

## ON COMMUTATIVITY OF ASSOCIATIVE RINGS

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In this paper we prove that if  $R$  is a ring with unity satisfying  $[xy - x^n y^m, x] = 0$ , for all  $x, y \in R$  and fixed integers  $m > 1$ ,  $n \geq 1$ , then  $R$  is commutative.

### 1. INTRODUCTION

The famous Jacobson theorem [4] that any ring in which for each ring element  $x$  there exists a positive integer  $n = n(x) > 1$  such that  $x^n = x$  is commutative, can be easily generalised as follows: if for each pair of elements in a ring  $R$  there exists a positive integer  $n = n(x, y) > 1$  such that  $(xy)^n = xy$ , then  $R$  is commutative [10]. Thus this class of rings includes the rings which satisfy the following identity:

(\*) For all  $x, y$  in  $R$  there is fixed integer  $n > 1$  such that  $x^n y^n = xy$ .

The object of this note is to investigate the commutativity of the rings satisfying condition (\*) which is certainly weaker than the condition  $(xy)^n = xy$ . In fact we prove rather a more general result:

**THEOREM.** *Let  $R$  be an associative ring with unity in which  $[xy - x^n y^m, x] = 0$  for all  $x, y \in R$  and fixed integers  $m > 1$ ,  $n \geq 1$ . Then  $R$  is commutative.*

**Remark 1.** The above theorem is also a generalisation of a theorem of Bell [2, Theorem 5], for rings with unity if  $n$  is assumed to be fixed.

**Remark 2.** It is trivial to see that not both  $m$  and  $n$  can be equal to 1 in the hypothesis of our theorem.

**Remark 3.** The ring of  $3 \times 3$  strictly upper triangular matrices over a ring provides an example showing that the existence of unity in the hypothesis of our theorem is essential.

In the remainder of the paper let us denote the centre of the ring  $R$  by  $Z(R)$ , the commutator ideal by  $C(R)$ , the set of nilpotent elements by  $N(R)$  and the set of all zero divisors in  $R$  by  $N'(R)$ . For any  $a, b \in R$ ,  $[a, b] = ab - ba$ , the well known Lie product.

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2.

The following results are pertinent in developing the proof of the above theorem. The proof of Lemma 1 can be found in [5, p. 221]. Although Lemma 2 has been proved in [6], we supply here an independent and more elementary proof.

LEMMA 1. *If  $x, y \in R$  and  $[x, y]$  commutes with  $x$ , then  $[x^n, y] = nx^{n-1}[x, y]$  for all positive integers  $n$ .*

LEMMA 2. *Suppose  $a$  and  $b$  are elements of  $R$  with unity 1, satisfying  $a^mb = 0$  and  $(1 + a)^mb = 0$  for some positive integer  $m$ . Then  $b = 0$ .*

PROOF: We have  $b = \{(1 + a) - a\}^{2m+1}b$ . On expanding the right hand side expression by the Binomial Theorem and using the fact that  $a^mb = 0$  and  $(1 + a)^mb = 0$ , we see that  $b = 0$ . ■

PROOF OF THEOREM: Since  $R$  is isomorphic to a subdirect sum of subdirectly irreducible rings  $R_\alpha$  each of which as a homomorphic image of  $R$  satisfies the property placed on  $R$ ,  $R$  itself can be assumed to be subdirectly irreducible. So  $S$ , the intersection of all non zero ideals, is non-zero. Now  $R$  satisfies  $[xy - x^ny^m, x] = 0$  for all  $x, y \in R$ , which is a polynomial identity with relatively prime integral coefficients. But if we consider  $x = e_{12}, y = e_{21}$ , we find that no ring of  $2 \times 2$  matrices over  $GF(p)$ ,  $p$  a prime, satisfies the identity. Hence by [1, Theorem 1] the commutator ideal  $C(R)$  of  $R$  is nil.

Using the hypothesis of our theorem, we get

$$(1) \quad x[x, y] = x^n[x, y^m] \text{ for all } x, y \in R.$$

By repeated use of (1), we see that for any positive integer  $p$ ,

$$\begin{aligned} x^p[x, y] &= x^{p-1}x^n[x, y^m] \\ &= x^{2n}x^{p-2}[x, (y^m)^m] \\ &= \dots \end{aligned}$$

and finally,

$$(2) \quad x^p[x, y] = x^{pn}[x, y^{m^p}].$$

Now if  $u$  is a nilpotent element of  $R$ , then  $u^{m^p} = 0$  for sufficiently large  $p$ . Using (2), we have  $x^p[x, u] = 0$ . Replace  $x$  by  $(1 + x)$ , to get  $(1 + x)^p[x, u] = 0$ . Then by Lemma 2, we get  $[x, u] = 0$  for all  $x \in R$  and hence

$$(3) \quad C(R) \subseteq N(R) \subseteq Z(R).$$

Now if  $n = 1$ , then replacing  $x$  by  $(1 + x)$  in (1) we get  $[x, y] = [x, y^m]$  that is  $[x, y - y^m] = 0$  for all  $x, y \in R$ . Hence  $R$  is commutative by a theorem of Herstein [3]. So we assume henceforth that  $n > 1$ , and we choose the positive integer  $q = 2^{n+1} - 2^2$ . Then by using (1), we have

$$\begin{aligned} qx[x, y] &= 2^{n+1}x[x, y] - 2^2x[x, y] \\ &= (2x)^n[2x, y^m] - 2x[2x, y] \\ &= 0 \end{aligned}$$

that is,  $qx[x, y] = 0$ . With  $1 + x$  in place of  $x$ , this yields  $q(1 + x)[x, y] = 0$ . On combining we get  $q[x, y] = 0$ . But since commutators are central, by employing Lemma 1, we have  $[x^q, y] = qx^{q-1}[x, y] = 0$ , which yields

$$(4) \quad x^q \in Z(R).$$

Replacing  $y$  by  $y^m$  in (1), we get

$$(5) \quad x[x, y^m] - x^n[x, (y^m)^m] = 0.$$

Since commutators are central,

$$\begin{aligned} x[x, y^m] &= [x, y^m]x \\ &= my^{m-1}[x, y]x, \text{ by (3) and Lemma 1,} \\ &= my^{m-1}x[x, y], \text{ by (3).} \end{aligned}$$

Again using (1) and (3) respectively, the above yields

$$x[x, y^m] = my^{m-1}[x, y^m]x^n.$$

Using similar techniques, we get

$$\begin{aligned} x^n[x, (y^m)^m] &= [x, (y^m)^m]x^n \\ &= m(y^m)^{m-1}[x, y^m]x^n \\ &= my^{m-1}y^{(m-1)^2}[x, y^m]x^n. \end{aligned}$$

Thus (5) gives  $my^{m-1}(1 - y^{(m-1)^2})[x, y^m]x^n = 0$ . Again the usual argument of replacing  $x$  by  $(1 + x)$ , etcetera in the last identity shows that  $my^{m-1}(1 - y^{(m-1)^2})[x, y^m] = 0$  and therefore finally we obtain,

$$(6) \quad my^{m-1}(1 - y^{q(m-1)^2})[x, y^m] = 0.$$

Next we claim that  $N'(R) \subseteq Z(R)$ . Let  $a \in N'(R)$ , then by (4)  $a^{q(m-1)^2} \in N'(R) \cap Z(R)$  and  $Sa^{q(m-1)^2} = (0)$ . Since by (6),  $ma^{m-1}(1 - a^{q(m-1)^2})[x, a^m] = 0$ , that is,  $(1 - a^{q(m-1)^2})ma^{m-1}[x, a^m] = 0$ . If  $ma^{m-1}[x, a^m] \neq 0$  then  $(1 - a^{q(m-1)^2}) \in N'(R)$  and so  $S(1 - a^{q(m-1)^2}) = 0$ , which leads to the contradiction that  $S = (0)$ . Hence  $ma^{m-1}[x, a^m] = 0$ . Using (1) and Lemma 1, we obtain

$$\begin{aligned} x^2[x, a] &= xx^n[x, a^m] \\ &= x^{2n}[x, (a^m)^m] \\ &= x^{2n}ma^{m(m-1)}[x, a^m] \\ &= x^{2n}a^{(m-1)^2}ma^{(m-1)}[x, a^m] \\ &= 0. \end{aligned}$$

This implies that  $x^2[x, a] = 0$ , and so the usual argument of replacing  $x$  by  $(1+x)$  etcetra shows that  $[x, a] = 0$  and hence,

$$(7) \quad N'(R) \subseteq Z(R).$$

Now for any  $x \in R$ ,  $x^q$  and  $x^{qm}$  are in  $Z(R)$  and for any  $y \in R$ , (1) yields

$$\begin{aligned} (x^q - x^{qm})x[x, y] &= x^qx[x, y] - x^{qm}x^n[x, y^m] \\ &= x[x, x^qy] - x^n[x, (x^qy)^m] \\ &= 0. \end{aligned}$$

Hence we have  $(x - x^{qm-q+1})x^q[x, y] = 0$ . If  $R$  is not commutative then by [3, Theorem 18], there exists an  $x \in R$  such that  $x - x^k \notin Z(R)$ , where  $k = qm - q + 1$ . Clearly  $x \notin Z(R)$ , hence neither  $x - x^k$  nor  $x$  is a zero divisor, thus  $(x - x^k)x^q$  is also not a zero divisor. Now for all  $y \in R$ ,  $(x - x^k)x^q[x, y] = 0$ , implies  $[x, y] = 0$ , which is a contradiction. Hence  $R$  is commutative. ■

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