Can. J. Math., Vol. XXX, No. 6, 1978, pp. 1215–1227

A NOTE ON UNITARY CROSS SECTIONS FOR OPERATORS

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1. Introduction. This note addresses the question of characterizing the elements of a C^*-algebra which have local unitary cross sections in the sense described below. Let $\mathscr{A}$ denote a C^*-algebra with identity and let $\mathscr{U}(\mathscr{A})$ denote the unitary group in $\mathscr{A}$. For an element X in $\mathscr{A}$, let $\mathscr{U}(X)$ denote the unitary orbit of X and let π_X denote the norm continuous mapping of $\mathscr{U}(\mathscr{A})$ onto $\mathscr{U}(X)$ defined by $\pi_X(U) = U^*XU$ (U in $\mathscr{U}(\mathscr{A})$). A *local cross section* for π_X is a pair $(\varphi_X, \mathscr{B})$ such that $\mathscr{B}$ is a relatively open subset of $\mathscr{U}(X)$ that contains X and $\varphi_X : \mathscr{B} \rightarrow \mathscr{U}(\mathscr{A})$ is a norm continuous function such that $\varphi_X(X) = 1$ and $\pi_X(\varphi_X(Y)) = Y$ for each Y in $\mathscr{B}$. If π_X has a local cross section, we say that X has a (*local unitary*) *cross section*, and in this case X clearly satisfies the following *sequential unitary lifting property*:

(P) If $\{U_n\} \subset \mathscr{U}(\mathscr{A})$ and $\lim ||U_n^*XU_n - X|| = 0$, then there exists a sequence $\{W_n\} \subset \mathscr{U}(\mathscr{A})$ such that $\lim ||W_n - 1|| = 0$ and such that $W_n^*XW_n = U_n^*XU_n$ for each n.

In [**7**] we began to study the problem of characterizing the Hilbert space operators which have cross sections or which satisfy (P). In the present note we establish the following structure theorem for operators satisfying the sequential unitary lifting property: each such operator is unitarily equivalent to an operator of the form $A \oplus B \oplus \cdots \oplus B \oplus \cdots$, where A and B are operators on finite dimensional Hilbert spaces (Theorem 2.2). This result, whose proof employs one of the deep results on C^*-algebras due to D. Voiculescu [**12**], enables us to extend the results of [**7**] in several directions.

Let $\mathscr{H}$ denote a separable infinite dimensional complex Hilbert space and let $\mathscr{A} = \mathscr{L}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. Each operator on a finite dimensional Hilbert space satisfies (P) [**7**, Corollary 2.4], and in Section 3 we prove that a compact operator in $\mathscr{L}(\mathscr{H})$ satisfies (P) if and only if it is of finite rank. It was proved in [**7**, Section 3] that a normal operator or isometry has a cross section if and only if its spectrum is finite. In Section 4 we extend this characterization to the class of hyponormal operators and we give an analogous result for hyponormal elements of the Calkin algebra. Section 5 contains a procedure for constructing non-normal operators with cross sections.

Received July 5, 1977 and in revised form, November 28, 1977. The author gratefully acknowledges support by a National Science Foundation Grant.

Theorem 2.2 also yields answers to several questions raised in [**7**]. Thus, we show that each operator satisfying (P) is algebraic and has a closed unitary orbit. On the other hand, not every algebraic operator satisfies (P) and, moreover, the class of operators satisfying (P) is not closed under similarity.

While our emphasis is on the case $\mathscr{A} = \mathscr{L}(\mathscr{H})$, we prove several results for general C^*-algebras and we point out the extent to which the results in the Hilbert space case extend to the more general setting. In this connection, the following results summarize some useful facts about the general problem. For X in $\mathscr{A}$, let $(X)'$ denote the commutant of X and let $\mathscr{V}(X) = \mathscr{U}(\mathscr{A}) \cap (X)'$; thus $\mathscr{V}(X)$ is a closed subgroup of $\mathscr{U}(\mathscr{A})$. Let p_X denote the canonical projection of $\mathscr{U}(\mathscr{A})$ onto the right coset quotient space $\mathscr{U}(\mathscr{A})/\mathscr{V}(X)$. If we endow $\mathscr{U}(\mathscr{A})/\mathscr{V}(X)$ with the quotient topology, then p_X is open and continuous. Let q_X: $\mathscr{U}(\mathscr{A})/\mathscr{V}(X) \rightarrow \mathscr{U}(X)$ be defined by $q_X(p_X(U)) = \pi_X(U)$; q_X is a well-defined continuous bijection, and we have the following commutative diagram:

$$\begin{array}{ccc} & \mathscr{U}(\mathscr{A}) & \\ \pi_X \swarrow & & \searrow p_X \\ \mathscr{U}(X) & \xleftarrow[p_X]{} & \mathscr{U}(\mathscr{A})/\mathscr{V}(X). \end{array}$$

The proof of the following result is routine diagram chasing and will be omitted; in view of our previous results, this result shows that q_X need not be a homeomorphism.

LEMMA 1.1. *The following are equivalent:*
i) *X satisfies the sequential unitary lifting property;*
ii) *π_X is an open mapping;*
iii) *q_X is a homeomorphism.*

COROLLARY 1.2. *If X has a local cross section, then p_X has a local cross section.*

We will show in Section 2 that the converse of the preceding corollary is false. Nonetheless, the existence of a local cross section for p_X can be utilized in some cases, as the following results show.

COROLLARY 1.3. *If X satisfies the sequential unitary lifting property, then X has a cross section if and only if p_X has a local cross section.*

COROLLARY 1.4. *If $\mathscr{H}$ is a finite dimensional Hilbert space and T is in $\mathscr{L}(\mathscr{H})$, then T has a local cross section.*

Proof. [**7**, Corollary 24] implies that T satisfies (P). Since $\mathscr{H}$ is finite dimensional, $\mathscr{U}(\mathscr{H})$ is a compact Lie group, so p_X has a local cross section (see [**3**, pg. 23–24]), and the result now follows from Corollary 1.3.

We conclude this section with some notation and terminology. For X in $\mathscr{A}$, $C^*(X)$ denotes the C^*-subalgebra of $\mathscr{A}$ generated by X and 1; $\sigma(X)$ denotes the spectrum of X. Let $\mathscr{S}(X)$ denote the set of all sequences $\{U_n\} \subset \mathscr{U}(\mathscr{A})$

such that $\lim U_n{}^*XU_n = X$. A sequence $\{W_n\} \subset \mathscr{U}(\mathscr{A})$ is said to *re-implement* $\{U_n\}$ for X if $\lim ||W_n - 1|| = 0$ and $W_n{}^*XW_n = U_n{}^*XU_n$ for each n. We include for completeness two additional criteria that are each equivalent to property (P):

P1) Given $\{U_n\}$ in $\mathscr{S}(X)$, there exists a subsequence $\{U_{n_k}\}$ that can be re-implemented.

P2) Given $\epsilon > 0$, there exists $\delta > 0$, such that if U is in $\mathscr{U}(\mathscr{A})$ and $||U^*XU - X|| < \delta$, then there exists W in $\mathscr{U}(\mathscr{A})$ such that $||W - 1|| < \epsilon$ and $W^*XW = U^*XU$.

The proofs of these equivalences are the same as in the Hilbert space case [**7**, Lemma 2.3]. For $\epsilon > 0$, we denote by $\mathscr{B}(X, \epsilon)$ the set $\{Y \in \mathscr{U}(X) : ||X - Y|| < \epsilon\}$. In the sequel we will use the fact that an orthogonal projection P always has a cross section of the form $(\rho, \mathscr{B}(P, 1))$ (see [**7**, Theorem 3.2] and its proof).

All Hilbert spaces that we consider are complex and separable, unless otherwise noted, and $\mathscr{H}$ denotes an infinite dimensional separable Hilbert space. For a closed subspace $\mathscr{M}$ of a Hilbert space, $P_{\mathscr{M}}$ denotes the projection onto $\mathscr{M}$. If $\mathscr{K}(\mathscr{H})$ denotes the ideal of all compact operators in $\mathscr{L}(\mathscr{H})$, then the Calkin algebra is the quotient $\mathscr{C} = \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, and we let $\pi:\mathscr{L}(\mathscr{H}) \to \mathscr{C}$ denote the canonical projection. For T in $\mathscr{L}(\mathscr{H})$, we set $\tilde{T} = \pi(T)$, $\sigma_e(T) = \sigma(\tilde{T})$ (the essential spectrum of T), and we let Re (T) denote the set of all reducing essential eigenvalues of T [**10**]. For an operator S on a Banach space, the range and null space will be denoted, respectively, by $\mathscr{R}(S)$ and ker (S).

The author is grateful to the referee for helpful comments and suggestions. The author also appreciatively acknowledges the contributions of R. J. Zimmer, who indicated the connections to Lie groups and who furnished the proof of Corollary 1.4 and the example following Corollary 2.3.

2. Structure of operators satisfying property (P). In this section we show that each operator satisfying the sequential unitary lifting property admits a direct sum decomposition of the form $A \oplus (B \otimes 1_{\mathscr{H}})$, where A and B are operators on finite dimensional Hilbert spaces. We begin with a result for the case of a general C^*-algebra $\mathscr{A}$. We show that if an element in $\mathscr{A}$ satisfies (P), then its unitary orbit is norm closed; this result answers [**7**, Question 2] affirmatively.

PROPOSITION 2.1. *If X is in $\mathscr{A}$ and X satisfies the sequential unitary lifting property, then $\mathscr{U}(X)$ is norm closed in $\mathscr{A}$.*

Proof. Since X satisfies (P), property P2) implies that for $\epsilon_n = 1/2^n$ there exists $\delta_n > 0$ such that if U is in $\mathscr{U}(\mathscr{A})$ and $||U^*XU - X|| < \delta_n$, then there exists W in $\mathscr{U}(\mathscr{A})$ such that $||W - 1|| < 1/2^n$ and $W^*XW = U^*XU$. Let Y be in $\mathscr{U}(X)^-$ and suppose that $\{U_k\}$ is a sequence in $\mathscr{U}(\mathscr{A})$ such that $\lim U_k{}^*XU_k = Y$. For $n > 0$, there exists $k_n > 0$ such that if $k \geqq k_n$, then

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$\|U_k{}^*XU_k - Y\| < (1/2)\delta_n$; moreover, we may assume that $k_{n+1} > k_n > \cdots > k_1$. For each n, since $k_{n+1} > k_n$, we have

$$\|U_{k_{n+1}}{}^*XU_{k_{n+1}} - U_{k_n}{}^*XU_{k_n}\| \leqq \|U_{k_{n+1}}{}^*XU_{k_{n+1}} - Y\| + \|Y - U_{k_n}{}^*XU_{k_n}\| < 2(1/2)\delta_n = \delta_n.$$

Thus

$$\|U_{k_n}U_{k_{n+1}}{}^*XU_{k_{n+1}}U_{k_n}{}^* - X\| < \delta_n,$$

so there exists W_n in $\mathscr{U}(\mathscr{A})$ such that $\|W_n - 1\| < 1/2^n$ and $W_n{}^*XW_n = U_{k_n}U_{k_{n+1}}{}^*XU_{k_{n+1}}U_{k_n}{}^*$. If $V_n = U_{k_n}{}^*W_nU_{k_n}$, then $\|V_n - 1\| < 1/2^n$ and $V_n{}^*U_{k_n}{}^*XU_{k_n}V_n = U_{k_{n+1}}{}^*XU_{k_{n+1}}$; thus, by repeated substitution, we have

$$U_{k_{n+1}}{}^*XU_{k_{n+1}} = V_n{}^*V_{n-1}{}^* \cdots V_1{}^*U_{k_1}{}^*XU_{k_1}V_1 \cdots V_{n-1}V_n.$$

Since $V_n = 1 + (V_n - 1)$ and $\sum_{n=1}^{\infty} \|V_n - 1\| < \infty$, then [**4**, Theorem 2, page 213] implies that

$$V = \prod_{n=1}^{\infty} V_n \quad \left(= \lim_{n\to\infty} (V_1 \cdots V_n)\right)$$

is norm convergent, so that V is unitary. Thus

$$Y = \lim U_{k_{n+1}}{}^*XU_{k_{n+1}} = \lim (V_n{}^* \cdots V_1{}^*)U_{k_1}{}^*XU_{k_1}(V_1 \cdots V_n) = V^*U_{k_1}{}^*XU_{k_1}V,$$

so that Y is in $\mathscr{U}(X)$, and the proof is complete.

In the Hilbert space case ($\mathscr{A} = \mathscr{L}(\mathscr{H})$), the preceding result leads to the above mentioned structure theorem for operators satisfying property (P).

THEOREM 2.2. *Let T be in $\mathscr{L}(\mathscr{H})$ and suppose that T satisfies the sequential unitary lifting property. Then there exist finite dimensional Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and operators A in $\mathscr{L}(\mathscr{H}_1)$, B in $\mathscr{L}(\mathscr{H}_2)$, such that T is unitarily equivalent to $A \oplus (B \otimes 1_{\mathscr{H}})$ $(= A \oplus B \oplus \cdots \oplus B \oplus \cdots)$.*

Proof. Since T satisfies (P), Proposition 2.1 implies that $\mathscr{U}(T)$ is closed in $\mathscr{L}(\mathscr{H})$. Now by a theorem of D. Voiculescu [**12**, Proposition 2.4], $\mathscr{U}(T)$ is closed if and only if $C^*(T)$ is finite dimensional. The latter condition is equivalent to the property that T is an orthogonal direct sum of copies of a finite number of finite dimensional operators $F_1, \ldots, F_n$ [**1**, section 3]. Suppose, after renumbering, that $F_1, \ldots, F_m (1 \leqq m \leqq n)$ are the operators amongst the F_i's that appear as direct summands infinitely many times in the above decomposition of T. We may then set $B = \sum_{i=1}^{m} \oplus F_i$ and let A be the direct sum of $F_{m+1}, \ldots, F_n$, each repeated as a direct summand according to its (finite) multiplicity as a summand in the original decomposition.

COROLLARY 2.3. *If T satisfies the sequential unitary lifting property, then T is reducible and algebraic, and* $\dim (C^*(T)) < \infty$.

Remark. From the preceding corollary and Lemma 1.1 it follows that if an operator T in $\mathscr{L}(\mathscr{H})$ is irreducible, then π_T is not an open mapping and q_T is not a homeomorphism; in particular, T does not have a local unitary cross section. On the other hand, in this case p_T does have a local cross section since

$$(T)' \cap \mathcal{U}(\mathcal{H}) = \{z1_{\mathcal{H}}: |z| = 1\},$$

a compact Lie group [**3**, Theorem 5.8, page 88] (cf., the remark following Corollary 1.2). In contrast to Corollary 2.3, not every algebraic operator satisfies (P); this is discussed more fully in Example 5.2 (below).

3. Compact operators. In this section we characterize the compact operators which satisfy the sequential unitary lifting property.

THEOREM 3.1. *A compact operator T in $\mathcal{L}(\mathcal{H})$ satisfies the sequential unitary lifting property if and only if it is of finite rank.*

Proof. Let T be a finite rank operator; let $\mathcal{H}_1 = (\ker (T))^\perp$ and $\mathcal{H}_2 = \ker (T)$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, T has an operator matrix of the form

$$\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix},$$

and since T has closed range, $J = A^*A + B^*B$ is invertible. Let $\{U_n\} \in \mathcal{S}(T)$; with respect to the above decomposition, U_n has an operator matrix of the form

$$\begin{bmatrix} X_n & Y_n \\ Z_n & W_n \end{bmatrix}.$$

Since $||TU_n - U_nT|| = ||TU_n^* - U_n^*T|| \to 0$, matrix calculations yield the following limits:

I-i) $AX_n - X_nA - Y_nB \to 0$;
ii) $BX_n - Z_nA - W_nB \to 0$;
iii) $AY_n \to 0$;
iv) $BY_n \to 0$;
II-i) $AX_n^* - X_n^*A - Z_n^*B \to 0$;
ii) $BX_n^* - Y_n^*A - W_n^*B \to 0$;
iii) $AZ_n^* \to 0$;
iv) $BZ_n^* \to 0$.

Now I-iii)-iv) imply that $A^*AY_n + B^*BY_n \to 0$, and since J is invertible, we have $Y_n \to 0$; similarly, using II-iii)-iv) it follows that $Z_n \to 0$. We thus have the limits $X_n^*X_n = 1 - Z_n^*Z_n \to 1$, $X_nX_n^* = 1 - Y_nY_n^* \to 1$, and, likewise, $W_n^*W_n \to 1$ and $W_nW_n^* \to 1$. For each n, $||X_n|| \leqq 1$, and $\mathcal{H}_1$ is finite dimensional; thus, by passing to a subsequence and using property P1), we may assume that $\{X_n\}$ is norm convergent to some operator X in $\mathcal{L}(\mathcal{H}_1)$. Since $X_n^*X_n \to 1$ and $X_nX_n^* \to 1$, X is unitary, and I-i) implies that X commutes with A.

Since the unit ball of $\mathcal{L}(\mathcal{H}_2)$ is compact and metrizable in the weak operator topology, by passing to a further subsequence we may assume there exists W

in $\mathcal{L}(\mathcal{H}_2)$ such that $W_n \to_w W$. Thus $W_nB \to_w WB$, and since $Z_n \to 0$, I-ii) implies the following relations:

III-i) $WB = BX$;

ii) $W_nB \to WB$.

Let P in $\mathcal{L}(\mathcal{H}_2)$ and Q in $\mathcal{L}(\mathcal{H}_1)$ denote, respectively, the projections onto the range of B and the initial space of B ($\mathcal{H}_1 \ominus \ker(B)$). We claim that

iii) $||(W_n - W)|P\mathcal{H}_2|| \to 0$.

Since B has closed range, there exists $\delta > 0$ such that $||BQt|| \geqq \delta||Qt||$ (t in $\mathcal{H}_1$). Now

$$||(W_n - W)Bt|| = ||(W_n - W)BQt|| \leqq ||(W_n - W)B|| \quad ||Qt||$$
$$\leqq ||(W_n - W)B||(1/\delta)||BQt|| = ||(W_n - W)B||(1/\delta)||Bt||,$$

and III-ii) implies the claimed convergence.

Taking adjoints, we have $W_n^* \to_w W^*$, and arguments analogous to those preceding (but using II-ii) give the following relations:

IV-i) $W^*B = BX^*$;

ii) $BX_n^* - W_n^*B \to 0$;

iii) $||(W_n^* - W^*)|P\mathcal{H}_2|| \to 0$.

Now $WB\mathcal{H}_1 = BX\mathcal{H}_1 \subset B\mathcal{H}_1$ and $W^*B\mathcal{H}_1 = BX^*\mathcal{H}_1 \subset B\mathcal{H}_1$, so the range of B reduces W; moreover, since $W^*WB = W^*BX = BX^*X = B$ and $WW^*B = WBX^* = BXX^* = B$, then $C = W|B\mathcal{H}_1$ is unitary.

Our aim is to use the unitary operators X and C in the construction of a re-implementing sequence for $\{U_n\}$. To this end, we set $\mathcal{H}_2 = \mathcal{H}_3 \oplus \mathcal{H}_4$, with $\mathcal{H}_3 = P\mathcal{H}_2$; then the operator matrices of W_n and W relative to this decomposition assume the form

$$W_n = \begin{bmatrix} C_n & D_n \\ E_n & F_n \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} C & 0 \\ 0 & F \end{bmatrix}.$$

Since $||(W_n - W)|\mathcal{H}_3|| \to 0$, $E_n \to 0$; similarly IV-iii) implies $||D_n|| = ||D_n^*|| \to 0$. Thus, since $W_n^*W_n \to 1$ and $W_nW_n^* \to 1$, we have $F_n^*F_n \to 1$ and $F_nF_n^* \to 1$, and we may assume that F_n is invertible. If $F_n = V_nP_n$ denotes the polar decomposition of F_n, then V_n is unitary and $||F_n - V_n|| = ||V_n(P_n - 1)|| = ||P_n - 1|| \to 0$. Let S_n be the unitary operator on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ defined by $S_n = X^* \oplus C^* \oplus V_n^*$, and let $R_n = C \oplus V_n$. Since X^* commutes with A and $R_n^*B = W^*B = BX^*$, a matrix calculation shows that S_n commutes with T.

The re-implementing sequence for $\{U_n\}$ may now be defined by $T_n = S_nU_n$; evidently $\{T_n\} \subset \mathcal{U}(\mathcal{H})$ and $T_n^*TT_n = U_n^*TU_n$. It thus suffices to verify that $T_n \to 1$. The operator matrix of $T_n = S_nU_n$ is of the form

$$\begin{bmatrix} X^*X_n & X^*Y_n \\ R_n^*Z_n & R_n^*W_n \end{bmatrix},$$

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and we have $X^*X_n \to 1$, $||X^*Y_n|| = ||Y_n|| \to 0$, and $||R_n^*Z_n|| = ||Z_n|| \to 0$. Moreover, the matrix of $R_n^*W_n$ is

$$\begin{bmatrix} C^*C_n & W^*D_n \\ V_n^*E_n & V_n^*F_n \end{bmatrix},$$

and we have $||W^*D_n|| = ||D_n|| \to 0$, $||V_n^*E_n|| = ||E_n|| \to 0$, and $||V_n^*F_n - 1|| = ||P_n - 1|| \to 0$. Finally, since $||(W_n - W)|\mathscr{H}_3|| \to 0$, $C_n \to C$, and since C is unitary we have $||C^*C_n - 1|| \to 0$, which completes the proof that T satisfies the sequential unitary lifting property.

For the converse, if T is an infinite rank compact operator, then it follows easily that $C^*(T)$ is infinite dimensional, so the result follows from Corollary 2.3.

The same argument yields the following result.

COROLLARY 3.2. *If T is a compact, infinite rank operator which is a direct summand of an operator S, then S does not satisfy the sequential unitary lifting property.*

Remark 3.3. In [**7**] it was proved that each irreducible compact operator fails to satisfy property (P). By a rather lengthy elaboration of the method of [**7**] it is possible to prove the "converse" direction of Theorem 3.1 without recourse to Corollary 2.3. Our original proof of Theorem 3.1 used this more elementary, if lengthier, approach. Indeed, the original proofs of the results in this section and the next provided the motivation for Theorem 2.2. and Corollary 2.3.

4. Hyponormal operators. In [**7**] it was proved that a normal operator or an isometry satisfies the sequential unitary lifting property if and only if its spectrum is finite, in which case it has a local unitary cross section. In the present section we extend this result to hyponormal operators and give an analogue of this result for hyponormal elements of the Calkin algebra.

PROPOSITION 4.1. *If* Re (T) *is infinite, then T does not satisfy the sequential unitary lifting property.*

Proof. Theorem 2.2 implies that the spectrum of each operator satisfying (P) is finite; since, from [**10**], Re $(T) \subset \sigma(T)$, the result follows immediately.

PROPOSITION 4.2. *If T in $\mathscr{L}(\mathscr{H})$ is hyponormal, then T satisfies* (P) *if and only if $\sigma(T)$ is finite, in which case T has a local unitary cross section.*

Proof. If $\sigma(T)$ is finite, then [**9**, Theorem 1] implies that T is normal, and the existence of a cross section follows from [**7**, Theorem 3.2]. If $\sigma(T)$ is infinite, then Theorem 2.2 implies that T does not satisfy (P), so the proof is complete.

To obtain analogues of the preceding results for the Calkin algebra we rely on the following lemma.

LEMMA 4.3. *If T is in $\mathcal{L}(\mathcal{H})$ and* Re (T) *is infinite, then there exists an element* $\{W_n\}$ *of* $\mathcal{S}(T)$ *such that if* $\tilde{V}$ *is unitary and* $\tilde{V}^*\tilde{T}\tilde{V} = \tilde{W}_n^*\tilde{T}\tilde{W}_n$ *for some* n, *then* $\|\tilde{V} - \lambda\| \geqq 1$ *for all scalars* λ *such that* $|\lambda| = 1$.

Proof. Let $\{\lambda_n\}$ denote a convergent sequence of distinct elements of Re (T). Let $\mathcal{H}_i$ denote a copy of $\mathcal{H}$, let

$$\mathcal{H}_\infty = \sum_{i=1}^{\infty} \oplus \mathcal{H}_i, \quad \text{and} \quad D = \sum_{i=1}^{\infty} \oplus \lambda_i.$$

Let S denote the operator on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_\infty$ given by $S = T \oplus D$; then [**10**, Theorem 4.6] implies that there exists a sequence of unitary operators $U_n : \mathcal{H} \to \mathcal{K}$ such that if $T_n = U_n^*SU_n$, then $\|T_n - T\| < 1/n$ and $K_n = T_n - T$ is compact. We now define the unitary operator X_n on $\mathcal{K}$ by the following relations: $X_ne_k = f_k$; $X_nf_k = e_k$, for each k, where $\{e_k\}$ and $\{f_k\}$ denote, respectively, orthonormal bases for $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$; also, X_n is the identity on $\mathcal{K} \ominus (\mathcal{H}_n \oplus \mathcal{H}_{n+1})$. If we set $W_n = U_n^*X_nU_n$, then

$$\|W_n^*T_nW_n - T_n\| = \|U_n^*X_n^*SX_nU_n - U_n^*SU_n\| = \|X_n^*SX_n - S\| = |\lambda_{n+1} - \lambda_n|.$$

Now

$$\|W_n^*TW_n - T\| \leqq \|W_n^*TW_n - W_n^*T_nW_n\| + \|W_n^*T_nW_n - T_n\| + \|T_n - T\| < 2/n + |\lambda_{n+1} - \lambda_n| \to 0$$

so that $\{W_n\}$ is a member of $\mathcal{S}(T)$.

Suppose that $\tilde{V}$ is unitary and that $V^*TV - W_n^*TW_n$ is compact for some n. Since $V^*T_nV - W_n^*T_nW_n = V^*(T + K_n)V - W_n^*(T + K_n)W_n$, then $\widetilde{V^*T_nV} = \widetilde{W_n^*T_nW_n}$, whence $\widetilde{VW_n^*}$ commutes with $\tilde{T}_n$. Since

$$VU_n^*X_nSU_n - U_n^*SU_nVU_n^*X_nU_n = VU_n^*X_n^*U_nU_n^*SU_n - U_n^*SU_nVU_n^*X_nU_n,$$

which is compact, then $U_nVU_n^*X_nS - SU_nVU_n^*X_n$ is compact. Let $W = U_nVU_n^*$; with respect to the decomposition $\mathcal{K} = \mathcal{L}_n \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1}$, the operator matrices of S, X_n, and W are, respectively, of the following form:

$$S = \begin{bmatrix} Y_n & 0 & 0 \\ 0 & \lambda_n & 0 \\ 0 & 0 & \lambda_{n+1} \end{bmatrix}; \quad X_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$W = (V_{ij})_{1 \leqq i,j \leqq 3}$. As $L = WX_nS - SWX_n$ is compact, a calculation of the row 2, column 3 entry in the matrix of L shows that $(\lambda_{n+1} - \lambda_n)V_{22} = V_{22}\lambda_{n+1} - \lambda_nV_{22}$ is compact; since the λ_n's are distinct, V_{22} is compact. Let $(K_{ij})_{1 \leqq i,j \leqq 3}$ denote the operator matrix of any compact operator K relative to the above decomposition of $\mathcal{K}$, and let λ be a scalar such that $|\lambda| = 1$. Now

$$\|W - \lambda + K\| \geqq \|V_{22} - \lambda + K_{22}\| \geqq \|\tilde{V}_{22} - \lambda\| = |\lambda| = 1;$$

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thus, for $|\lambda| = 1$ we have $\|\tilde{V} - \lambda\| = \|\widetilde{U_n V U_n^*} - \lambda\| \geqq 1$, and the proof is complete.

COROLLARY 4.4. *If* Re (T) *is infinite, then* $\tilde{T}$ *does not satisfy the sequential unitary lifting property relative to the Calkin algebra.*

We state for ease of reference the following result of [**7**]. (Although this result was stated in [**7**, Theorem 3.2] only for the separable case, it is easy to see that the same proof holds regardless of the dimension.)

LEMMA 4.5. *Let* N *be a normal operator on the complex Hilbert space* $\mathscr{K}$. *If* $\sigma(N)$ *is finite, then* N *has a local cross section* $(\rho, \mathscr{B})$ *such that for each* S *in* $\mathscr{B}$, $\rho(S)$ *is contained in the norm closed algebra generated by* N, S, *and* 1.

PROPOSITION 4.6. *Let* $\mathscr{A}$ *be a* C^**-algebra with identity. If* X *is a normal element of* $\mathscr{A}$ *and* $\sigma(X)$ *is finite, then* X *has a local unitary cross section.*

Proof. Let $\tau : \mathscr{A} \to \mathscr{L}(\mathscr{K})$ denote an isometric C^*-isomorphism of $\mathscr{A}$ onto a C^*-subalgebra $\mathscr{E}$ of $\mathscr{L}(\mathscr{K})$ (where $\mathscr{K}$ is a Hilbert space of suitable dimension). Let $\mathscr{U}$ denote the unitary group in $\mathscr{A}$; set $\mathscr{V} = \tau(\mathscr{U})$ and $\mathscr{W} = \tau(\mathscr{U}(X))$ (note that $\mathscr{V} = \mathscr{U}(\mathscr{K}) \cap \mathscr{E}$). Let $Z = \tau(X)$; since $\sigma(Z) = \sigma(X)$, Lemma 4.5 implies that Z, as a normal operator on $\mathscr{K}$, has a local unitary cross section $(\psi, \mathscr{D})$ such that for each Y in $\mathscr{D}$, $\psi(Y)$ is in the norm closed algebra generated by Z, Y, and 1. In particular, if Y is in $\mathscr{W} \cap \mathscr{D}$, then $\psi(Y)$ is in $\mathscr{V}$. Thus, since $\tau^{-1} : \mathscr{E} \to \mathscr{A}$ is an isometric C^*-isomorphism, a local unitary cross section for X may be defined by $(\rho, \mathscr{B})$, where $\mathscr{B} = \tau^{-1}(\mathscr{W} \cap \mathscr{D})$ and $\rho = \tau^{-1}\psi\tau|\mathscr{B}$.

We note that the converse of Proposition 4.6 is false; each element of a commutative C^*-algebra has a cross section. For T in $\mathscr{L}(\mathscr{H})$, $\tilde{T}$ is hyponormal if $\pi(T^*T - TT^*)$ is positive.

COROLLARY 4.7. *If* $\tilde{T}$ *is hyponormal, then* $\tilde{T}$ *satisfies property* (P) (*relative to the Calkin algebra*) *if and only if* $\sigma_e(T)$ *is finite, in which case* $\tilde{T}$ *has a local unitary cross section.*

Proof. Since $\tilde{T}$ is hyponormal, [**10**, Theorem 3.10] implies that bdry $(\sigma_e(T)) \subset$ Re (T); thus, if $\sigma_e(T)$ is infinite, Corollary 4.4 implies that $\tilde{T}$ does not satisfy (P). Conversely, if $\sigma_e(T)$ is finite, then it follows (e.g., by representing the Calkin algebra as an operator algebra and applying [**9**, Theorem 1]) that $\tilde{T}$ is normal, so the result follows from Proposition 4.6.

Remark. The proof of Proposition 4.6 shows that the cross sections of Proposition 4.6 and Corollary 4.7 also have the additional property possessed by the cross section $(\rho, \mathscr{B})$ of Lemma 4.5.

Let T be an essentially normal operator and let α_T denote the restriction of π to $\mathscr{U}(T)$. A question of Brown, Douglas, and Fillmore [**5**, page 121] asks whether α_T is an open mapping. It is easily verified that α_T is open if and only if T satisfies the following property:

(R) If $\{U_n\} \subset \mathscr{U}(\mathscr{H})$ and $\pi(U_n{}^*TU_n) \rightarrow \pi(T)$, then there exists a sequence $\{V_n\} \subset \mathscr{U}(\mathscr{H})$ such that $V_n{}^*TV_n \rightarrow T$ and $\pi(V_n{}^*TV_n) = \pi(U_n{}^*TU_n)$ for each n.

For the case when $\sigma_e(T)$ if finite, we next show that T satisfies a property that is stronger than property (R).

PROPOSITION 4.8. *Let T be an essentially normal operator whose essential spectrum is finite. If $\{U_n\} \subset \mathscr{L}(\mathscr{H})$, $\pi(U_n)$ unitary for each n, and $\pi(U_n{}^*TU_n) \rightarrow \pi(T)$, then there exists $\{V_n\} \subset \mathscr{U}(\mathscr{H})$ such that $V_n \rightarrow 1$ and $\pi(V_n{}^*TV_n) = \pi(U_n{}^*TU_n)$; in particular, α_T is an open mapping.*

Proof. Suppose that $\{U_n\}$ satisfies the above hypotheses; since $\sigma_e(T)$ is finite, Corollary 4.7 implies that there exists a sequence $\{W_n\}$ of essentially unitary operators such that $\lim \pi(W_n) = 1$ and $\pi(W_n{}^*TW_n) = \pi(U_n{}^*TU_n)$ for each n. In particular, there exists a sequence $\{K_n\} \subset \mathscr{K}(\mathscr{H})$ such that $W_n + K_n \rightarrow 1$. If $W_n + K_n = V_nP_n$ denotes the polar decomposition of $W_n + K_n$, we may thus assume that V_n is unitary and P_n is invertible. Since $P_n{}^2 = (W_n + K_n)^*(W_n + K_n) \rightarrow 1$ then $P_n \rightarrow 1$, and so $P_n{}^{-1} \rightarrow 1$, which in turn implies that $V_n \rightarrow 1$. Now $\pi(P_n{}^2) = \pi(P_nV_n{}^*V_nP_n) = \pi(W_n{}^*W_n) = 1$, and it follows that there is a compact operator J_n such that $P_n = 1 + J_n$. Thus $\pi(W_n) = \pi(V_n)\pi(1 + J_n) = \pi(V_n)$, so $\pi(V_n{}^*TV_n) = \pi(W_n{}^*TW_n) = \pi(U_n{}^*TU_n)$, which completes the proof.

Remark. Some of the results of this section are analogues for operators with cross sections of certain results pertaining to operators which generate an inner derivation having closed range. Thus Proposition 4.1 and Corollary 4.4 are analogues of [**2**, Proposition 1] and [**2**, Proposition 2] respectively, while Proposition 4.2 corresponds to [**11**, Theorem 1]. Moreover, Theorem 3.1 parallels [**2**, Theorem 2]. However, as we show in Section 5, this correspondence breaks down for arbitrary operators.

5. Some non-normal operators with cross sections. In [**7**, Theorem 2.7] it was proved that if T_i is in $\mathscr{L}(\mathscr{H}_i)$, $1 \leqq i \leqq n$, and $\sigma(T_i) \cap \sigma(T_j) = \emptyset$ for $i \neq j$, then $\sum_{i=1}^n \oplus T_i$ has a cross section if and only if each T_i has a cross section. In this section we give an additional procedure for constructing operators with cross sections. For operators T in $\mathscr{L}(\mathscr{H}_1)$ and S in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that $S^*S + T^*T$ is invertible, let $M = M(T, S)$ denote the (possibly non-normal) operator on $\mathscr{H}_1 \oplus \mathscr{H}_2$ whose operator matrix is of the form

$$\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}.$$

Of course, the above hypotheses on S and T are equivalent to the conditions that $\mathscr{R}(M)$ is closed and $\ker(M) = \mathscr{H}_2$. Note also that each unitary operator that commutes with M is reduced by $\mathscr{H}_1$.

THEOREM 5.1. *Suppose S has closed range and S^*S commutes with every unitary operator that T commutes with. If T has a local unitary cross section, then so does M.*

Remark. The hypothesis on S^*S is equivalent to the condition that S^*S belongs to the von Neumann algebra generated by T [**6**, page 4]. If S is right invertible and an element of $C^*(T)$, the theorem reduces to [**7**, Theorem 4.2]; the case when S is an isometry corresponds to [**7**, Theorem 4.3].

Before proving Theorem 5.1 we give an example which shows that some hypothesis on S is required in order to conclude that M has a cross section.

Example 5.2. Let S be such that $C^*(S^*S)$ is infinite dimensional and let $T = 1_{\mathscr{H}_1}$. Obviously T has a cross section, but we assert that M does not. Although direct (but lengthy) proofs of this can be given for specific examples of such operators S, we prefer to rely on Corollary 2.3. Indeed, since $C^*(S^*S)$ is infinite dimensional, then so is $C^*(M^*M)$. Thus $C^*(M)$ is infinite dimensional and it follows that M does not satisfy the sequential unitary lifting property.

It is instructive to compare this example and Theorem 5.1 to the results of [**1**]. In [**1**] C. Apostol characterized the class of operators T which generate an inner derivation $\Delta(T)$ having closed range in $\mathscr{L}(\mathscr{H})$. Although a hyponormal or compact operator is in this class if and only if it has a cross section (see the remark at the end of Section 4), this class actually differs from the class of operators having cross sections. Indeed, the former class is clearly closed under similarity; however, the operator M of Example 5.2 is similar to an operator with a cross section, namely, the projection onto $\mathscr{H}_1$. Further, [**1**, Lemma 3.3] states that $\mathscr{R}(\Delta(M(T, S)))$ is closed if and only if $\mathscr{R}(\Delta(T))$ is closed, while Example 5.2 shows that the analogue of this result for operators with cross sections is false. In showing that algebraic operators need not satisfy (P), Example 5.2 answers [**7**, Question 4] negatively.

Proof of Theorem 5.1. Since the case when $S = 0$ may be proved by a straightforward modification of the following argument, we assume that $S \neq 0$. Let $Q = P_{\ker(S^*)}$ and let $(\gamma, \mathscr{B}(Q, 1))$ denote a cross section for Q. We first note a property of γ that will be used somewhat later in the proof. Since SS^* has closed range, the mapping $g: \mathscr{U}(SS^*) \rightarrow \mathscr{L}(\mathscr{H})$, defined by $g(R) = P_{\ker(R)}$, is norm continuous (see e.g. [**7**, Lemma 4.1]). Thus there exists $\delta_1 > 0$ such that if R is in $\mathscr{B}(SS^*, \delta_1)$, then $||g(R) - Q|| < 1$, so that $\gamma(g(R))$ is defined.

Proceeding as in the proof of [**7**, Theorem 4.3] we show how to continuously replace an element of $\mathscr{U}(M)$ (sufficiently close to M) by a nearby element of $\mathscr{U}(M)$ whose kernel is exactly $\mathscr{H}_2$ ($= \ker(M)$). Let $(\varphi, \mathscr{B}(P, 1))$ denote a cross section for the projection, P, onto $\mathscr{H}_2$. Let $(\psi, \mathscr{B}(T, \epsilon))$ denote a cross section for T with $\epsilon < \delta_1/(2||S||)$. Since M has closed range it follows as above that there exists $\delta_2 > 0$ such that if X is in $\mathscr{B}(M, \delta_2)$, then $||P_{\ker(X)} - P|| < 1$, so that $V = \varphi(P_{\ker(X)})$ is defined. Now we have

$$||VXV^* - M|| \leqq 2||V - 1|| \, ||M|| + ||X - M||;$$

thus, since the composite mapping $X \rightarrow \varphi(P_{\ker(X)})$ (X in $\mathscr{B}(M, \delta_2)$) is norm continuous, there exists $\delta > 0$, with $\delta < \delta_2$, such that if X is in $\mathscr{B}(M, \delta)$, then $||VXV^* - M|| < \epsilon$. It follows as in the proof of [**7**, Theorem 4.3] that $\ker(VXV^*) = \ker(M) = \mathscr{H}_2$ and that there exist unitary operators Z and Y, acting on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, such that $VXV^* = M(T_1, S_1)$, with $T_1 = Z^*TZ$ and $S_1 = Y^*SZ$.

Now $||Z^*TZ - T|| \leqq ||VXV^* - M|| < \epsilon$, and thus $\psi(T_1)$ is defined; set $U = Y^*SZ\psi(T_1)^*$. We define a mapping W_1: $\mathscr{R}(S) \rightarrow \mathscr{R}(U)$ by the formula $W_1(St) = Ut$ (t in $\mathscr{H}_1$). Since $Z\psi(T_1)^*$ commutes with T, it commutes with S^*S, and it follows from this that W_1 is well-defined, linear, isometric, and onto. We next define a suitable unitary extension of W_1 to all of $\mathscr{H}_2$. We have $(\mathscr{R}(S))^\perp = \ker(S^*) = \ker(SS^*)$ and $(\mathscr{R}(U))^\perp = \ker(U^*) = \ker(UU^*) = \ker(Y^*SS^*Y)$. Now

$$||Y^*SS^*Y - SS^*|| = ||S_1S_1^* - SS^*|| \leqq 2||S_1 - S|| \, ||S|| < 2\epsilon ||S|| < \delta_1,$$

and thus

$$\gamma(P_{\ker(U^*)})^* P_{\ker(S^*)} = P_{\ker(U^*)} \, \gamma(P_{\ker(U^*)})^*.$$

Since $\gamma(P_{\ker(U^*)})^*$ maps $(\mathscr{R}(S))^\perp$ isometrically onto $(\mathscr{R}(U))^\perp$, $\gamma(P_{\ker(U^*)})^*$ may be used to extend W_1 to a unitary operator W in $\mathscr{L}(\mathscr{H}_2)$ such that $WS\psi(T_1) = Y^*SZ = S_1$(*).

We set $\tau(X) = (\psi(T_1) \oplus W)\varphi(P_{\ker(X)})$ and we claim that $(\tau, \mathscr{B}(M, \delta))$ is a cross section for M. Easy matrix calculations (using (*)) show that $\pi_M\tau = 1_{\mathscr{B}(M,\delta)}$ and that $\tau(M) = 1_{\mathscr{H}_1 \oplus \mathscr{H}_2}$. The proof of the continuity of τ will be omitted since it is very similar to the proof when S is an isometry [**7**, Theorem 4.3]. (To modify the argument given in [**7**] we need only note that S, having closed range, is bounded below on $\mathscr{H}_1 \ominus \ker(S)$. It follows readily from this fact that $W|\mathscr{R}(S)$ varies continuously with X; the rest of the proof is identical to that given in [**7**].)

6. Conclusion. We note that by using our previous results, the converse of Theorem 2.2 can be established for certain choices of (finite dimensional) operators A and B. Thus if A and B are normal, then $T = A \oplus (B \otimes 1_{\mathscr{H}})$ is a normal operator with finite spectrum and thus T has a cross section. Moreover, T is a finite rank operator if and only if $B = 0$, so in this case T satisfies property (P). In [**7**, Section 4] it was proved that if A is absent and B acts on a one or two dimensional space, then T has a cross section. Using Theorem 5.1 it is not difficult to prove a similar result in case B acts on a three dimensional space. Based on these results we conjecture the following characterization of operators with cross sections.

CONJECTURE. *For T in $\mathscr{L}(\mathscr{H})$, the following are equivalent:*
1) *T satisfies the sequential unitary lifting property;*
2) *T has a local unitary cross section;*
3) *$\mathscr{U}(T)$ is closed in $\mathscr{L}(\mathscr{H})$.*

Of course, other conditions that are equivalent to (1) are given in Lemma 1.1. Also, as mentioned in the proof of Theorem 2.2, condition (3) is known to be equivalent to each of the following:

4) $C^*(T)$ is finite dimensional [**12**];
5) T is unitarily equivalent to $A \oplus (B \otimes 1_{\mathscr{H}})$ where A and B are operators on finite dimensional Hilbert spaces [**1**].

We also observe that the analogue of the above conjecture for arbitrary C^*-algebras is false. In the case of the Calkin algebra, Corollary 4.7 shows that if $\tilde{T}$ is normal, then $\tilde{T}$ satisfies (P) if and only if $\sigma(\tilde{T})$ is finite, in which case $\tilde{T}$ has a cross section However, it follows from a result of Brown, Douglas, and Fillmore concerning the index invariant for essentially normal operators [**5**, page 63], that if $\tilde{T}$ is normal, then $\mathscr{U}(\tilde{T})$ is closed in the Calkin algebra; thus 2) and 3) are inequivalent.

Added in Proof. In a forthcoming paper by Don Deckard and the author, we prove the preceding Conjecture, thereby characterizing the Hilbert space operators having unitary cross sections.

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https://doi.org/10.4153/CJM-1978-101-8 Published online by Cambridge University Press