DIMENSION AND FINITE CLOSURE

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(Received 7 June 1976)

Abstract

If \( M \) is a model with dimension and finite closure, then \( T(M) \) is \( \aleph_0 \)-categorical. If \( M \) is atomic, has dimension and finitely many algebraic elements, then \( M \) has finite closure or a finite basis. If \( M \) has finite closure, satisfies the Exchange Lemma, and one-one maps between independent subsets are elementary, then \( M \) has dimension.

In Crossley & Nerode (1974, p. 44), the authors assume that the theories which they treat are \( \aleph_0 \)-categorical, but note that it is sufficient, for their purposes, to consider a complete theory \( T \) for which each \( B_n(T) \) is atomistic and every model has finite closure. A large part of their work concerns models with dimension. We show, in Section 1, that for a complete theory \( T \) with an infinite model which can be covered by finitely many minimal formulae, in particular with a model with dimension, \( T \) must be \( \aleph_0 \)-categorical for its model to have finite closure. We also show, in Section 2, that if a model is atomic, has dimension and finitely many algebraic elements, then it has either a finite basis or finite closure.

If \( M \) has dimension, then \( M \) satisfies the Exchange Lemma and one-one maps between independent subsets of \( M \) are elementary (see Propositions 1 and 3). We show, in Section 3, that if \( M \) has these two properties and finite closure, then \( M \) has dimension. No form of the axiom of choice is used.

Section 0 gives the notation and conventions we follow, as well as the necessary definitions and propositions from Crossley & Nerode (1974).

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MODELS. For a model, \( M \), we use \( M \) to denote the domain of \( M \), if no ambiguity arises. We use \( a, b \), etc. to denote elements of \( M \) and \( A, B \) etc. to
denote subsets of $\mathcal{M}$. $T(\mathcal{M})$ denotes the complete theory of $\mathcal{M}$. We do not assume that the language of a model is countable but we do assume that the language contains a symbol for equality and that $\mathcal{M}$ is a normal model. Thus we can define in the language quantifiers $\exists^k v_0 \cdots$ meaning "there exist $< k$ $v_0, \cdots$" and $\exists^* v_0 \cdots$ meaning "there exist exactly $k$ $v_0, \cdots$". $\chi(v_0, \cdots, v_n)$ will always denote a formula of the language of $\mathcal{M}$ with all its free variables among $v_0, \cdots, v_n$. For $a_0, \cdots, a_n \in \mathcal{M}$ we write $\mathcal{M} \models \chi(a_0, \cdots, a_n)$ or $\mathcal{M} \models \chi[a_1, \cdots, a_n]$ if $a_0, \cdots, a_n$ satisfies $\chi$ in $\mathcal{M}$. If $T$ is a complete theory, $B_n(T)$ denotes the boolean algebra of equivalence classes of formulae of the language of $T$ with all their free variables among $v_0, \cdots, v_n$. $(v_0, \cdots, v_n)$ will always denote a formula of the language of $\mathcal{M}$ with all its free variables among $v_0, \cdots, v_n$. We use $\chi$ to denote the equivalence class containing $\chi$ as no ambiguity arises. A model $\mathcal{M}$ is atomic, if for every $n$-tuple $(a_0, \cdots, a_{n-1})$ of $\mathcal{M}$ there is an atom $x$ of $B_n(T(\mathcal{M}))$ such that $\mathcal{M} \models \chi[a_0, \cdots, a_{n-1}]$. We say $\mathcal{M}$ is covered by the formulae $\chi_1(v_0), \cdots, \chi_n(v_0)$ if $\mathcal{M} \models \forall v_0 (\chi_1 \lor \cdots \lor \chi_n)$. $a$ is a solution of $\chi(v_0)$ if $\mathcal{M} \models \chi[a]$. 

**Algebraic Closure.** We follow chapters 4 and 6 of Crossley & Nerode (1974). $a$ is algebraic over $A$ if for some $a_1, \cdots, a_n \in A$, $\chi(v_0, \cdots, v_n)$ and natural number $k$, $\mathcal{M} \models (\exists^k v_0 \chi(v_0, \cdots, v_n) \land \chi(v_0, \cdots, v_n)) [a, a_1, \cdots, a_n]$. $a$ is algebraic if it is algebraic over $\phi$. The algebraic closure of $A$, $\text{cl} A$, is the set of all elements of $\mathcal{M}$ algebraic over $A$. Clearly $A \subseteq \text{cl} A$. $\mathcal{M}$ has finite closure if $\text{cl} A$ is finite whenever $A$ is finite. $A$ is independent if for all $a \in A$, $a \not\in \text{cl}(A \setminus \{a\})$. We write $(a_1, \cdots, a_n)$ is independent if $\{a_1, \cdots, a_n\}$ is independent and the $a_i$ are distinct. $A$ is a basis of $\mathcal{M}$ if $A$ is independent and $\mathcal{M} = \text{cl} A$. $\phi(v_0)$ is a minimal formula for $\mathcal{M}$ if $\phi$ has infinitely many solutions in $\mathcal{M}$ and for each $\psi(v_0, \cdots, v_n)$ and $a_1, \cdots, a_n \in \mathcal{M}$ either $\phi(v_0) \land \psi(v_0, a_1, \cdots, a_n)$ or $\phi(v_0) \land \neg \psi(v_0, a_1, \cdots, a_n)$ has finitely many solutions in $\mathcal{M}$. Clearly if $\phi(v_0)$ is minimal and $\psi(v_0)$ has only finitely many solutions then $\phi \lor \psi$ and $\phi \land \neg \psi$ are minimal. $\text{Min}(\mathcal{M})$ is the set of solutions of minimal formulae. $\mathcal{M}$ has dimension if for some minimal formula $\phi$, $\mathcal{M} \models \phi[a]$ for every non-algebraic element, $a$, of $\mathcal{M}$. If $\mathcal{M}, \mathcal{M}'$ have the same language $\mathcal{L}$, $A \subseteq \mathcal{M}$, $A' \subseteq \mathcal{M}'$ and $p : A \to A'$, then $p$ is an elementary monomorphism if for all $a_0, \cdots, a_n \in A$ and for all $\chi(v_0, \cdots, v_n) \in \mathcal{L}$ 

$$\mathcal{M} \models \chi[a_0, \cdots, a_n] \text{ if and only if } \mathcal{M}' \models \chi[p a_0, \cdots, p a_n]$$

($p$ is one-one as $\mathcal{L}$ contains equality and $\mathcal{M}$ is a normal model). We use the following propositions.
PROPOSITION 1. (Crossley & Nerode (1974), Lemma 6.4(1b)). (Exchange Lemma) For any model, $\mathcal{M}$, if $\{a_1, \ldots, a_n\} \subseteq \mathcal{M}$ is independent but $\{a_1, \ldots, a_{n-1}\}$ is not, and $a_{n+1} \in \text{Min} (\mathcal{M})$, then $a_{n+1} \in \text{cl}\{a_1, \ldots, a_n\}$.

PROPOSITION 2. (Crossley & Nerode (1974), Lemma 6.4(ii)). For any model $\mathcal{M}$, suppose $A, B \subseteq \text{Min} (\mathcal{M})$, $\text{cl} A \subseteq \text{cl} B$ and $A$ is independent. Then

(a) $\text{card} A \leq \text{card} B$

(b) there is a subset $B_0$ of $B$ such that $A \cup B_0$ is independent and $\text{cl} (A \cup B_0) = \text{cl} B$.

An obvious and trivial modification of the proof of Crossley & Nerode (1974), Lemma 6.9, gives:

PROPOSITION 3. Let $\mathcal{M}, \mathcal{M}'$ be models of a complete theory $T$, $A \subseteq \text{Min} (\mathcal{M})$, $B \subseteq \mathcal{M}'$ independent sets and $p : A \rightarrow B$ a one-one map such that for $a \in A$ there is some minimal formula, $\phi (v_0)$ for $\mathcal{M}$ such that $\mathcal{M} \models \phi [a]$ and $\mathcal{M}' \models \phi [p(a)]$. Then $p$ is an elementary monomorphism.

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We can prove our first result immediately.

THEOREM 4. Suppose a complete theory $T$ has an infinite model $\mathcal{M}$ with finite closure which is covered by minimal formulae $\phi_1, \ldots, \phi_n$. Then $B_m (T)$ is finite for all $m$ and $T$ is $\aleph_0$-categorical.

PROOF. We may assume that $\mathcal{M} \models \bigwedge_{i \neq i} \forall v_0 \neg (\phi_i (v_0) \& \phi_j (v_0))$. For if, for $i \neq j$, $\mathcal{M} \models (\phi_i \& \phi_j)[a]$ for infinitely many $a$ then replace $\phi_i$ (say) with $\phi_i \lor \phi_j$ and delete $\phi_j$. Now $\mathcal{M} \models (\phi_i \& \neg \phi_i)[a]$ for finitely many $a$, so $\phi_i \lor (\phi_i \& \neg \phi_i)$ (i.e. $\phi_i \lor \phi_j$) is again minimal. If $\mathcal{M} \models (\phi_i \& \phi_j)[a]$ for finitely many $a$ and $i < j$ replace $\phi_j$ with $\phi_i \& \neg \phi_i$, which is again minimal. In both cases the new $\phi$'s cover $\mathcal{M}$, so a simple induction validates the assumption.

Let $D_i = \{a \in \mathcal{M} : \mathcal{M} \models \phi_i [a]\}$. By the definition of a minimal formula each $D_i$ is infinite.

Suppose $m \in \omega$. Then there is an independent subset $C$ of $\mathcal{M}$ such that $\text{card} C \cap D_i = m$ for all $i$, for if not, let $r$ be the least $m$ for which it fails. Then $r > 0$, and there is an independent $C'$ such that $\text{card} C' \cap D_i = r - 1$. As $C'$ is finite and $D_i$ is infinite, there is $c_i \in D_i$ such that $c_i \notin \text{cl} C'$. So by Proposition 1, $C' \cup \{c_i\}$ is independent. Thus we can construct by induction $C''$ such that $\text{card} C'' \cap D_i = r$ for $i = 1, \ldots, n$, contradicting the choice of $r$.

If $\chi \in B_n (T)$, $\chi \neq 0$, then $T \models \exists v_1, \ldots, v_m \chi (v_1, \ldots, v_m)$. So there are $a_1, \ldots, a_m \in \mathcal{M}$ such that $\mathcal{M} \models \chi [a_1, \ldots, a_m]$. By Proposition 2 there is an independent set $A = \{a_1', \ldots, a_m'\} \subseteq \mathcal{M}$
\{a_1, \ldots, a_m\} such that \(\text{cl}\{a_1, \ldots, a_m\} = \text{cl} A\). Thus \(\{a_1, \ldots, a_m\} \subseteq \text{cl} A\). As \(\text{card} D \cap A \leq m\) there is a one-one map \(p : A \to C\) which satisfies the hypothesis of Proposition 3 and so is elementary.

As \(\{a_1, \ldots, a_m\} \subseteq \text{cl} A\), for \(i = 1, \ldots, m\), there are formulae \(\psi_i(v_1, \ldots, v_m)\), \(\sigma_i(v_1, \ldots, v_m)\) and natural numbers \(k_i\) such that

\[
\sigma_i(v_1, \ldots, v_m) = (\psi_i(v_1, \ldots, v_m) & \exists^{k_i} v_0 \psi_i(v_1, \ldots, v_m))
\]

and

\[
\mathcal{M} \models \sigma_i[a_1, \ldots, a_m].
\]

Hence \(\mathcal{M} \models (\exists u_1, \ldots, u_m (\chi(u_1, \ldots, u_m) & \land \bigwedge_{i=1}^m \sigma_i(u_1, v_1, \ldots, v_m))) [a_1, \ldots, a_m] \)

and so

\[
\mathcal{M} \models (\exists u_1, \ldots, u_m (\chi(u_1, \ldots, u_m) & \land \bigwedge_{i=1}^m \sigma_i(u_1, v_1, \ldots, v_m))) [p(a_1'), \ldots, p(a_m')].
\]

As \(p(a_i') \in C\), there exist \(c_1', \ldots, c_m \in \text{cl} C\) such that \(\mathcal{M} \models \chi[c_1, \ldots, c_m]\).

The map \(q : B_m(T) \to \mathcal{P}(\text{cl} C)^m\) given by

\[
q(\chi) = \{(c_1, \ldots, c_m) \in (\text{cl} C)^m : \mathcal{M} \models \chi[c_1, \ldots, c_m]\}
\]

is one-one, for suppose \(\chi_1, \chi_2 \in B_m(T)\) and \(\chi_1 \neq \chi_2\). Then we may assume \(\chi_1 & \neg \chi_2 \neq 0\). So by the above, there are \(c_1, \ldots, c_m \in C\) such that \(\mathcal{M} \models \chi_1 & \neg \chi_2[c_1, \ldots, c_m]\) and therefore \(q(\chi_1) \neq q(\chi_2)\). But \(\mathcal{P}(\text{cl} C)^m\) is finite as \(\text{cl} C\) is, whence \(B_m(T)\) is finite.

So by Ryll-Nardzewski (1959), \(T\) is \(\mathfrak{K}_n\)-categorical. We note that this direction of Ryll-Nardzewski's proof does not require the axiom of choice. \(\square\)

Regarding the converse of Theorem 4, if \(\mathcal{M} \models T\) and \(B_m(T)\) is finite for all \(m\), indeed just for \(m = 1\), then \(\mathcal{M}\) can have at most finitely many minimal formulae, as it has only finitely many inequivalent 1-place formulae. However \((Q, \leq)\) is a model of an \(\mathfrak{K}_n\)-categorical theory and has no minimal formulae.

**Corollary 5.** If \(T\) is a complete theory with a model \(\mathcal{M}\) with dimension and finite closure, then \(B_m(T)\) is finite for each \(m\) and \(T\) is \(\mathfrak{K}_n\)-categorical.

**Proof.** As \(\mathcal{M}\) has finitely many algebraic elements, \(v_0 = v_0\) is a minimal formula which covers \(\mathcal{M}\). \(\square\)

**Corollary 6.** If \(T\) is a complete theory with a model \(\mathcal{M}\) with dimension and finite closure then every model \(\mathcal{N}\) of \(T\) has dimension and finite closure.

**Proof.** By Corollary 5, \(B_m(T)\) is finite for each \(m\) and so by Crossley & Nerode (1974), Lemma 5.9, \(\mathcal{N}\) has finite closure. Furthermore \(\mathcal{N}\) is atomic.
As $\mathcal{M}$ has dimension and finite closure, $v_0 = v_0$ is a minimal formula for $\mathcal{M}$. We will show $v_0 = v_0$ is a minimal formula for $\mathcal{M}$.

Let $a_1, \cdots, a_n \in \mathcal{M}$ and $\chi(v_0, \cdots, v_n)$ be any formula. Let $\psi(v_0, \cdots, v_{n-1})$ be the atom satisfied by $a_1, \cdots, a_n$ and let $b_1, \cdots, b_n \in \mathcal{M}$ satisfy $\psi$. As $v_0 = v_0$ is a minimal formula for $\mathcal{M}$,

$$\mathcal{M} \models \exists^k v_0 \sigma(v_0, \cdots, v_n)[b_1, \cdots, b_n],$$

for some finite $k$, where $\sigma$ is $\chi$ or $\neg \chi$. Hence

$$T \vdash \forall v_1, \cdots, v_n (\psi(v_1, \cdots, v_n) \rightarrow \exists^k v_0 \sigma(v_0, \cdots, v_n))$$

as $\psi$ is an atom and so

$$\mathcal{M} \models \exists^k v_0 \sigma(v_0, \cdots, v_n)[a_1, \cdots, a_n].$$

So $v_0 = v_0$ is minimal for $\mathcal{M}$ whence $\mathcal{M}$ has dimension. \(\square\)

We first prove a theorem from which our second claim follows readily:

**Theorem 7.** Suppose $\mathcal{M}$ is an atomic model of a complete theory $T$ and $\phi(v_0)$ is a minimal formula for $\mathcal{M}$. Then for all $n$ such that $\mathcal{M}$ contains an independent set with $\geq n + 1$ solutions of $\phi$, there is a formula $\rho_{n+1}$, an atom of $B_{n+1}(T)$, such that for any model $\mathcal{M}'$ of $T$:

$$\mathcal{M}' \models \rho_{n+1}[a_0', \cdots, a_n'] \text{ if and only if } (a_0', \cdots, a_n') \text{ is independent and } \mathcal{M}' \models \phi[a_i'] \text{ } i = 0, \cdots, n.$$ 

**Proof.** Suppose $(a_0, \cdots, a_n) \subseteq \mathcal{M}$ is independent and $\mathcal{M} \models \phi[a_i] \text{ } i = 0, \cdots, n$. Then the $a_i$ are distinct. As $\mathcal{M}$ is atomic, there is an atom $\rho_{n+1}$ of $B_{n+1}(T)$ such that $\mathcal{M} \models \rho_{n+1}[a_0, \cdots, a_n]$. We show that it has the desired property.

Suppose $(a_0', \cdots, a_n') \subseteq \mathcal{M}'$ is independent and $\mathcal{M}' \models \phi[a_i'] \text{ } i = 0, \cdots, n$. Then by Proposition 3, $p : a_i \mapsto a_i'$ is elementary whence $\mathcal{M}' \models \rho_{n+1}[p(a_0), \cdots, p(a_n)]$ which is precisely $\mathcal{M}' \models \rho_{n+1}[a_0', \cdots, a_n']$.

Conversely, suppose $\mathcal{M}' \models \rho_{n+1}[a_0', \cdots, a_n']$ and $(a_0', \cdots, a_n')$ is independent. We may assume, without loss of generality, that $a_0' \in \text{cl} \{a_1', \cdots, a_n'\}$. So there is a formula $\psi(v_0, \cdots, v_n)$ and natural number $k$, such that

$$\mathcal{M}' \models (\psi(v_0, \cdots, v_n) \& \exists^k v_0 \psi(v_0, \cdots, v_n))[a_0', \cdots, a_n'].$$

So

$$T \vdash \exists v_0, \cdots, v_n (\rho_{n+1} \& \psi \& \exists^k v_0 \psi).$$
\[ T \vdash \forall v_0, \ldots, v_n (\rho_{n+1} \rightarrow (\psi \land \exists^k v_0 \psi)). \]

But \( \rho_{n+1} \) is an atom of \( B_{n+1}(T) \).

So \[ M \models \rho_{n+1}[a_0, \ldots, a_n], \quad \text{so} \quad M \models (\psi \land \exists^k v_0 \psi)[a_0, \ldots, a_n]. \]

Hence \( a_0 \in \text{cl}\{a_1, \ldots, a_n\} \) which contradicts the independence of \( (a_0, \ldots, a_n) \).

So \( (a_0', \ldots, a_n') \) is independent.

It remains to show that \( M' \models \phi[a'] \) \( i = 0, \ldots, n \).

\[ M' \models \left( \rho_{n+1} \land \bigwedge_{i=0}^{n} \phi(v_i) \right)[a_0, \ldots, a_n]. \]

Hence \( T \vdash \forall v_0, \ldots, v_n (\rho_{n+1} \rightarrow \bigwedge_{i=0}^{n} \phi(v_i)) \) as \( \rho_{n+1} \) is an atom of \( B_{n+1}(T) \), and so \( M' \models \bigwedge_{i=0}^{n} \phi(v_i)[a_0', \ldots, a_n'] \) as \( M' \models \rho_{n+1}[a_0', \ldots, a_n'] \). \( \square \)

**Corollary 8.** Suppose \( M \) is an atomic model of a complete theory \( T \), \( \phi(v_0) \) is a minimal formula and \( D = \{ a : M \models \phi(a) \} \). If there are arbitrarily large finite independent subsets of \( D \), then for any finite \( A \subseteq D \), \( D \cap \text{cl} A \) is finite.

**Proof.** Suppose for some \( A \subseteq D \), that \( A \) is finite but \( D \cap \text{cl} A \) is infinite. Let \( A = \{ a_1, \ldots, a_n \} \). We may assume that \( A \) is independent, for by Proposition 2, there is an independent \( A' \subseteq A \) such that \( \text{cl} A' = \text{cl} A \) (No choice is needed as \( A \) is finite). By Theorem 7 and the hypothesis there is an atom of \( B_{n+1}(T) \), \( \rho_{n+1}(v_0, \ldots, v_n) \), such that for \( d \in D \), \( M \models \rho_{n+1}[d_0, \ldots, d_n] \) if and only if \( (d_0, \ldots, d_n) \) is independent. By Proposition 1, if \( (d_1, \ldots, d_n) \) is independent, \( M \models \rho_{n+1}[d_0, \ldots, d_n] \) if and only if \( d_0 \not\in \text{cl}\{d_1, \ldots, d_n\} \).

As \( D \cap \text{cl} A \) is infinite then \( \phi(v_0) \land \neg \rho_{n+1}(v_0, a_1, \ldots, a_n) \) has infinitely many solutions in \( M \), and as \( \phi \) is minimal, \( \phi(v_0) \land \rho_{n+1}(v_0, a_1, \ldots, a_n) \) has finitely many solutions, \( d_1, \ldots, d_k \) say.

Hence \( D \subseteq \text{cl}\{a_1, \ldots, a_n, d_1, \ldots, d_k\} \) and \( \{a_1, \ldots, a_n, d_1, \ldots, d_k\} \subseteq \text{Min}(M) \). So if \( \{a_1', \ldots, a_m'\} \subseteq D \) is independent, by Proposition 2,

\[ m = \text{card}\{a_1', \ldots, a_m'\} \leq \text{card}\{a_1, \ldots, a_n, d_1, \ldots, d_k\} \leq n + k, \]

which contradicts the hypothesis of the corollary.

Hence \( D \cap \text{cl} A \) is finite for all finite \( A \subseteq D \). \( \square \)

The main result of this section is:

**Corollary 9.** If \( M \) is an atomic model with dimension and finitely many algebraic elements then \( M \) has a finite basis or finite closure.
PROOF. The following are clear, as is the deduction of Corollary 9 from them and Corollary 8.

If \( \mathcal{M} \) has dimension and \( \text{cl} \phi \) is finite there is a minimal formula \( \phi \) for which \( D = \mathcal{M} \). If \( \mathcal{M} \) does not have a finite basis then it has arbitrarily large independent subsets. \( \square \)

We can find atomic models with dimension and finitely many algebraic elements with a finite basis but not finite closure \(((Z, S))\) where \( S(n) = n + 1 \) and with no finite basis but finite closure \(((N, =))\). Models with a finite basis and finite closure are finite and so do not have dimension, as we require a minimal formula to have infinitely many solutions.

If we do not assume that \( \mathcal{M} \) is atomic, then Corollary 9 is false. If we take \( \mathcal{M} = (V, +, f_\lambda)_{\lambda \in F} \) where \( V \) is an infinite dimensional vector space over an infinite field \( F \), and \( f_\lambda : v \mapsto \lambda v \) is a unary function, then \( \mathcal{M} \) has dimension but neither a finite basis nor finite closure, as algebraic closure is closure in the usual vector space sense. \( \mathcal{M} \) is not atomic, for if \( \{a_0, \cdots, a_n\} \) is independent, \( \mathcal{M} \models a_0 \neq \lambda_1 a_1 + \cdots + \lambda_n a_n \) for all \( \lambda_1, \cdots, \lambda_n \in F \), whereas \( B_{n+1}(T) \) is generated by \( \{\lambda_0 v_0 = \lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \in F\} \) as \( T(\mathcal{M}) \) admits elimination of quantifiers. Hence there is no atom satisfied by \( (a_0, \cdots, a_n) \).

Combining Corollaries 5 and 9 we obtain:

**Corollary 10.** If \( T \) is a complete theory with an atomic model with dimension but no finite basis and finitely many algebraic elements then \( T \) is \( N_0 \)-categorical.

### 3

**Theorem 11.** Suppose \( \mathcal{M} \) has the following properties.

1. \( \mathcal{M} \) has finite closure.
2. If \( \{a_1, \cdots, a_n\} \subseteq \mathcal{M} \) is independent and \( \{a_1, \cdots, a_{n+1}\} \) is not, then \( a_{n+1} \in \text{cl}\{a_1, \cdots, a_n\} \). (\( \mathcal{M} \) satisfies the Exchange Lemma).
3. If \( A, B \subseteq \mathcal{M} \) are independent and \( p : A \to B \) is one-one, then \( p \) is elementary.

Then \( \mathcal{M} \) has dimension.

**Proof.** We prove the following by induction on \( n \):

4. If \( \{a_1, \cdots, a_m\} \) is independent, \( b_1, \cdots, b_n \in \text{cl}\{a_1, \cdots, a_m\} \) then there exist \( c_1, \cdots, c_p \) such that \( \{a_1, \cdots, a_m, c_1, \cdots, c_p\} \) is independent and for any formula \( \psi(v_1, \cdots, v_{m+n+p+1}) \) and for any \( d_1, d_2 \notin \text{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\} \),

\[
\mathcal{M} \models \psi[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_1] \text{ if, and only if,}
\]

\[
\mathcal{M} \models \psi[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_2].
\]
Suppose \( n = 0 \). If \( d_1, d_2 \not\in \text{cl}\{ a_1, \ldots, a_m \} \) then, by (2), \{ a_1, \ldots, a_m, d_1 \} and \{ a_1, \ldots, a_m, d_2 \} are independent, and (4) holds by (3).

Suppose (4) holds for some \( n \) and \( b_{n+1} \in \text{cl}\{ a_1, \ldots, a_m \} \). Then \( b_{n+1} \in \text{cl}\{ a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p \} \) and there is a formula \( \chi_0(v_0, \ldots, v_{m+n+p}) \) and a natural number \( k_0 \geq 1 \), such that
\[
\mathcal{M} \models \chi_0 \land \exists v_0 \chi_0[b_{n+1}, a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p].
\]

We construct sequences \( k_0 > k_1 > \cdots > k_q \geq 1 \), \( \chi_0, \ldots, \chi_q, c_{p+1}, \ldots, c_{p+q} \), such that for any formula \( \psi(v_1, \ldots, v_{m+n+p+1}) \) and \( d_1, d_2 \not\in \text{cl}\{ a_1, \ldots, a_m, c_1, \ldots, c_p \} \)
\[
\mathcal{M} \models \psi[a_1, \ldots, a_m, b_1, \ldots, b_{n+1}, c_1, \ldots, c_p, d_1]
\]
if, and only if,
\[
\mathcal{M} \models \psi[a_1, \ldots, a_m, b_1, \ldots, b_{n+1}, c_1, \ldots, c_p, d_2].
\]
(5)

If (5) holds with \( q = 0 \), we are done. If not, there is a formula \( \psi(v_1, \ldots, v_{m+n+p+1}) \) and \( d_1, d_2 \not\in \text{cl}\{ a_1, \ldots, a_m, b_1, \ldots, b_{n+1}, c_1, \ldots, c_p \} \) such that
\[
\mathcal{M} \models \psi[a_1, \ldots, a_m, b_1, \ldots, b_{n+1}, c_1, \ldots, c_p, d_1]
\]
and
\[
\mathcal{M} \models \neg \psi[a_1, \ldots, a_m, b_1, \ldots, b_{n+1}, c_1, \ldots, c_p, d_2].
\]
(7)

Put \( c_{p+1} = d_i \) and put
\[
\chi_1(v_0, \ldots, v_{m+n+p+2}) = \chi_0(v_0, \ldots, v_{m+n+p}) \land \neg \psi(v_1, \ldots, v_{m+n+p}, v_0, v_{m+n+1}, \ldots, v_{m+n+p+1}).
\]

By (2), \{ a_1, \ldots, a_m, c_1, \ldots, c_{p+1} \} is independent. Clearly
\[
\mathcal{M} \models \chi_1[b_{n+1}, a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_{p+1}].
\]
And
\[
\mathcal{M} \models \exists v_0(\chi_0 \land \neg \chi_1)[a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_{p+1}]
\]
for suppose otherwise. Then
\[
\mathcal{M} \models \forall v_0(\chi_0 \rightarrow \chi_1)[a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p, d_1]
\]
whence, by (4),
\[
\mathcal{M} \models \forall v_0(\chi_0 \rightarrow \chi_1)[a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p, d_2].
\]
But \( \mathcal{M} \models \chi_0[b_{n+1}, a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p] \) and so
\[
\mathcal{M} \models \chi_1[b_{n+1}, a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p, d_2]
\]
and therefore
\[ \mathcal{M} \models \psi[a_1, \ldots, a_m, b_1, \ldots, b_{n+1}, c_1, \ldots, c_p, d_2] \]

which contradicts (7). Thus (8) holds and
\[ \mathcal{M} \models \exists^{k} v_0 \chi[v_0, a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_{p+1}] \]

where \( 1 \leq k_i < k_0 \).

We can choose \( \chi, c_i \) is a similar fashion until (5) holds, which is when (4) holds for \( n + 1 \).

Thus (4) holds for all \( n \).

Now suppose \( b_1, \ldots, b_n \in \mathcal{M} \) and \( \psi(v_0, \ldots, v_n) \) is any formula. Using (2), we can choose \( a_1, \ldots, a_m \in \{b_1, \ldots, b_n\} \) such that \( \{a_1, \ldots, a_m\} \) is independent and \( b_1, \ldots, b_n \in \text{cl}\{a_1, \ldots, a_m\} \). By (4), there exist \( c_1, \ldots, c_p \) such that for all \( \chi(v_0, \ldots, v_n) \) and \( d_1, d_2 \notin \{a_1, \ldots, a_m, c_1, \ldots, c_p\} \) \( \mathcal{M} \models \chi[d_1, b_1, \ldots, b_n] \) if, and only if \( \mathcal{M} \models \chi[d_2, b_1, \ldots, b_n] \).

If \( \psi(v_0, b_1, \ldots, b_n) \) has infinitely many solutions in \( \mathcal{M} \), then \( \mathcal{M} \models \psi[d, b_1, \ldots, b_n] \) for some \( d \notin \text{cl}\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \) as \( \text{cl}\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \) is finite by (1). Hence \( \mathcal{M} \models \psi[d, b_1, \ldots, b_n] \) for all \( d \notin \text{cl}\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \) and \( \mathcal{M} \models \neg \psi[d, b_1, \ldots, b_n] \) for at most \( d \in \text{cl}\{a_1, \ldots, a_m, c_1, \ldots, c_p\} \). Thus \( \neg \psi(v_0, b_1, \ldots, b_n) \) has finitely many solutions, and so \( v_0 = v_0 \) is a minimal formula.

Therefore \( \mathcal{M} \) has dimension. \( \square \)

Conditions (2) and (3) are not sufficient for \( \mathcal{M} \) to have dimension. Consider the model \( \mathcal{N} = (Z \times Z, <, S) \) where

\[ (n_1, m_1) < (n_2, m_2) \text{ if } n_1 = n_2 \text{ and } m_1 < m_2 \]

and

\[ S((n, m)) = (n, m + 1). \]

It is easy to see the following:

(a) \( \text{cl}\{(n_1, m_1), \ldots, (n_t, m_t)\} = \{n_1, \ldots, n_t\} \times Z. \)

(b) \( \{(n_1, m_1), \ldots, (n_t, m_t)\} \) is independent if and only if \( n_1, \ldots, n_t \) are distinct, and therefore (2) holds.

(c) If \( A, B \subseteq \mathcal{N} \) are finite and independent and \( p: A \rightarrow B \) is one-one, then \( p \) extends to an automorphism of \( \mathcal{N} \) and so is elementary. Therefore (3) holds.

(d) \( \mathcal{N} \) has no algebraic elements and \( v_0 < a_1, \neg v_0 < a_1 \) both have infinitely many solutions in \( \mathcal{N} \). Thus \( \mathcal{N} \) does not have dimension.

By Corollary 5, if \( \mathcal{W} \) satisfies (1), (2) and (3), then \( B_*(T(\mathcal{W})) \) is finite for each \( n \), and so \( \mathcal{W} \) is atomic. However Theorem 11 does not hold if we replace (1) by "\( \mathcal{M} \) is atomic", for the model \( \mathcal{W} \) provides a counter-example.
\[ \psi(v_{11}, \ldots, v_{1k}, \ldots, v_{l1}, \ldots, v_{lk}) \]

\[ = \bigwedge_{i_1 \neq j_2} \left( \neg (v_{i_1} < v_{j_2}) \land \neg (v_{i_2} < v_{j_1}) \land \bigwedge_{i = 1, \ldots, j_2 \leq j_1} v_{i_2} = S^{h^2}(v_{i_1}) \right) \]

is a formula satisfied by

\[ ((n_1, m_{11}), \ldots, (n_1, m_{1k}), \ldots, (n_l, m_{11}), \ldots, (n_l, m_{kl})) \]

where \( n_1, \ldots, n_l \) are distinct, and is an atom, as can be seen by extending the map \( a_i \mapsto a'_i \), where \( \mathfrak{N} \models \psi(a_{11}, \ldots, a_{ik}) \) and \( \mathfrak{N} \models \psi(a'_{11}, \ldots, a'_{ik}) \), to an automorphism of \( \mathfrak{N} \).

References


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