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A HILBERT-SCHMIDT NORM INEQUALITY ASSOCIATED WITH THE FUGLEDE-PUTNAM THEOREM

TAKAYUKI FURUTA

Dedicated in deep sorrow to the memory of the
late Professor Teishirô Saitô

The familiar Fuglede-Putnam theorem asserts that $AX = XB$ implies $A^*X = XB^*$ when A and B are normal. We prove that A and B^* be hyponormal operators and let C be a hyponormal commuting with A^* and also let D^* be a hyponormal operator commuting with B respectively, then for every Hilbert-Schmidt operator X, the Hilbert-Schmidt norm of $AXD + CXB$ is greater than or equal to the Hilbert-Schmidt norm of $A^*XD^* + C^*XB^*$. In particular, $AXD = CXB$ implies $A^*XD^* = C^*XB^*$. If we strengthen the hyponormality conditions on A, B^*, C and D^* to quasinormality, we can relax Hilbert-Schmidt operator of the hypothesis on X to be every operator and still retain the inequality under some suitable hypotheses.

1. Introduction

An *operator* means a bounded linear operator on a separable infinite dimensional Hilbert space H. Let $B(H)$ and C_2 denote the class of all bounded linear operators acting on H and the Hilbert-Schmidt class in $B(H)$ respectively. C_2 forms a two-sided ideal in the algebra $B(H)$ and C_2 is itself a Hilbert space for the inner product

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$$(X, Y) = \sum (Xe_j, Ye_j) = \mathrm{Tr}(Y^*X) = \mathrm{Tr}(XY^*)$$

where $\{e_j\}$ is any orthonormal basis of H and $\mathrm{Tr}(T)$ denotes the trace. In what follows, $\|T\|_2$ denotes the Hilbert-Schmidt norm.

An operator T is called *quasinormal* if T commutes with T^*T, *subnormal* if T has a normal extension and *hyponormal* if $[T^*, T] \geq 0$ where $[S, T] = ST - TS$. The inclusion relation of the classes of non-normal operators listed above is as follows:

$$\text{Normal} \subsetneqq \text{Quasinormal} \subsetneqq \text{Subnormal} \subsetneqq \text{Hyponormal} ;$$

the above inclusions are all proper [6, Problem 160, p. 101].

In [2], Berberian shows the following result.

THEOREM A [2]. *If A and B^* are hyponormal, then $AX = XB$ implies $A^*X = XB^*$ for an operator X in the Hilbert-Schmidt class.*

On the other hand, in [3] we have shown Theorem B which is an extension of the Fuglede-Putnam theorem.

THEOREM B [3]. *If A and B^* are subnormal and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

Recently Weiss has obtained the following result.

THEOREM C [11]. *Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ denote commuting pairs of normal operators and let $X \in B(H)$. Then*

$$\|A_1XB_1+A_2XB_2\|_2 = \|A_1^*XB_1^*+A_2^*XB_2^*\|_2 .$$

In this paper we prove Theorem 1 which is an extension of Theorem A and also we prove a slightly stronger Theorem 2 by integrating Theorem B and Theorem C.

2.

First of all we show the following theorem.

THEOREM 1. *Let A and B^* be hyponormal on H. Let C be a hyponormal commuting with A^* and also D^* be a hyponormal commuting with B respectively. Then*

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(i)

$$(*) \qquad \|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X *in the Hilbert-Schmidt class. Equality in* (*) *holds for every* X *in the Hilbert-Schmidt class when* A, B, C *and* D *are all normal.*

(ii) If X *is an operator in the Hilbert-Schmidt class such that* $AXD = CXB$, *then* $A^*XD^* = C^*XB^*$.

Proof. Define an operator T on C_2 as follows:

$$TX = AXD + CXB .$$

Then, if we view C_2 as an underlying Hilbert space, then T^* exists and T^* is given by the formula $T^*X = A^*XD^* + C^*XB^*$ since we easily see from

$$\begin{aligned}(T^*X, Y) = (X, TY) &= (X, AYD+CYB) = \operatorname{Tr}(XD^*Y^*A^*) + \operatorname{Tr}(XB^*Y^*C^*) \\ &= \operatorname{Tr}(A^*XD^*Y^*) + \operatorname{Tr}(C^*XB^*Y^*) = \operatorname{Tr}\big((A^*XD^*+C^*XB^*)Y^*\big) \\ &= (A^*XD^*+C^*XB^*, Y) .\end{aligned}$$

Also

$$\begin{aligned}(T^*T-TT^*)X &= A^*(AXD+CXB)D^* + C^*(AXD+CXB)B^* \\ &\qquad - A(A^*XD^*+C^*XB^*)D - C(A^*XD^*+C^*XB^*)B \\ &= (A^*AXDD^*-AA^*XD^*D) + (C^*CXBB^*-CC^*XB^*B) \\ &\qquad + A^*CXBD^* - AC^*XB^*D + C^*AXDB^* - CA^*XD^*B \\ &= (A^*A-AA^*)XDD^* + AA^*X(DD^*-D^*D) + (C^*C-CC^*)XBB^* + CC^*X(BB^*-B^*B) \\ &\qquad + (A^*CXBD^*-CA^*XD^*B) + (C^*AXDB^*-AC^*XB^*D)\end{aligned}$$

and the fifth and the sixth terms in the formula above are both zero since the hypotheses $CA^* = A^*C$ and $D^*B = BD^*$ hold, so that

$$(1) \quad (T^*T-TT^*)X = (A^*A-AA^*)XDD^* + AA^*X(DD^*-D^*D) + (C^*C-CC^*)XBB^* + CC^*X(BB^*-B^*B) .$$

Left and right multiplication acting on C_2 as the Hilbert space by a positive operator is itself a positive operator. Since $T^*T - TT^*$ is the sum of four positive operators by the hyponormality of A, B^*, C and D^* , T is hyponormal. Therefore

$$\|TX\|_2 \geq \|T^*X\|_2 ;$$

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that is,

$$\|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2 \tag{2}$$

and the proof of equality easily follows by (1) and (2). If an operator T is hyponormal, then $-T$ is also hyponormal, so the proof of *(ii)* easily follows by (*) in *(i)*.

COROLLARY 1. *Let A and B^* be hyponormal on H. Let C be a normal commuting with A and also let D be a normal commuting with B respectively. Then*

(i)

$$\|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2 \tag{*}$$

holds for every X in the Hilbert-Schmidt class. Equality in (*) holds for every X in the Hilbert-Schmidt class when A and B are both normal.

(ii) If X is an operator in the Hilbert-Schmidt class such that $AXD = CXB$, then $A^*XD^* = C^*XB^*$.

Proof. The hypotheses $CA = AC$ and $DB = BD$ imply $CA^* = A^*C$ and $DB^* = B^*D$, that is, $D^*B = BD^*$ by the original Fuglede-Putnam theorem [*1*], [*6*], [*7*], [*8*], so the proof follows by Theorem 1.

REMARK 1. We remark that Weiss [*10*, Theorem 4] shows the case of the equality in *(i)* of Corollary 1 when $A = B$ is normal and $C = D = I$ the identity operator on H, by a different method and also Corollary 1 is an extension of Theorem A.

3.

If we strengthen the hyponormality conditions to quasinormality, then we can relax the Hilbert-Schmidt operator of the hypothesis on X to be every operator in $B(H)$ in Theorem 1 and still retain the inequality under some suitable hypotheses.

DEFINITION 1. Let N_T denote a normal extension on $H \oplus H$ of a subnormal operator T on H. In fact, for every subnormal operator T, there exists a normal extension N_T on $H \oplus H$ whose restriction to $H \oplus \{0\}$ is T [5].

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LEMMA. *Let* A *and* B^* *be subnormal on* H. *Let* C *be subnormal such that* N_C *commutes with* N_A *and also* D^* *be subnormal such that* N_{D^*} *commutes with* N_{B^*} *respectively. Then*

(i)

$$(**)\qquad \|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X *in* $B(H)$. *Equality in* (**) *holds for every* X *in* $B(H)$ *when* A, B, C *and* D *are all normal.*

(ii) If X *is an operator such that* $AXD = CXB$, *then* $A^*XD^* = C^*XB^*$.

Proof. By Definition 1, N_A and N_C are given by

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad N_C = \begin{pmatrix} C & C_{12} \\ 0 & C_{22} \end{pmatrix}$$

acting on $H \oplus H$ whose restriction to $H \oplus \{0\}$ are A and C respectively and also N_{B^*} and N_{D^*} are given by the same reason as follows on $H \oplus H$;

$$N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix} \quad \text{and} \quad N_{D^*} = \begin{pmatrix} D^* & D_{12} \\ 0 & D_{22} \end{pmatrix}.$$

For X acting on H, we consider $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ acting on $H \oplus H$. $\{N_A, N_C\}$ and $\{N_{D^*}^*, N_{B^*}^*\}$ are commuting pairs of normal operators on $H \oplus H$. Then, by Theorem C, we have

$$\left\| \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ D_{12}^* & D_{22}^* \end{pmatrix} + \begin{pmatrix} C & C_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ B_{12}^* & B_{22}^* \end{pmatrix} \right\|_2$$

$$= \left\| \begin{pmatrix} A^* & 0 \\ A_{12}^* & A_{22}^* \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D^* & D_{12} \\ 0 & D_{22} \end{pmatrix} + \begin{pmatrix} C^* & 0 \\ C_{12}^* & C_{22}^* \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix} \right\|_2 ;$$

that is,

$$\left\| \begin{pmatrix} AXD+CXB & 0 \\ 0 & 0 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} A^*XD^*+C^*XB^* & A^*XD_{12}+C^*XB_{12} \\ A_{12}^*XD^*+C_{12}^*XB^* & A_{12}^*XD_{12}+C_{12}^*XB_{12} \end{pmatrix} \right\|_2$$

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so that

$$(3)\quad \|AXD+CXB\|_2^2 = \|A^*XD^*+C^*XB^*\|_2^2 + \|A^*XD_{12}+C^*XB_{12}\|_2^2 + \|A_{12}^*XD^*+C_{12}^*XB^*\|_2^2 + \|A_{12}^*XD_{12}+C_{12}^*XB_{12}\|_2^2$$

whence we have

$$\|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2$$

which is the desired norm inequality (**). When A, B, C and D are all normal, then $A_{12} = 0$, $B_{12} = 0$, $C_{12} = 0$ and $D_{12} = 0$ in (3), so that equality in (**) holds and the proof is complete.

We remark that the sum of second, third and fourth of the right hand in (3) can be considered as a "perturbed terms" measures the deviation of subnormality from normality.

DEFINITION 2. Let $[S, T]_*$ denote the following "*-commutator":

$$[S, T]_* = ST - TS^* ;$$

this *-commutator is completely different from the usual commutator $[S, T]$.

DEFINITION 3. Let S_T denote the positive square root of $[T^*, T]$ for the hyponormal operator T .

THEOREM 2. *Let* A *and* B^* *be quasinormal on* H . *Let* C *be a quasinormal such that commutes with* A *and satisfies* $[A, S_C]_* = [C, S_A]_*$ *and also let* D^* *be a quasinormal such that commutes with* B^* *and satisfies* $[B^*, S_{D^*}]_* = [D^*, S_{B^*}]_*$ *respectively. Then*

(i)

$$(**)\qquad \|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X *in* $B(H)$. *Equality in* (**) *holds for every* X *in* $B(H)$ *when* A, B, C *and* D *are all normal.*

(ii) If X *is an operator such that* $AXD = CXB$, *then* $A^*XD^* = C^*XB^*$.

Proof. Let $A = UP$ be the polar decomposition of A , where U is a partial isometry and P is a positive operator such that $P^2 = A^*A$. A normal extension N_A of A can be written as follows [6, p. 308],

$$N_A = \begin{pmatrix} A & S(A) \\ 0 & A^* \end{pmatrix}$$

acting on $H \oplus H$, where $S(A) = (I-UU^*)P$. Since A is quasinormal, then $A = UP = PU$ [6, Problem 108]. As UU^* is projection and P commutes with U and U^*, then

$$(4) \qquad S(A) = (I-UU^*)P = \left[(I-UU^*)P^2\right]^{\frac{1}{2}} = \left(P^2-UPU^*P\right)^{\frac{1}{2}} = (A^*A-AA^*)^{\frac{1}{2}} = S_A .$$

Similarly normal extensions of C, B^* and D^* are also given as follows:

$$N_C = \begin{pmatrix} C & S_C \\ 0 & C^* \end{pmatrix}, \quad N_{B^*} = \begin{pmatrix} B^* & S_{B^*} \\ 0 & B \end{pmatrix} \text{ and } N_{D^*} = \begin{pmatrix} D^* & S_{D^*} \\ 0 & D \end{pmatrix}.$$

Hypotheses imply that $\{N_A, N_C\}$ and $\{N^*_{D^*}, N^*_{B^*}\}$ are pairs of commuting normal operators, so that the desired relations follow by the lemma.

COROLLARY 2. *Let A and B^* be quasinormal on H. Let C be a normal commuting with A and also D be a normal commuting with B respectively. Then*

(i)

$$(**) \qquad \|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X in $B(H)$. Equality in (**) *holds for every X in $B(H)$ when A, B, C and D are all normal.*

(ii) If X is an operator such that $AXD = CXB$, then $A^*XD^* = C^*XB^*$.

Proof. Take $N_C = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ in the proof of Theorem 2 since C is normal. Then the hypothesis $CA = AC$ implies $CA^* = A^*C$ by the original Fuglede-Putnam theorem [1], [6], [7], [8], so that we have $CS_A^2 = S_A^2C$; that is, $CS_A = S_AC$ holds, whence N_A in the proof of Theorem 2 commutes with $N_C = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ since (4) holds. Similarly $N^*_{D^*} = \begin{pmatrix} D^* & 0 \\ 0 & D^* \end{pmatrix}$ commutes with $N^*_{B^*}$ in the proof of Theorem 2, so that the proof is complete by the lemma.

REMARK 2. If we strengthen on X to be in Hilbert-Schmidt class in Corollary 2, then we can relax quasinormality of the hypotheses on A and B^* to hyponormality and still retain the inequality; that is, just Corollary 1.

COROLLARY 3. *Let* A *and* B^* *be hyponormal satisfying* $[A^*, S_A]_* = 0$ *and* $[B, S_{B^*}]_* = 0$ *respectively. Let* C *be a hyponormal which commutes with* A *and satisfies* $[C^*, S_C]_* = 0$ *and* $[A, S_C]_* = [C, S_A]_*$ *and also let* D^* *be a hyponormal which commutes with* B^* *and satisfies* $[D, S_{D^*}]_* = 0$ *and* $[B^*, S_{D^*}]_* = [D^*, S_{B^*}]_*$ *respectively. Then*

(i)

$$\|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2 \tag{**}$$

holds for every X *in* $B(H)$. *Equality in* (**) *holds for every* X *in* $B(H)$ *when* A, B, C *and* D *are all normal.*

(ii) If X *is an operator such that* $AXD = CXB$, *then* $A^*XD^* = C^*XB^*$.

Proof. The hypotheses imply that A, B^*, C and D^* are all subnormal and $N_A = \begin{pmatrix} A & S_A \\ 0 & A^* \end{pmatrix}$ and similarly N_{B^*}, N_C and N_{D^*} are also given in the similar forms [4, Theorem 1]. As stated in the proof of Theorem 2, the hypotheses imply that $\{N_A, N_C\}$ and $\{N_{D^*}^*, N_{B^*}^*\}$ are pairs of commuting normal operators, so that the proof is complete by the lemma.

Can quasinormality be replaced by subnormality (or further hyponormality) in Theorem 2 and Corollary 2? Partial and modest answers to this question are cited in [2], [3], [9]. Theorem 1 is a modest result and Corollary 3 is in this direction.

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Department of Mathematics,
Faculty of Science,
Hirosaki University,
Bunkyo-Cho 3,
036 Aomori, Japan.