \otimes j) = \epsilon(u \otimes k) = 0,$$

and put $U'_{\mathbf{R}} = \ker \epsilon\kappa$. We get a canonical inclusion $\varphi' : U'_{\mathbf{R}} \to V'_{\mathbf{R}}$, and then define

$$C^+(U, V, \varphi) = (U', V', \varphi').$$

Now, having the Coxeter functors available, it is rather easy to determine the endomorphism rings, the homomorphisms (and the extensions). For example, End $\boldsymbol{A}(2m-1)$ is isomorphic to End $\boldsymbol{A}(1) = \mathbf{H}$, in view of $\boldsymbol{A}(1) = C^{+(m-1)}\boldsymbol{A}(2m-1)$; and similarly, End $\boldsymbol{B}(n) \approx$ End $\boldsymbol{B}(1) = \mathbf{R}$ etc. In order to find Hom $(\boldsymbol{A}(2m-1), \boldsymbol{B}(q))$, we apply the functor $C^{+(m-1)}$ and deduce that

$$\text{Hom}\,(C^{+(m-1)}\boldsymbol{A}(2m-1), C^{+(m-1)}\boldsymbol{B}(q)) = \text{Hom}\,(\boldsymbol{A}(1), \boldsymbol{B}(q-m+1)).$$

Now, in order to verify that the R-dimension of the homomorphism space is $4\ (q - m + 1)$, we only observe that Hom $(\boldsymbol{A}(1),\ (U,\ V))$ is isomorphic as a vector space to V. By the same argument,

$$\text{Hom}\ (\boldsymbol{A}(2m - 1), \boldsymbol{A}(2p - 1)) \approx \text{Hom}\ (\boldsymbol{A}(1), \boldsymbol{A}(2(p - m) + 1)),$$

$$\text{Hom}\ (\boldsymbol{A}(2m - 1),\ \boldsymbol{C}(q, g) \approx \text{Hom}\ (\boldsymbol{A}(1),\ \boldsymbol{C}(q, g)),$$

$$\text{Hom}\ (\boldsymbol{A}(2m - 1), \boldsymbol{D}(q)) \approx \text{Hom}\ (\boldsymbol{A}(1), \boldsymbol{D}(q + m - 1),$$

$$\text{Hom}\ (\boldsymbol{A}(2m - 1), \boldsymbol{E}(2p - 1) \approx \text{Hom}\ (\boldsymbol{A}(1), \boldsymbol{E}(2(p + m) - 3),$$

and thus the respective $\mathbf{R}$-dimensions are $8(p - m) + 4$, $4q$, $4(q - m) + 4$, $4(q + m) - 4$, $8(p + m) - 12$.

Similarly, applying $C^{+(n-1)}$, we get

$$\text{Hom}\ (\boldsymbol{B}(n), \boldsymbol{A}(2p - 1)) \approx \text{Hom}\ (\boldsymbol{B}(1), \boldsymbol{A}(2(p - n) + 1)),$$

and using the fact that Hom $(\boldsymbol{B}(1),\ (U,\ V))$ is isomorphic as a vector space to U, we conclude that the $\mathbf{R}$-dimension equals $4(p - n)$.

Summarizing this procedure, we can always reduce the calculation, by applying the Coxeter functors, to a situation in which one of the pairs is either $\boldsymbol{A}(1)$, $\boldsymbol{B}(1)$, $\boldsymbol{D}(0)$ or $\boldsymbol{E}(1)$, and then, in addition to the above observations on Hom $(\boldsymbol{A}(1),\ (U,\ V))$ and Hom $(\boldsymbol{B}(1),\ (U,\ V))$, we use the fact that Hom $((U,\ V), \boldsymbol{D}(0))$ and Hom $((U,\ V),\ \boldsymbol{E}(1))$ are isomorphic as vector spaces to U and V, respectively.

Finally, we want to prove the geometrical characterizations of the pre-projective pairs. Let $(U,\ V)$ be a pre-projective pair. Assuming that there exists a one-dimensional subspace $W_{\mathbf{H}}$ of $V_{\mathbf{H}}$ with $\dim\ (U \cap W)_{\mathbf{R}} \geqq 2$, we get a non-zero homomorphism

$$(U \cap W,\ W) \rightarrow (U,\ V),$$

which is impossible. Similarly, assuming that there is a non-zero element $u \in U \cap Uh$ with $h \in \mathbf{H}\backslash\mathbf{R}$, we get $\dim\ (U \cap u\mathbf{H}) \geqq 2$, a contradiction. Finally, let $U \cap Uh = 0$ for all $h \in \mathbf{H}\backslash\mathbf{R}$. Then Hom $(\boldsymbol{C}(1,\ h),\ (U,\ V)) = 0$ for all $h \in \mathbf{H}\backslash\mathbf{R}$ and therefore $(U,\ V)$ has neither regular nor pre-injective direct summands.

Added in proof (September, 1978). In this paper, we have shown that the simple regular pairs can be parametrized by the points of the real projective plane $P_2(\mathbf{R})$. A better understanding of this fact can be accomplished as follows: Note that $P_2(\mathbf{R})$ is the projective variety corresponding to the graded ring $\mathbf{R}[X,\ Y,\ Z]/(X^2 + Y^2 + Z^2)$. Then, considering the function ring $A = \mathbf{R}[X,\ Y]/(X^2 + Y^2 + 1)$, its maximal spectrum corresponds to $P_2(\mathbf{R})$ with one point omitted.

Given an A-module M_A, observe that $M \oplus M$ can be endowed with the structure of an $\mathbf{H}$-space by defining the action of i and j in terms of

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } j = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}.$$

https://doi.org/10.4153/CJM-1978-102-6 Published online by Cambridge University Press

Thus, we can define the functor Γ from the category of all right A-modules to the category of all pairs by

$$\Gamma(M_A) = ((M \oplus 0)_{\mathbf{R}}, (M \oplus M)_{\mathbf{H}}).$$

This functor induces an equivalence between the category of all right A-modules of finite $\mathbf{R}$-dimension and the full subcategory of $\mathscr{L}({}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}})$ of all pairs (U, V) satisfying $U \oplus Ui = V$. The indecomposable pairs with this property are just the pairs $C(n, h)$ with $h \notin \mathbf{R} + \mathbf{R}i$.

References

1. I. N. Bernštejn, I. M. Gel'fand and V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspechi Mat. Nauk *28* (1973), 19–33; translated in Russian Math. Surveys *28* (1973), 17–32.
2. W. Burau, *Mehrdimensionale projektive une höhere Geometrie* (Deutscher Verlag der Wissenschaften, Berlin, 1961).
3. W. C. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, (Interscience Publ., New-London, 1962).
4. V. Dlab and C. M. Ringel, *On algebras of finite representation type*, J. Algebra *33* (1975), 306–394.
5. ——— *Indecomposable representations of graphs and algebras*, Memoirs of Amer. Math. Society No. *173* (Providence, 1976).
6. ——— *Normal forms of real matrices with respect to complex similarity*, Linear Algebra and Appl. *17* (1977), 107–124.
7. P. Gabriel, *Indecomposable representations II*, Symposia Math. Ist. Nat. Alta Mat. *11* (1973), 81–104.
8. C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra *41* (1976), 269–302.

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